Maximizing or Minimizing Polynomials Using Algebraic Inequalities

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Abstract: In mathematics, several methods can be used to maximize or minimize polynomials and other types of functions. One such method uses techniques in calculus, such as the derivatives. However, there are several types of functions which can be maximized or minimized without resorting to methods of calculus. For example, quadratic functions can be maximized or minimized using the method of completing the square. In this paper, we will discover a brand-new method for optimizing certain types of functions. The new method that we will describe is based on a famous inequality in algebra, known as the Inequality of the Means. The proposed method can be considered as a surprising application of the above mentioned inequality. One can check the results obtained from this method by using calculus-based methods, or by using computer algebra systems such as *Mathematica*.

1. Introduction

We will first recall a well-known inequality in algebra, known as the Inequality of the Means or the Theorem of the Means (see [1] and [3]). Given a sequence of positive real numbers $a_1, a_{2,} ..., a_n$ where n is a positive integer, its arithmetic mean A and the geometric mean G are defined as follows:

$$A = \frac{a_1 + a_2 + ... a_n}{n} \tag{1.1}$$

$$G = \sqrt[n]{a_1 a_2 ... a_n} \tag{1.2}$$

The Inequality of the Means compares the above two means, A and G, as given below:

Theorem 1.1 (Inequality of the Means)
The arithmetic mean of a sequence of positive numbers $a_1, a_{2,} ..., a_n$ where n is a positive integer is greater than or equal to its geometric mean. In other words,

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n} \tag{1.3}$$

The equality occurs if and only if all the numbers a_i, $i = 1, 2, ..., n$ are equal to one another.

Proof. The proof is based on induction on n, the number of positive numbers. The complete proof can be found in [1]. ■

Example 1 The truth of the above theorem can easily be demonstrated for the case $n = 2$. In this case, the above theorem asserts that for any positive real numbers x and y, the following inequality is true, with equality holding if and only if $x = y$:

$$\frac{x+y}{2} \geq \sqrt{xy} \tag{1.4}$$

In order to see why this must be the case, note that (1.4) is equivalent to the inequality $(x+y)^2 \geq 4xy$, which in turn is equivalent to $(x-y)^2 \geq 0$. However, this last inequality is obviously true since x and y are real numbers. Also notice that $(x-y)^2 = 0$ if and only if $x = y$. Thus the inequality (1.3) is true with equality holding if and only if $x = y$. This proves Theorem 1.1 for the case $n = 2$.

The inequality (1.4) has a nice geometric interpretation: It is equivalent to the obvious geometric fact that the radius *DE* of any semicircle is greater than or equal to the length of a half-chord *CF*. See the following figure:

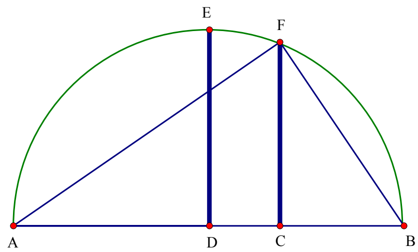

Figure 1.1 A Geometric Interpretation of the Inequality of the Means for $n = 2$

In order to explain this geometric connection, let us consider the line segment *AB*, with *AC* and *CB* representing the positive real numbers x and y respectively. Let *D* be the midpoint of the line segment *AB*. Therefore, *AD* represents the quantity $(x+y)/2$, which is the arithmetic mean of x and y. One now needs to construct the quantity $\sqrt{xy}$. With the point *D* as the center and *AB* as the diameter, construct a semi circle. Draw a perpendicular line through *C* meeting the semicircle at *F*. Observe that the two right triangles *ACF* and *BCF* are similar triangles. So using the properties of similar triangles, one obtains $AC/CF = CF/BC$, implying $CF = \sqrt{AC.BC} = \sqrt{xy}$. Also, through *D*, draw a line perpendicular to *AB* meeting the semicircle at *E*. Therefore, the arithmetic mean of x and y is represented by the segment *DE* while the geometric mean is represented by the segment *CF*. Since the length of the segment *DE* is equal to half of the diameter and the length of the segment *CF* is equal to the half of a chord, clearly *DE* must be greater than or equal to *CF*. Furthermore, the lengths of the segments *DE* and *CF* are equal if and only if the midpoint *D* coincides with the point *C*, i.e. if and only if $x = y$. ■

2. Optimizing Second Degree Polynomials Using the New Method

We are now in a position to illustrate our new method of optimization. Before describing the method for the general second-degree polynomial, we will first consider a specific example:

Example 2.1 Consider the second degree polynomial $f(x) = -2x^2 + 3x + 4$. The graph of f is a parabola opening downwards, since the coefficient of the second degree term is negative. Therefore, we know the function f has an absolute maximum (a local maximum as well). In order to find this maximum value, we proceed as follows: By ignoring the constant term of f, consider the function $g(x) = -2x^2 + 3x = x(3-2x)$. This means that $g(x)$ is a product of two factors, x and $(3-2x)$. We would like to use the Inequality of the Means (1.3) for these two factors. However, in doing so we would like the left-hand side of the inequality to be a constant, so that one can produce a least upper bound for the function *g*. Therefore, instead of the original factors x and $(3-2x)$, we would consider the modified factors $2x$ and $3-2x$, because the sum of these latter two factors is a constant. Also notice that for $0 < x < 3/2$, both quantities $2x$ and $3-2x$ are positive, so we are in a position to apply (1.3).

Thus, the Inequality of the Means (1.3) implies for $0 < x < 3/2$

$$\frac{2x + (3-2x)}{2} \geq \sqrt{2x(3-2x)} \tag{2.1}$$

where the equality occurs if and only if $2x = 3-2x$, i.e. if and only if $x = 3/4$. The inequality (2.1) implies that for any x such that $0 \leq x \leq 3/2$, $2x(3-2x) \leq (3/2)^2$, or equivalently, $x(3-2x) \leq 9/8$. The nature of the graph of *g* implies that for all real values of *x*, $g(x) = x(3-2x) \leq 9/8$ with equality occurring if $x = 3/4$. This immediately implies that for all real values of *x*, $f(x) = -2x^2 + 3x + 4 \leq (9/8) + 4 = 41/8$ with equality occurring if $x = 3/4$. Therefore, the maximum value of the function $f(x) = -2x^2 + 3x + 4$ is equal to 41/8 and this maximum is achieved when $x = 3/4$. ■

Note how different this method of optimizing the second degree polynomials is from the method of completing the square. The algebra involved in the new method is a lot simpler! Let us also check the answer to the previous problem using calculus: One can calculate that $f'(x) = -4x + 3$, so $f'(x) = 0$ if and only if $x = 3/4$. Using the First or Second Derivative Tests, it is clear that $x = 3/4$ corresponds to a local maximum, as well as an absolute maximum (see [4]). The maximum value is given by $f(3/4) = -2(3/4)^2 + 3(3/4) + 4 = 41/8$. This agrees with our previous calculations.

One can now illustrate the method for the general second degree polynomial: Consider an arbitrary second degree polynomial $f(x) = ax^2 + bx + c$ where *a*, *b*, and *c* are arbitrary real numbers with $a < 0$. The condition $a < 0$ guarantees that f has an absolute maximum. Without loss of generality, we will assume that $b \neq 0$, since for $b = 0$, the maximum value of *f* is trivially equal to c. By dropping the constant term "*c*" in the polynomial *f*, consider the polynomial $g(x) = ax^2 + bx = x(b + ax)$. We will consider two cases:

Case 1 ($b > 0$): As suggested by the foregoing example, we will use the Inequality of the Means (1.3) for the modified factors $-ax$ and $b + ax$, because the sum of these two factors is a constant, independent of *x*. Also notice that both quantities $-ax$ and $b + ax$ are positive for $0 < x < -b/a$. Then the Inequality of the Means (1.3) implies that for $0 < x < -b/a$

$$\frac{-ax + (b + ax)}{2} \geq \sqrt{-ax(b + ax)} \tag{2.2}$$

In the above (2.2), the equality holds if and only if $-ax = b + ax$, i.e. if and only if $x = -b/(2a)$. This implies that, for all real *x*-values $-ax(b+ax) \le (b/2)^2$, or equivalently $x(b+ax) \le -b^2/(4a)$ with equality occurring if and only if $x = -b/(2a)$.

Case 2 ($b < 0$): One can write $g(x) = x(b+ax) = -x(-b-ax)$. Note that the quantities ax and $-b-ax$ add to form a constant, and they are each positive for $-b/a < x < 0$. Therefore, the Inequality of the Means (1.3) implies that for $-b/a < x < 0$

$$\frac{ax + (-b-ax)}{2} \ge \sqrt{-ax(b+ax)} \tag{2.3}$$

In the above (2.3), the equality holds if and only if $ax = -b - ax$, i.e. if and only if $x = -b/(2a)$. Therefore, as in Case 1 we obtain that for all real *x*-values, $x(b+ax) \le -b^2/(4a)$ with equality occurring if and only if $x = -b/(2a)$.

Therefore, in either case one can say that for all *x*-values $f(x) = ax^2 + bx + c \;\le\; -b^2/(4a) + c = (4ac - b^2)/(4a)$. Therefore, the maximum value of the function f is equal to $(4ac - b^2)/(4a)$ and it is achieved when $x = -b/(2a)$.

A similar method can be used to find the *maximum* value of a second degree polynomial $f(x) = ax^2 + bx + c$ where *a*, *b*, and *c* are arbitrary real numbers with $a > 0$. The trick here is to consider the polynomial $-f(x)$ and to find its minimum value as described before.

The optimization method described in this section can be extended to other types of polynomials. In the next section, we will attempt to optimize higher degree polynomials using the new method, provided the maximum or minimum values exist.

3. Optimizing Certain Higher Degree Polynomials

In this section we will attempt to find the maximum or minimum values of higher degree polynomials using the Inequality of the Means, provided they exist. Rather than challenging ourselves with the general *n*th degree polynomial, we will first consider several types of specific higher-degree polynomials. In the process, some interesting strategies will be observed!

Example 3.1 Consider the third degree polynomial $f(x) = x^2(1-x)$. The zeros of f are clearly, $x = 0$ with multiplicity two (even) and $x = 1$ with multiplicity one (odd). The graph of f can be roughly constructed as follows by considering the nature of the multiplicities of the zeros:

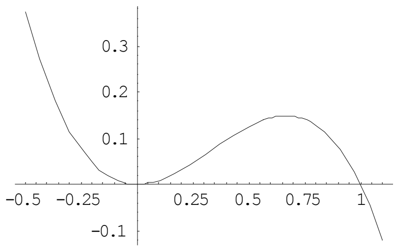

Figure 3.1 The Graph of $f(x) = x^2(1-x)$

The above function f has an absolute maximum for $0 \le x \le 1$. Our task is to find this maximum without using the methods of calculus:

The idea is to break the product $x^2(1-x)$ into several components, and use the inequality (1.3). However, as we set up the inequality, we want the left-hand side of (1.3) to be constant. Let us try to break the product $x^2(1-x)$ in to three components "x", "x", and "$1-x$" with suitable coefficients. Proceeding this way, for any $0 < x < 1$ we obtain the following inequality, by using (1.3):

$$\frac{x + x + 2(1-x)}{3} \ge \sqrt[3]{2x^2(1-x)} \tag{3.1}$$

In the above, the equality holds if and only if $x = 2(1-x)$, i.e. if and only if $x = 2/3$. Note that we have chosen the coefficients 1, 1, and 2 for the components "x", "x", and "$1-x$", so that the left-hand side of the inequality (3.1) is a constant. Using this constant left-hand side, we will able to manufacture a least upper bound for $f(x)$ for $0 \le x \le 1$. The inequality (3.1) implies that for any $0 \le x \le 1$, $2x^2(1-x) \le (2/3)^3$, or equivalently, $x^2(1-x) \le 4/27$, with equality holding if and only if $x = 2/3$. Therefore, the maximum value of the function $f(x) = x^2(1-x)$ for $0 \le x \le 1$ is 4/27, and this maximum value is achieved when $x = 2/3$. ■

Let us check the above answers using calculus: Since $f(x) = x^2(1-x) = x^2 - x^3$, it follows that $f'(x) = 2x - 3x^2$. Therefore, $f'(x) = 0$ if and only if $x = 0$ or $x = 2/3$. Using the derivative tests, one can show that $x = 0$ corresponds to a local minimum while $x = 2/3$ corresponds to a local maximum. It is not hard to show that for $0 \le x \le 1$, an absolute maximum for f occurs at $x = 2/3$. Therefore, this absolute maximum value is given by $f(2/3) = (2/3)^2(1-2/3) = 4/27$. These agree with the results obtained in the above Example 3.1.

One can also use a computer algebra system such as *Mathematica* to check our answers (see [5]). The *Mathematica* command "**FindMinimum**" numerically calculates the minimum of a variety of functions. The following *Mathematica* commands enable us to find the maximum of the function $f(x) = x^2(1-x)$ for $0 \le x \le 1$. The idea is to use the "**FindMinimum**" command for the negative of the given function. For example the command **FindMinimum[-f[x], {x, 1/2, 0, 1}]** calculates the minimum value of the function $-f(x)$ between the x-values 0 and 1 starting with the seed $x = 1/2$.

Program 3.1

```
f[x_]:= x^2
FindMinimum[ -f[x], {x, 1/2, 0, 1} ]
```

The above command can be executed by pressing "**Shift-Enter**". The output is $\{\{-0.148148, \{x \to 0.666667\}\}$. Reverting back to the original function, this means that the approximate *maximum* value of the function $f(x) = x^2(1-x)$ for $0 \le x \le 1$ is 0.148148 and this occurs at $x = 0.666667$. These values agree with the answers obtained in Example 3.1 because $4/27 \approx 0.148148$ and $2/3 \approx 0.666667$.

Remark One must be careful in using the *Mathematica* command "**FindMinimum**". To illustrate the point, let us consider the function $f(x) = x(x-1)(x-2)(x-4)$ whose graph is given below.

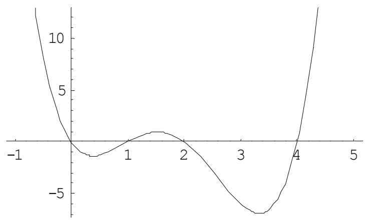

Figure 3.2 The Graph of $f(x) = x(x-1)(x-2)(x-4)$

Suppose we want to find the absolute minimum value of f on the interval $[0, 4]$. One must bear in mind that "**FindMinimum**" command normally produces local minimum values. Depending on the initial seed used, one might get different local minimums. For the above function, with the initial seed $x = 1.54$, "**FindMinimum[f[x], {x, 1.54, 0, 4}]**" produces $\{-1.38275, \{x \to 0.392748\}\}$ as the output. However, -1.38275 is not the absolute minimum of the function on the interval $[0,4]$. However, with the initial seed $x = 1.55$, "**FindMinimum[f[x],{x,1.55,1,4}]**" produces $\{-6.9141, \{x \to 3.32635\}\}$ as the output. The value -6.9141 is a local minimum, but is also the absolute minimum of the function f on the interval $[0, 4]$. Therefore, one must be careful interpreting the answers obtained from "**FindMinimum**" command. In general, technology gives us enormous power to do mathematics, but one must be careful interpreting the results.

Example 3.2 As suggested by Example 3.1, one can use the Inequality of the Means to find the minimum or maximum values of any function of the type $f(x) = k(a-x)^n (b-x)^m$ where *m*, *n* are arbitrary positive integers, *a*, *b* are any two distinct real numbers, and *k* is any nonzero real number.

As another example of this type, consider finding the maximum or minimum values of the function $f(x) = (1-x)^3 (4-x)^2$. As suggested by the multiplicities of its roots, its graph is given below:

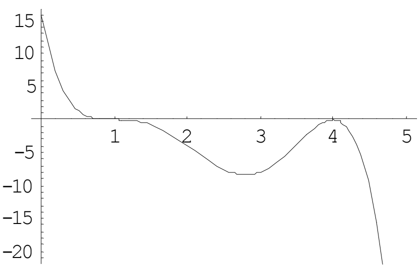

Figure 3.3 The Graph of $f(x) = (1-x)^3 (4-x)^2$

Suppose we want to find the absolute minimum of f on the interval [1,4] without calculus methods. The problem is equivalent to finding the absolute maximum of the function $-f(x) = (x-1)^3 (4-x)^2$ on the interval $[1,4]$. Using the key inequality (1.3), for any $1 < x < 4$ we obtain the following:

$$\frac{(2/3)(x-1)+(2/3)(x-1)+(2/3)(x-1)+(4-x)+(4-x)}{5} \geq \sqrt[5]{(2/3)^3(x-1)^3(4-x)^2} \quad (3.2)$$

In the above (3.2), the equality occurs if and only if $(2/3)(x-1)=(4-x)$, i.e. if and only if $x=14/5$. The inequality (3.2) implies that for any $1 \le x \le 4$, $(x-1)^3(4-x)^2 \le (6/5)^5(3/2)^3 = 26244/3125$. Therefore, for any $1 \le x \le 4$, $f(x) \ge -26244/3125$, with the equality occurring if and only if $x=14/5$. In other words, the absolute minimum of the function $f(x)=(1-x)^3(4-x)^2$ over the interval [1,4] is equal to $-26244/3125$ and this value is achieved when $x=14/5$. ■

The "**FindMinimum**" command of *Mathematica* is indeed a convenient way of checking the above answers. The output for " **FindMinimum[(1-x)^3(4-x)^2, {x, 2, 1, 4}]** " is $\{-8.39808, \{x \to 2.8\}\}$ which agrees with our answers.

Example 3.3 Let us consider the problem of finding the maximum or minimum values of the function $f(x)=x(4-x^2)$ by using the Inequality of the Means.

As suggested by its graph, the above function f has an absolute maximum on the interval $[0,2]$, and suppose we want to find it. The technique described in Examples 3.1 and 3.2 is not helpful here because of the "x^2" term in the factor $4-x^2$. Let us consider the square of the function $f(x)$ and try to maximize it. Therefore, consider the function $g(x)=x^2(4-x^2)^2$ on $[0,2]$. The inequality (1.3) implies that for any $0<x<2$, we have the following:

$$\frac{2x^2+(4-x^2)+(4-x^2)}{3} \geq \sqrt[3]{2x^2(4-x^2)^2} \quad (3.3)$$

In the above, the equality occurs if and only if $2x^2 = 4-x^2$, i.e. if and only if $x=2/\sqrt{3}$ (Note that the other negative solution does not lie between 0 and 2). The inequality (3.3) implies that for any $0 \le x \le 2$, $x^2(4-x^2)^2 \le 256/27$ with equality occurring if and only if $x=2/\sqrt{3}$. By taking the square roots, it follows that the absolute maximum of the function $f(x)=x(4-x^2)$ on $[0,2]$ is $16/(3\sqrt{3})$ and this maximum value is achieved when $x=2/\sqrt{3}$. These answers can be again confirmed by the *Mathematica* command "**FindMinimum**". ■

Example 3.4 One can use the idea used in Example 3.3 to optimize any function of the type $f(x)=k\,x^m(a-x^n)^l$ where *l*, *m*, and *n* are positive integers, *a* is a positive real number, and *k* is any nonzero real number.

As a specific example, let us consider the function $f(x)=x^4(2-x^3)^2$. The roots of f are $x=0$ with multiplicity 4 (even), and $x=\sqrt[3]{2}$ with multiplicity 2 (even). As suggested by the graph of *f*, f has an absolute maximum over the interval $[0, \sqrt[3]{2}]$ and we wish to find it. Learning from the Example 3.3, let us raise the function *f* to the third power, because of the term "x^3 " in the factor $(2-x^3)$. So let $g(x)=[f(x)]^3 = x^{12}(2-x^3)^6$, and find the absolute maximum of the function *g* on the interval $[0, \sqrt[3]{2}]$. The inequality (1.3) implies the following for any $0<x<\sqrt[3]{2}$:

$$\frac{(3/2)x^3 + (3/2)x^3 + (3/2)x^3 + (3/2)x^3 + \sum_{i=1}^{6}(2-x^3)}{10} \geq \sqrt[10]{(3/2)^4 g(x)} \qquad (3.4)$$

In the above, the equality occurs if and only if $(3/2)x^3 = 2 - x^3$, i.e. if and only if $x = \sqrt[3]{4/5}$. Using the inequality (3.4), it is not hard to show that the absolute maximum of the function $f(x) = x^4(2-x^3)^2$ on the interval $[0, \sqrt[3]{2}]$ is equal to $(144/125)\sqrt[3]{4/5}$ and it occurs at $x = \sqrt[3]{4/5}$. These answers indeed agree with those obtained using the "**FindMinimum**" command of *Mathematica*. ■

To summarize, in this section we learnt how to optimize the following two types of polynomials, using the Inequality of the Means.

I $f(x) = k(a-x)^n(b-x)^m$ where *m*, *n* are arbitrary positive integers, *a*, *b* are any two distinct real numbers, and *k* is any nonzero real number. It can be shown that for *x*-values between *a* and *b*, the absolute extremum of *f* occurs at $x = (ma+nb)/(m+n)$.

II $f(x) = k\,x^m(a-x^n)^l$ where *l*, *m*, and *n* are positive integers, *a* is a nonzero positive real number, and *k* is any nonzero real number.

However, the above are not the only types of polynomials that can be optimized by using the Inequality of the Means. The next section carries more information.

4. Optimizing Third Degree Polynomials

In this section we will consider the general third degree polynomial $f(x) = ax^3 + bx^2 + cx + d$ where *a*, *b*, *c*, *d* are real constants with $a \neq 0$. It is very interesting to observe that under a very simple condition, *f* can always be optimized without explicitly computing the derivatives! We will summarize this discovery in the following theorem.

Theorem 4.1 Consider the general third degree polynomial $f(x) = ax^3 + bx^2 + cx + d$ where *a*, *b*, *c*, *d* are real constants with $a \neq 0$. If $3a^2 - b^2 < 0$, the local maximum and minimum values of *f* can be found by using the Inequality of the Means, without explicitly computing the derivatives.

Proof Let us consider the reduced cubic polynomial given by $g(x) = f(x - b/(3a))$ (see [1]). One can readily check that *g* is given by the following:

$$g(x) = ax^3 + x[c - b^2/(3a)] + [d - bc/(3a) + 2b^3/(27a^2)] \qquad (4.1)$$

Since the graph of *g* is just a horizontal translation of the graph of *f*, they share the same optimum values, if such exist. In order to optimize the function *g*, ignore its constant term to consider the polynomial *h* given by the following:

$$h(x) = ax^3 + x[c - b^2/(3a)] = ax[x^2 + (3ac - b^2)/(3a^2)] \qquad (4.2)$$

If $3ac-b^2<0$, then the polynomial h given by (4.2) has three distinct real roots, 0 and $\pm\alpha$, where $\alpha=\sqrt{(b^2-3ac)}/(a\sqrt{3})$, each with multiplicity one. Therefore, by inspecting its graph, it will have a local maximum as well as a local minimum. We can find both of them using the Inequality of the Means. We will assume that $a>0$, since the case $a<0$ is very similar. As implied by its graph, the function $h(x)=ax(x^2-\alpha^2)$ has an absolute maximum on $[-\alpha,0]$ and suppose we want to find it. See the diagram below:

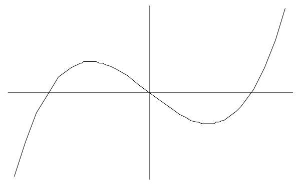

Figure 4.1 The Graph of $h(x)=ax(x^2-\alpha^2)$ where $a>0$

We will use the same technique as in Example 3.3. Square the function h to consider the new function $h^2(x)=a^2x^2(\alpha^2-x^2)^2$. For any $-\alpha<x<0$, using (1.3) we obtain the following inequality:

$$\frac{2x^2+(\alpha^2-x^2)+(\alpha^2-x^2)}{3}\geq\sqrt[3]{2x^2(\alpha^2-x^2)^2} \quad (4.3)$$

In the above the equality occurs if and only if $2x^2=\alpha^2-x^2$, i.e. if and only if $x=-\alpha/\sqrt{3}$. The inequality (4.3) implies that for any $-\alpha\leq x\leq 0$, $x^2(\alpha^2-x^2)^2\leq 4\alpha^6/27$. By taking the square root, we obtain that for any $-\alpha\leq x\leq 0$, $ax(x^2-\alpha^2)\leq 2a\alpha^3/(3\sqrt{3})=2(b^2-3ac)^{3/2}/(27a^2)$ with equality occurring if and only if $x=-\alpha/\sqrt{3}$. This means that the absolute maximum of the function $h(x)$ over the interval $[-\alpha,\ 0]$ is equal to $2(b^2-3ac)^{3/2}/(27a^2)$ with the maximum value occurring if $x=-\alpha/\sqrt{3}$. But the functions g and h only differ by the constant $d-bc/(3a)+2b^3/(27a^2)$, as seen by (4.1) and (4.2). Thus the absolute maximum for the function $g(x)$ over the interval $[-\alpha,\ 0]$ is equal to $[2b^3+2(b^2-3ac)^{3/2}]/(27a^2)+d-bc/(3a)$ where this value is achieved when $x=-\alpha/\sqrt{3}$. However, since $g(x)=f(x-b/(3a))$, the function g is a horizontal translation of f by $b/(3a)$ units. Therefore, the absolute maximum of f is still equal to the quantity $[2b^3+2(b^2-3ac)^{3/2}]/(27a^2)+d-bc/(3a)$, but over the interval $[-\alpha-b/(3a),\ -b/(3a)]$, and this value is achieved when $x=-\alpha/\sqrt{3}-b/(3a)$. By inspecting the nature of the graphs of h, g, and f, this discussion means that the function f has a local maximum at $x=-\alpha/\sqrt{3}-b/(3a)=(-b-\sqrt{b^2-3ac})/(3a)$ where the local maximum value is equal to $[2b^3+2(b^2-3ac)^{3/2}]/(27a^2)+d-bc/(3a)$, assuming $a>0$. Similarly, one can also find the local minimum of f. ■

Because of the space limitations of the paper, we are unable to include a concrete example to illustrate above Theorem 4.1.

Conclusion In this paper, we discussed one of the most surprising applications of the Inequality of the Means, i.e. as a tool for optimizing certain types of functions. As illustrated by our examples, we used more than one strategy in achieving the task. Thus, one can optimize many types of functions without using the methods of calculus. Even though this paper only concerns with polynomials, the author has investigated other types of functions for which the method works as well. These results will be published elsewhere.

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