A K-12 MATHEMATICS CURRICULUM WITH CAS: WHAT IS IT AND WHAT WOULD IT TAKE TO GET IT?

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Abstract

Computer algebra systems (CAS) in school mathematics are in their infancy. Yet we already know that they can enable many people to utilize algebra who in the past would be crippled by algebra. We have a moral obligation to utilize CAS if we are serious about bringing significant mathematical competence to all our students. However, to achieve this we must change societal attitudes towards algebra. We need to convince people that algebra is important for all students, that it can be learned by all students, and algebra is too important to be used as a gatekeeper. CAS enables us to be freed from a pasteurized and homogenized algebra in which the manipulation is restricted to simple coefficients and only includes forms amenable to paper-and-pencil work. A few examples are given of the use of CAS at the primary and secondary school levels.

I am interested in the best possible education that we can give to our children. I believe strongly that an educated person is more likely to make wise decisions than one who is not educated. I want our children to make wise decisions in their day-to-day activities. I want our children to understand their place in the world in both time and space, and to realize that events happen because of cause and effect, and sometimes due to randomness. I want our children to see the beauty of structure and form that permeates the universe and the noble inventions of people who have helped us see these structures and forms and, for some to contribute their own inventions and discoveries. I believe strongly that mathematics and the mathematical sciences are critical in all these broad goals and for that I want the best possible mathematics curriculum.

Computer algebra systems in school mathematics are in their infancy. But it helps perhaps to be reminded of how recently any calculator and computer technology has become available. I remember when the first hand-held calculators appeared in the United States in 1971; I was already a faculty member at the University of Chicago. I couldn't believe what I saw. Finally something to take the drudgery out of arithmetic! In the 1973-74 school year I wrote an algebra text entitled *Algebra Through Applications with Probability and Statistics* that used calculators. I felt then and still feel that the elementary school mathematics curriculum will ultimately embrace calculators, from kindergarten on up. Future generations will wonder how anyone could have been against their use. And so I spoke and wrote on the subject of "What Happens to Arithmetic Now that There Are Calculators?"

Twenty-one years ago, in 1983, I first spoke about CAS in a major talk. It was at a session at the annual meeting of our National Council of Teachers of Mathematics that no longer exists – the secondary level general session. I considered being invited to do that session as one of the greatest honors that could be given to someone in our field, and I ventured to give a talk that would lay out what I thought should be the high school mathematics curriculum. The title I had chosen a full year before giving the talk was: "We Need Another Revolution in School Mathematics".[1] I had no idea that, by the time I gave the talk, we knew we would start at the University of Chicago that fall the largest university-based curriculum project since the new math, the project that we titled the University of Chicago School Mathematics Project, or UCSMP for short.[2]

One of the reasons I gave for the revolution in that 1983 talk was the existence of muMath, a descendant of MACSYMA and the mother of Derive. I thought then that it would take only a few years before muMath would appear in a reasonably inexpensive and easy-to-use form, and that there would be some other similar systems. It turned out to be 12 years before the TI-92 appeared and another few years before Casio came out with a competitor. And so the UCSMP curriculum, even in its second edition, while being influenced by the existence of what we now call CAS software, could not assume students would have access to CAS, and only in the course *Precalculus and Discrete Mathematics*, designed to precede calculus, do we have sections using it.

In the meantime, my belief that technology helps in the learning of mathematics has not waned one iota. I believe that technology has to be a partner if we are serious as a society about bringing significant mathematical competence to all our students. If we truly wish to improve the use of mathematics in society, we have a moral obligation to further the use of instruments that can give so much power to people.

But, although in the world we herald the many advances in technology, in most mathematics classrooms we are not close to using technology to the extent that we ought to be using it. So in planning this talk I wavered between talking about the practical, political and philosophical issues and ideologies that are causing many people to question the use of technology if not delay its use altogether, or talking about what a curriculum thoroughly infused with CAS might resemble. In either case, I realized that there are so many experts here that I would likely say nothing that would be new. So I have decided to try speak about both of these ideas – a curriculum that uses CAS and the obstacles that would have to be surmounted before such a curriculum could be implemented on a wide scale in the United States. I will concentrate on grades K-12, and in particular, on the learning of algebra. I hope that my remarks will also be felt to apply to your countries and that some of what I say will be applicable to other branches of mathematics.

The algebra we teach

Let us begin by examining what we mean by "algebra". The algebra we teach to students can be traced back through the Greek mathematician Diophantus to the Babylonians over 3500 years ago, but the language we use in today's algebra is relatively new as mathematics goes, dating back only to the French mathematician Viète in 1591 and the systematization of the content by Euler in 1770. That language involves variables, usually represented by single letters of the Latin and Greek alphabets. These variables are used in a variety of ways, of which the following are the most common:

generalized arithmetic
unknowns
constants
parameters
arguments of functions
arguments of algorithms

placeholders

In the United States as recently as 1910, less than 15% of the age cohort entered high school and only they studied any algebra.[3] Through the 20th century the situation changed dramatically. By 1972, virtually all students entered high school and about 72% of them were taking one year of algebra. About half of these students took a second year of algebra. Algebra was still viewed as a high school subject and introduced before 9th grade only to perhaps 10% of students. But there has been a major change in the last 30 years. By 2003, algebra was being introduced to virtually all students before high school, over 90% of students were taking one year of algebra by the end of the 9th grade, and 2/3 of students were studying a second year of algebra in 10th or 11th grade.[4]

In the first year of the canonical algebra curriculum, we teach linear equations and inequalities, graphing of lines, operations with polynomial expressions (including factoring), laws of positive integer powers, linear systems, square roots, and maybe quadratics and rational expressions. In the second year, students are most likely to review first-year algebra and be taught rational exponents and nth roots, operations with matrices, general notions of functions, linear and quadratic functions, logarithms and exponential functions, sequences, and perhaps some combinatorics and probability.

Notice that I say that students are *taught* these subjects. I did not say that they *learn* them. Many colleges require these subjects but many students have to take remedial algebra courses in college. Despite these difficulties, there is a movement in the U.S. for all students, including those not planning to go to college, to take two years of algebra. What's more, many of the supporters of this movement scrupulously avoid any use of technology to do algebraic manipulation, and use graphing calculators only grudgingly. The logic is that many students have trouble with algebra, so let's require more of it in the hope that they might learn a little by struggling even more than they have struggled in the past.

Computer algebra systems take away the struggle and thus disrupt this logic, even though they can make it possible for algebra to be understood by students who never before understood the subject.

Societal attitudes towards algebra

Because CAS makes it possible for students to do algebra virtually automatically, its use with all students requires that we believe that all students should learn algebra. This in turn rubs against a very common belief in society that algebra is not for everyone. It is not for everyone either because (a) not everyone *needs* algebra, or (b) not everyone *can understand* the language of algebra (even if they can do it). That is, to many people in the west, algebra is like the Latin language. Today Latin is not spoken but is the language of Roman Catholic priests. Similarly, to many people algebra is the language of mathematical priests and others interested in the religion of mathematics and although certain words may come into our vocabulary, algebra is only a curiosity to most people. And calculus – it is like Babylonian cuneiform – undecipherable except by experts.

In contrast, people realize that they need to know arithmetic. Whole numbers, fractions, decimals, and percents are everywhere. Just pick up a newspaper or magazine, open to any page at random, and count the numbers on it. I have examined the uses of numbers in newspapers around the world and have found that in almost every country, a daily newspaper has a *median* of about 125 numbers on its pages.

Society's treatment of algebra is different. Scan a daily newspaper in any language and you are not likely to see any algebraic formulas. Items for sale in stores have lots of numbers on them, but no algebra. Even scientific publications meant for the public and with lots of numbers in them tend to have little if any algebra. Adults may be handicapped by a lack of knowledge of arithmetic, but lack of knowledge of algebra does not seem too debilitating.

When people are required to learn a subject that they feel they do not need, will not use, and that requires work to learn, there are predictable results. Many of them will dislike the subject and they will be proud of their dislike. No one makes fun of reading. No one makes fun of arithmetic. But people make fun of algebra (a cartoon will be shown). This attitude towards algebra is passed on from generation to generation.

And so, it seems to me that one way to get CAS into schools is to convince people that CAS changes algebra from being like a dead language to being like their mother tongue, that it makes algebra comprehensible**.** In this regard, we can say that we would like CAS to do for algebra what graphing calculators have done for functions. Graphing calculators have made it possible for all students to work with functions that only a certain few worked with before. We might argue that rational expressions and other complicated manipulations have been taken out of the curriculum in order to accommodate the increased number of students in our courses, and that CAS can help us get them back.

It is worth wondering why there is quite a difference between the public's view towards 4-function calculators and their view towards graphing calculators. Many in the public do not want simple calculators in elementary school because they feel that these calculators take away understanding. We know that calculators don't necessarily take away understanding, but it is a common view. On the other hand, there is a very tolerant view towards graphing calculators in high schools because the public never understood functions in the first place and they realize that graphs and lists and automatic calculation of function values do assist in an understanding that most people have never gotten without this technology. In order to get CAS into the mathematics curriculum, we must emphasize the use of CAS to do things that cannot easily be done without it. Specifically, CAS enables students to work with symbolic manipulation that we have avoided in the curriculum because of its difficulty.

Algebra has a reputation that is different from functions, even though we may think of functions as a part of algebra. Consider the mathematical language that has made its way into educated discourse: exponential growth, direct and inverse variation, rate of change, periodic nature, approaching asymptotically, extrapolation, and so on. All these words are associated with functions. Other language that is not necessary associated with functions has also made its way into discourse, including formula, unknown, and equation, but much of the language of what CAS does well has not made it: expressions such as binomial or polynomial; properties such as distributivity or commutativity; structures such as group or ring or field. The word *algorithm* has made it into educated discourse and it may be the word that should be highlighted to the public when touting CAS. CAS does algorithms better than anybody – better than any body.

The ease with which CAS does what the public perceives as algebra gets in the way of one of the main reasons for learning algebra, at least in the United States. Algebra is a gatekeeper. In the U.S., without a knowledge of algebra, a person cannot go to any college that has any selectivity. A person is kept from doing many jobs or even entering many job-training programs. Algebra is such a significant gatekeeper that one reformer, Bob Moses, has called the study of algebra a *civil right*.[5] Recently, algebra – not just arithmetic – is beginning to appear on high school graduation tests. So, for the first time in the United States, there are places where students cannot graduate without a knowledge of algebra. Yet I know of none of these exit exams that allows use of a calculator as powerful as a graphing calculator. When CAS is universally used, algebra is no longer a gatekeeper, so in order to get its use, the idea of algebra being a gatekeeper must be removed. The civil right is not merely the study of algebra, but the use of any available technology that might assist that study.

Thus I believe that three attitudes about algebra need to be changed if we are going to have CAS for all students: the belief that algebra is not needed by all students; the belief that algebra is too hard to be learned by all students; and the use of algebra as a gatekeeper. And to change those beliefs we need to stress that CAS makes some algebra accessible to all that has only been accessible to a few, and that CAS empowers students to do important mathematics that they could not do easily without it.

The reasons for learning algebra

What is that important mathematics? And how is CAS related to it? For this, it is helpful to examine algebra in some detail and to ask why algebra is so important that its study is required of all educated people worldwide. Allow me to review what you most probably have heard many times: the reasons why all people should learn algebra.

Algebra is the language of generalization. As a result, it enables a person to answer *all* the questions of a particular type at one time. Algebra is also the language of relationships among quantities. This makes algebra a powerful language for solving certain types of problems. Algebra is also the study of structures with certain properties.

Through its descriptions of generalizations and structures, algebra – more than any other branch of knowledge created by humans - shows that our universe possesses order. And because it has all these properties, algebra is the basis for virtually all mathematics and is a fundamental tool for dealing with the generalizations, relationships, problems, and structures in many other disciplines. This is why algebra is important.

Algebra in schools

Algebra in many classrooms does not at all present the picture of the vibrant, widely applicable subject that I have described. Instead of being taught as a living language with a logical structure and many connections between its topics and other subjects, algebra is taught as a dead language with a myriad of rules that seem to come from nowhere. That so many well-educated adults wonder why they studied algebra is testimony to the disjuncture between such an important subject and the way it is presented in schools. So if CAS is used in a traditional algebra classroom, taught without applications and with algebraic manipulations taught only for their own sake, it will not be much of an advance.

If algebra is so important, how is it that people can and do live without algebra? They do, but they cannot appreciate as much of what is going on around them. They may not be eligible for a job they would like to have or the training program or courses they would like to take. They cannot participate fully in our technological society. They are more likely to make unwise financial decisions and will find themselves with less control over their lives than others who know algebra well. They are like me when I travel to China or Japan or Malaysia. I am in the same country as those who can speak the native language, but I understand very little of what I hear, I do not understand as much of what I see, I miss out on many opportunities and even when I am present at an event, I do not appreciate what is going on around me as much of the natives.

In the past, when only a very small percent of the population needed to learn algebra, we could be content with the algebra course as a gatekeeper. But today, when a good deal of algebra knowledge is expected from anyone who wants to continue in school after high school graduation or plans to study any of a number of trades, we cannot afford to weed out so many students. Algebra needs to *turn on* students rather than turn them off. CAS in the curriculum must be exciting, and it must help students understand why they are learning algebra.

With all this in mind, let me now turn to the curriculum. I will begin with a few broad comments.

Today's pasteurized and homogenized algebra

The algebra that we teach in grades K-12 is purposely pasteurized and homogenized. By pasteurized I mean it is stripped of difficult coefficients and cleansed of non-integer solutions. By homogenized I mean that the problems are of certain predictable forms. So, for example, if a problem begins with the statement "A train leaves a station…", it places in the minds of both teacher and student an expectation about the type of problem to be solved. In a section on the solution of quadratic equations, we know that there will be certain kinds of problems: some in the form $ax^2 + bx + c = 0$; some in the form $a(x - h)^2 + k = 0$, and perhaps some easy variants of these forms. The letter x will be the most common variable for which we solve. There are rarely equations of the form $\frac{x+5}{2x-3} = \frac{9-x}{5x}$ or equations involving irrational coefficients or complex coefficients. And there are almost never equations for which there does not exist a standard algorithm.

By speaking of the pasteurization and homogenization of algebra, I do not mean to deride these ideas. There are good reasons that we use canonical forms, that we keep things simple, and that we *want* students to have things in mind when they read certain words or see certain symbols. But this cleansing and simplification of algebra means that, for the standard algebra that is taught to most students, there is not very much need for the power of CAS. It is as if we are teaching arithmetic using numbers with no more than 3 digits and we come in with a calculator that manipulates 12-digit numbers. The need for the power is not particularly clear.

Thus, in order to get CAS into the K-12 algebra curriculum, expectations have to change. And I do not mean just the conceptual expectations, such as an earlier emphasis on functions, but also the procedural expectations, namely, the algorithms students are expected to master.

And the last of my general observations is that we need to take into account the fact that technology itself needs to be learned. A child needs to begin with simple technology in the early grades and then gradually be introduced to more sophisticated technology in later grades.

I have split my comments on opportunities for CAS into four grade levels: K-3, 4-6, 7-9, and 10-12. Obviously I cannot describe anything close to what CAS might do at each of these levels. There is time for only one example at each level.

CAS at Grades K-3

Every child needs to have a piece of hardware – calculator or computer or whatever you want to call it – that has numbers and letters on it and that can graph. The technology needs a screen that one can write on just as one writes on a slate when signing for a charge at some stores these days. That is, we need to be able to record paper-and-pencil work as well as computer work. I mean " ree cord' " in the sense that any work the child does can be brought up as a " reh' cord".

Whatever is done at grades K-3 has to be a part of literacy. Fortunately, there is widespread software that can be viewed as rudimentary CAS: the spreadsheet. A large percentage of people who use a computer today work with spreadsheets. Most spreadsheet users do not think they are doing algebra when they use Excel and other spreadsheets because this does not look like school algebra. To most people, algebra is supposed to be hard. Algebra is supposed to be something you do not understand. But because spreadsheets do not look like algebra, elementary school teachers do not have the fear of them that they would have of technology that more obviously incorporates algebra.

But working with a spreadsheet *is* algebra. The names of the variables are AI, A2, A3, ..., B1, B2, ... Each time we ask the spreadsheet to calculate something and put it in another cell we are writing an algebraic formula. For example, if in cell C1 we type "= A1 + B1", then we are naming the variable C1. The spreadsheet does the calculation for us, of course. Each time that we copy a formula to other cells, we are creating a function. The function may be explicitly defined or it may be an iterative sequence, but it is a function either way. Most spreadsheet programs allow us to graph the explicitly defined functions. With either the graph or by successive approximation, we can solve many equations. Also, almost all spreadsheet programs have a form of summation notation that makes for a very easy transition to traditional □-notation. Sum(A1:A5) is so similar to $\sum_{i=1}^{5} A_i$ Thus algebra capability is in most every computer, on virtually every business desk. And people who think they know no algebra are using this capability.

There are many things spreadsheets can do. Charts can automatically change as entries are changed, thus showing variability and patterns. (This will be shown by a move to a spreadsheet program.) There is an entire talk that could be given here. The point is that the introduction of a spreadsheet is a very nice way to introduce CAS quite early. In the United States, the large commercial publishers already have rudimentary spreadsheets as a tool in the primary school programs available on line and this could spur them to upgrade that technology.

CAS in Grades 4-6

In grades 4-6, we can obviously do more. Here is a use of a variable as an argument in an algorithm. Think of an integer. Add 2. Now multiply it by 3. Now add 4. Now multiply your number by 2. Now subtract 8. Last, divide by 6 and tell me what you got. Then I can tell you what you started with (2 less than what you got at the end). Here the first number is a variable acting not as an unknown but as an input number in an algorithm. By putting the variable in a spreadsheet, it has become a placeholder. Then the variable becomes an unknown because I am playing a game.

The use of variables as inputs is perhaps the easiest use of variables for students to learn. Almost all students find it very easy to evaluate algebraic expressions. It is something for us to take advantage of.

A more conceptual use of placeholders has to do with place value. The notion that a two-digit number with tens digit t and units digit u has value $10t + u$ is difficult for many students. But with a spreadsheet it is easy.

CAS in Grades 7-9

As students get older, it gets easier to use CAS. I will do the next example without CAS. But I believe it to be a very rich problem whose richness is usually never seen by students and I believe CAS can bring out that richness in a variety of ways.

How many ounces of a 90% solution of alcohol should be added to 6 ounces of a 50% alcohol solution to create an 80% solution?

Students are typically expected to write and solve the equation

$$.50 \bullet 6 + .90x = .80(x + 6).$$

They solve as follows:

$$3 + .90x = .80x + 4.8.$$
$$.10x = 1.8$$
$$x = 18$$

So18 ounces should be added. This is where the problem usually stops. Some teachers are satisfied if their students can get the right answer, regardless of whether students know what they are doing.

Good teachers generally feel satisfied if students realize that the three products in the equation are the amount of alcohol at the beginning, the amount of alcohol added, and the final amount of alcohol. Few teachers ask the questions Polya would ask in the "looking back" step: Does the answer make sense? Can you check it? Can the solution be generalized? Are there other problems you can now solve?[6] This step is where CAS can enter the scene.

It is instructive to generalize the problem. Suppose m ounces of the 50% solution originally existed. Now how many ounces of the 90% solution are needed to change it to an 80% solution? The answer to this new question will not be a single number but an expression in m. We now modify what we had before, replacing 6 by m.

$$.50 \bullet m + .90x = .80(x + m),$$

from which

$$50m + .90x = .80x + .80m$$
$$.10x = .30m$$
$$m = 3x.$$

If we do not simplify so quickly to obtain the penultimate equation, we have

$$(.90 - .80)x = (.80 - .50)m.$$

Thus m is 3 times x because .50 is 3 times as far from the desired .80 as .90 is. Now we see that this is a problem of weighted averages. We can visualize the problem on a number line with .80 as the fulcrum. We have a weight of 6 units at .50 and an unknown weight at .90 that causes the number line to balance at .80. (Show diagram.) The products on the two sides of the equation become moments, the product of a weight and the distance of that weight from the fulcrum in order for the number line to balance at .80. The diagram shows this clearly. Thus I envision the use of CAS to show dynamically how the steps of the equation get modified as the equation changes coupled with a moving balance as the weight x changes. This turns a problem for which few students have any intuition into a problem whose solution becomes intuitive.

Now suppose that we generalize in a different but quite reasonable way. Again begin with 6 ounces of the 50% solution. If we add x ounces of 90% solution, we obtain 6+x ounces of a y% solution. How are x and y related?

Now the answer is a function. But what is this function? We already know some of its values. When x = 0, y = 50. When x = 6, then y = 70. When x = 18, y = 80. But what is a formula for y in terms of x? Now we solve 6•50 + x•90 = (6 + x)y, so y = $\frac{300+90x}{6+x}$. This formula enables us to ask how the percentage of alcohol in the solution changes as we add more and more of the 90% solution.

This is a rational function, but what kind of rational function? Here some algebraic manipulation can help. First we rewrite the formula as

$$6y + xy - 300 - 90x = 0.$$

We would like to factor the left side so that there is at most a constant left to be added to both sides. The left side will be the product of two binomials

$$(\qquad)(\qquad) = 0.$$

We need a product xy, so make x and y the first terms.

$$(x \qquad)(y \qquad) = 0$$

Now x has to be multiplied also by –90, and y has to be multiplied by 6. So the left side will be

$$(x + 6)(y - 90) =$$

and we find the right side by realizing that the product of the constant terms is now –540, which is 240 less than the constant in the formula. So

$$(x + 6)(y - 90) = -240.$$

We now see that the graph of this function is congruent to the graph of xy = -240, for it is the image of that graph under the translation that maps points 6 units to the left and 90 units up. The lines x = -6 and y = 90 are asymptotes.

We have already seen why y = 90 is an asymptote, for the percent of alcohol in the final solution gets closer to 90% as more and more of the 90% solution is added. Again a dynamic connection between the equation and the graph is useful. This can be done by graphing calculator technology, but if we change the 6 or the 90 we can see how the graph changes as the number of ounces or the strength of the added solution vary. The meaning of the asymptote at x = -6 may be a little harder to see. But it is due to the fact that we already had 6 ounces of a solution when we began.

This problem has other contexts. Suppose that a basketball player has a lifetime free throw percentage of 90%. Now suppose that the player has started off a new season by making only 3 of the first 6 free throws attempted. The solution we have just found also answers the question of how her percentage of free throws made changes if the player now starts making free throws at a 90% rate. In particular, the original problem tells us that after 18 more free throws at a 90% rate, the total percentage will be up to 80%. In the free-throw situation, the asymptote at x = –6 is telling us that there were 6 free throws attempted before we began; the y-intercept determines the accuracy rate before the player again starts making free throws at a 90% success rate.

Or we could use still another context. Suppose that a car has been traveling for 6 minutes at a speed of 50 km/hr. How long does the car need to travel at 90 km/hr in order to have an average speed of 80 km/hr for the total trip?[7]

I have belabored this example not just because I believe it is rich for CAS but also because I wish to emphasize that we need to use CAS to get insights into and beyond the mathematics we teach, and not just use CAS because it can do manipulation more quickly and accurately. And we must do things we could not do before. There is no sense using CAS for the algebraic equivalent of multiplying 11 by 13. Over the years, I have heard a number of speakers allude to how technology can improve paper-and-pencil skill. I know I may not be politically correct, but the goal of mathematics is not to me the ability to do things with paper and pencil. The goals of mathematics learning are competence with problems, understanding of the mathematics, communication of that mathematics to others, and the ability to use deduction to demonstrate the validity of mathematical results. Skill, obtained by any technology, paper-and-pencil or electronic, is only a means to those ends.

CAS in Grades 10-12

Once students have had some algebra and geometry, the opportunities for CAS greatly increase. The dynamic algebra is naturally applied in dynamic geometry software such as Cabri and Geometer's Sketchpad. CAS can nicely show the analytic geometry that is behind geometry drawing programs. It seems certainly a fruitful area for exploration.

Another area that is rich in history and used to be important in the education of future mathematicians is the area of polynomial solutions. In the UCSMP course *Precalculus and Discrete Mathematics* that I mentioned earlier, we show an example of how the real and non-real roots of a 4^{th} degree polynomial equation move in the complex plane as the constant term increases. There is rich mathematics here not just in the graphical representation of the roots but also in the relationships between the roots and the coefficients of the polynomial and in various methods used to approximate these roots. When I was a student a standard college mathematics course entitled "Theory of Equations" was devoted to this material. The manipulation required for this mathematics is formidable but now can easily be done.

The coda

I know I have only scratched the surface, but the time has come for the coda. In music, the coda to a movement of a sonata or symphony often recapitulates the themes.

Paper-and-pen and, more recently, paper-and-pencil, has been the technology that we have used for the past 400 years to do arithmetic. Paper-and-pen became dominant not because people understood why the algorithms work. People still don't understand! Paper-and-pencil won because its algorithms were more widely applicable. Now I believe we are in the same situation with computers and algebra. Ultimately, not in my lifetime but perhaps in some of yours, the computer algorithms embodied in spreadsheets and automatic graphers and CAS systems will become the algebra that everyone uses and recognizes. Spreadsheets will be employed to help introduce the language of algebra to students, and CAS systems will be used to perform the complex manipulations.

This does not mean that paper-and-pencil algebra will become obsolete. It will not become obsolete any more than mental arithmetic has become obsolete. People will still need to know how to solve simple equations and do simple manipulations by hand. And, perhaps more important, they will need to be able to translate from real and fanciful situations to mathematics and vice-versa. Algebra is more important than ever. CAS does not make school algebra obsolete, but school algebra needs to change to touch the everyday lives of students and their families in order to be relevant to all students, and in order for the public to be comfortable with technology that makes algebra accessible to all.

Many if not most mathematics teachers in the world do not have much experience with this algebra. Even some talented mathematicians and many of the best mathematics teachers in the United States and those most involved with technology do not view algebra in this light. So, in our task to have our students learn the best mathematics we can deliver, we have a number of jobs to do: to work out the best mathematics curriculum we can using widely available technology; to encourage the development of inexpensive technology and materials for teachers and students for that curriculum; to help our colleagues and future teachers become able to deal with these ideas; and to get messages to the public that will make them support our use of such a curriculum. We should be able to say: What happens to algebra now that there is CAS? We and our students can devote more time to the mathematical underpinnings, to the applications, and to the representations that provide a deeper understanding of the subject. What happens to algebra and other mathematics now that there is CAS? It gets better!

Notes

[1] Zalman Usiskin, "We Need Another Revolution in Secondary School Mathematics", in Christian Hirsch and Marilyn Zweng (eds.), *The Secondary School Mathematics Curriculum*", 1985 Yearbook of the National Council of Teachers of Mathematics (Reston, Virginia, USA: NCTM, 1985), pp. 1-21.

[2] The UCSMP curriculum for secondary schools is currently published by Prentice-Hall, Upper Saddle River, New Jersey, USA. For details, visit www.phschool.com and search UCSMP. for details.

[3] Claudia Goldin, The Human Capital Century, www.educationnext.org/20031/73.html.

[4] For a summary of curricula used in the United States, see Zalman Usiskin and John Dossey, *Mathematics Education in the United States: A Capsule Summary* (Reston, Virginia, USA: National Council of Teachers of Mathematics, 2004).

[5] Bob Moses directs a project called "The Algebra Project", designed to prepare inner city middle school students for algebra. The name of the project is somewhat misleading, because there is very little algebra in any of the materials of the project. The name is more a political statement about the importance of preparing inner city children for algebra. It reflects the difference between the view of algebra that exists in our suburban schools and the view of algebra in city schools. In the suburbs, algebra is viewed as the gateway to all of mathematics, a language understandable by virtually everyone. In the inner city, algebra is viewed as an arcane code needed by only a few.

[6] George Polya, *How To Solve It* (Princeton, New Jersey, USA: Princeton University Press, 1952).

[7] In-depth analyses of this type beginning with other standard problems can be found in Zalman Usiskin, Anthony Peressini, Elena Marchisotto, and Dick Stanley, *Mathematics for High School Teachers: An Advanced Perspective* (Upper Saddle River, New Jersey, USA: Prentice-Hall, 2003).