TEACHING GRAPHS OF FUNCTIONS IN THE CARTESIAN PLANE: SHOULD WE RE-THINK OUR APPROACH IN SECONDARY MATHS?

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Abstract. At a time when technology is available to help in the teaching, learning and understanding of mathematics I express some concern that such technologies are being used to reinforce traditional methods in ways that often appear to be inconsistent and illogical. Some graphic calculators in current use appear to perpetuate this problem. The seeming inconsistencies pervade much of the mathematics taught in secondary schools and are a potential source of confusion. Many students see mathematics as a series of rules and procedures to be memorised. The end result is that students dislike mathematics and are often turned off the subject. One example of this is the current approach to the teaching of functions and graphs in the Cartesian plane. Much of the structure and nature of the mathematics taught in secondary schools relies upon a clear understanding of the nature of graphs in the Cartesian plane. So why is it that many students have so much difficulty with this work? One possible explanation is that many of us teach as we ourselves were taught and we only use that technology which enhances and supports the way we, the teachers, learnt to "understand" the mathematics. This paper suggests a possible alternative approach to the way in which graphs in the co-ordinate plane are currently taught in Secondary Schools. What is proposed, though not new, will require a re-think of the way we currently teach functions and their graphs. The approach suggested provides a unifying link between Coordinate Geometry, Geometric Transformations, Matrix Transformations, the Multiplication of Matrices, Vectors, and Transformation of Functions. It is presented in a way that is consistent and easily pictured by the students. By using appropriate technology/software, this presentation will attempt to demonstrate how a more meaningful approach to the concept of functions and their graphs can be adopted in the Secondary mathematics classrooms. The relationship between coefficients of various functions and the associated Transformations is, hopefully, made clearer.

1. THE CARTESIAN CO-ORDINATE SYSTEM – A Two-Dimensional Model $\mathbb{R}^2$

This paper suggests one possible approach for introducing the plotting of points and functions in the Cartesian plane. The suggested approach requires the use of a computer with suitable software.

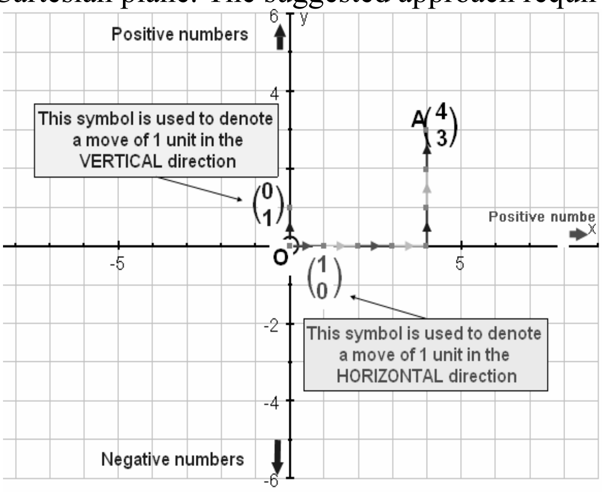


Figure 1

As in the conventional Cartesian co-ordinate system the horizontal axis is labelled the *x*-axis, and the vertical axis is labelled the *y*-axis.
The **unit** along the (horizontal) *x*-axis is labelled

$\begin{pmatrix}1\\0\end{pmatrix}$. This may also be written as $\overset{x}{\begin{pmatrix}1\\0\end{pmatrix}}$

This format suggests a *horizontal move* with no vertical move.

The **unit** along the *y*-axis is labelled $\begin{pmatrix}0\\1\end{pmatrix}$ or $\overset{y}{\begin{pmatrix}0\\1\end{pmatrix}}$

This format suggests a *vertical move* only.
This convention will be used throughout this document to describe the *x* and *y* units.

The coordinates of any point in the plane will be given as a pair of numbers in the column form $\begin{pmatrix}x\\y\end{pmatrix}$

This is traditionally written as **(*x*,*y*)**. but for now we shall stay with using $\begin{pmatrix}x\\y\end{pmatrix}$!

The symbol $\mathbb{R}^2$ indicates that we are in a system where all elements are *pairs of real numbers*.

Any given point P$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$ can be mapped *uniquely* to a fixed point in the Cartesian plane. Starting from the origin any point can be located by moving horizontally x units, followed by a vertical movement of y units. The first number of the pair x, indicates the number of x-units $\overset{x}{\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}}$ while the second number in the pair, y, indicates how many y-units $\overset{y}{\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}}$ are used to locate the point.

Example: To locate the point A$\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ in the Cartesian plane using this approach: (shown in figure 1),

The process, as can be summarised as

$$\boxed{4\times\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} + 3\times\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{4} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{3} \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}}$$

Since $\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$ are the basic units in the Cartesian coordinate system we shall denote the use of the Cartesian system by the *matrix* $\begin{pmatrix} \overset{x}{\mathbf{1}} & \overset{y}{\mathbf{0}} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$. This fits well with the use of *ordered pairs* such as (x,y) or (p,q) to represent oints in the plane. Thus the point $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$, [or alternatively $(x,y) = (-5,2)$], could be viewed as follows:

$$\begin{aligned}
\begin{pmatrix} \overset{x}{\mathbf{1}} & \overset{y}{\mathbf{0}} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}\begin{pmatrix} -5 \\ 2 \end{pmatrix} &= -5\overset{x}{\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}} + 2\overset{y}{\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}} \\
&= \begin{pmatrix} -5\times\mathbf{1} \\ -5\times\mathbf{0} \end{pmatrix} + \begin{pmatrix} 2\times\mathbf{0} \\ 2\times\mathbf{1} \end{pmatrix} \\
&= \begin{pmatrix} -5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} -5 \\ 2 \end{pmatrix}
\end{aligned}$$

That is $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ indicates that you must move as follows:

- ***Horizontally -5(x-units).*** That is $(-5)\times\overset{x}{\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}}$

followed by

- ***a vertical movement of 2(y-units).*** That is $(2)\times\overset{y}{\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}}$

The final position is the "*unique*" point A$\begin{pmatrix} -5 \\ 2 \end{pmatrix}$

Colour 'coding' is used throughout to help students to follow and understand what is taking place!
In the initial stages (Year 8), this approach will make it easier for students to correctly plot points.
Some later flow-on consequences of using this approach include:

- *The concept of the identity matrix is introduced quite naturally and in a meaningful context.*
- *The operation of "multiplying two matrices" will be easier to develop and explain. This should result in a better understanding of the multiplication process, than that offered by the conventional 'definition' approach so often used to introduce or teach matrix multiplication.*
- *The introduction of vectors and the use of vector notation flow on from this approach.*

2. HORIZONTAL & VERTICAL STRETCHES (SCALING FACTORS)

(2.1) Horizontal Stretch by scaling factor b: A Geometric approach.

Figure 2 shows the *parent* or *base* geometric figure ABCDE which is to be stretched horizontally.

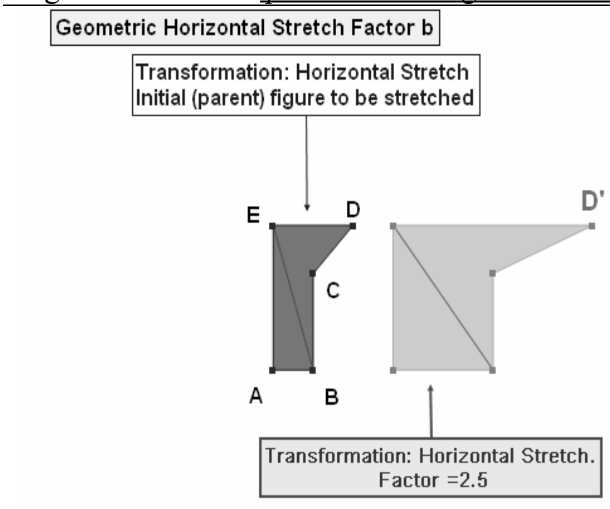


Figure 2

Experiment with different stretch factors **b**.

Compare the horizontal "lengths" of the stretched figure with those of the parent figure

Investigate what happens when the stretch factor is:

- **(a)** A value greater than 1?
- **(b)** A fraction between 0 and 1? (a shrink)
- **(c)** A negative number?
- **(d)** What occurs when the stretch factor is:

  (i) 1?0 (ii) 0? (iii) 1 ≤b≤ 0?

Note that vertical distances remain fixed or invariant.

(2.2) The Algebraic Analysis of the horizontal stretch by factor b (where b > 0)

If the horizontal or ***x-stretch factor*** is **2** then, using the notation introduced above:
Each x-unit is now stretched horizontally by a factor of 2.

Thus the x-unit $\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$ now becomes 2 x $\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ \mathbf{0} \end{pmatrix}$ while the y-unit $\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$ stays the same.

The coordinates of any points **P** $\begin{pmatrix} p \\ q \end{pmatrix}$ of the figure to be stretched are found by applying the

transformation $\begin{pmatrix} \mathbf{2} & 0 \\ \mathbf{0} & 1 \end{pmatrix}$ to the point P $\begin{pmatrix} p \\ q \end{pmatrix}$. This transformation process is written as $\begin{pmatrix} \mathbf{2} & 0 \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$

An ***x*-stretch** (with scaling factor 2) on the point $\begin{pmatrix} p \\ q \end{pmatrix}$ could be described algebraically as follows:

$$\begin{pmatrix} \mathbf{2} & 0 \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix} = \boldsymbol{p}\begin{pmatrix} \mathbf{2} \\ \mathbf{0} \end{pmatrix} + \boldsymbol{q}\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{p}\times 2 \\ \boldsymbol{p}\times 0 \end{pmatrix} + \begin{pmatrix} \boldsymbol{q}\times 0 \\ \boldsymbol{q}\times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times \boldsymbol{p} + 0\times \boldsymbol{q} \\ 0\times \boldsymbol{p} + 1\times \boldsymbol{q} \end{pmatrix}$$

$$= \begin{pmatrix} 2\boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix}$$

$\begin{pmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$ is called the *Transformation matrix*.

For this particular horizontal scaling this matrix contains the full details of the x-scaling transformation:
Namely ***a horizontal stretch factor of 2.***

If $\begin{pmatrix} p \\ q \end{pmatrix} \in y = \mathrm{f}(x)$ then the transformation matrix $\begin{pmatrix} \mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$ maps the function $\boldsymbol{y} = \mathbf{f}(\boldsymbol{x})$ to $y = f(\frac{x}{b})$

In figure 2 enter the *x*–stretch transformation matrix in the form $\begin{pmatrix} \overset{x}{\mathbf{b}} & \overset{y}{0} \\ \mathbf{0} & 1 \end{pmatrix}$ then vary the value of **b**.

Compare the *geometric stretch* with the corresponding stretch using the *transformation matrix*. The connection between the two provides the link to the algebraic analysis of the process..

Example: 1 To analyse the *x*-stretch transformation *algebraically,* superimpose a Cartesian grid onto figure 2. This is shown in figure 3.

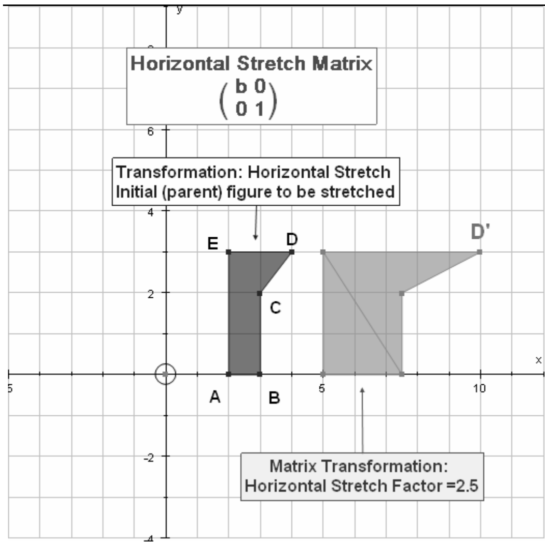


Figure 3

Using an *x*-stretch (or dilation) factor of 2.5:

$$\begin{pmatrix} \overset{x}{\mathbf{2.5}} & \overset{y}{\mathbf{0}} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4\begin{pmatrix} \overset{x}{\mathbf{2.5}} \\ \mathbf{0} \end{pmatrix} + 3\begin{pmatrix} \overset{y}{\mathbf{0}} \\ \mathbf{1} \end{pmatrix}$$

$$= \begin{pmatrix} 4 \times 2.5 \\ 4 \times 0 \end{pmatrix} + \begin{pmatrix} 3 \times 0 \\ 3 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2.5 \times 4 + 0 \times 3 \\ 0 \times 4 + 1 \times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ 3 \end{pmatrix}$$

The point D$\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is transformed to D'$\begin{pmatrix} 10 \\ 3 \end{pmatrix}$

That is **4 (*x*-units)** or $4\times\begin{pmatrix} \mathbf{2.5} \\ \mathbf{0} \end{pmatrix}$ followed by **3 (*y*-units).**

If this same transformation is carried out in turn on each of the vertices A, B, C, D, E the result is

$$\begin{pmatrix} \overset{x}{\mathbf{2.5}} & \overset{y}{0} \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} \overset{A}{2} & \overset{B}{3} & \overset{C}{3} & \overset{D}{4} & \overset{E}{2} \\ 0 & 0 & 2 & 3 & 3 \end{pmatrix} \xrightarrow[T]{} \begin{pmatrix} \overset{A'}{5} & \overset{B'}{7.5} & \overset{C'}{7.5} & \overset{D'}{10} & \overset{E'}{5} \\ 0 & 0 & 2 & 3 & 3 \end{pmatrix}$$

Compare the co-ordinates of ABCDE with the co-ordinates of the transformed *image* A'B'C'D'E'

Example: 2 A *horizontal stretch factor of 3* is applied to rectangle ABCD in figure 4. Use the same steps as outlined in **example 1** above. The rectangle ABCD is transformed into the figure A'B'C'D'. Each vertex of the parent figure can have its transformation described algebraically as follows:

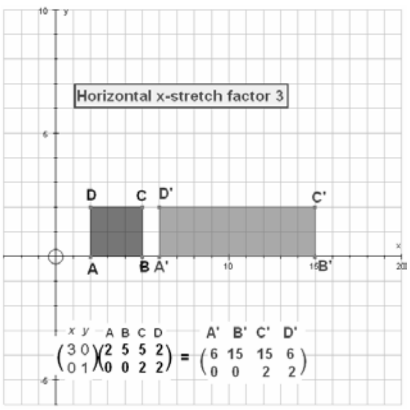


Figure 4

$$\begin{pmatrix} \overset{x}{\mathbf{3}} & \overset{y}{0} \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} \overset{A}{2} \\ 0 \end{pmatrix} = 2\begin{pmatrix} \mathbf{3} \\ \mathbf{0} \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{3}\times 2 \\ \mathbf{0}\times 2 \end{pmatrix} + \begin{pmatrix} 0\times 0 \\ 1\times 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \overset{A'}{6} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \overset{x}{\mathbf{3}} & \overset{y}{0} \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} \overset{B}{5} \\ 0 \end{pmatrix} = 5\begin{pmatrix} \mathbf{3} \\ \mathbf{0} \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\times 5 + 0\times 0 \\ 0\times 5 + 1\times 0 \end{pmatrix} = \begin{pmatrix} 15 + 0 \\ 0 + 0 \end{pmatrix} = \begin{pmatrix} \overset{B'}{15} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \overset{x}{\mathbf{3}} & \overset{y}{0} \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} \overset{C}{5} \\ 2 \end{pmatrix} = 5\begin{pmatrix} \mathbf{3} \\ \mathbf{0} \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{3}\times 5 + 0\times 2 \\ \mathbf{0}\times 5 + 1\times 2 \end{pmatrix} = \begin{pmatrix} 15 + 0 \\ 0 + 2 \end{pmatrix} = \begin{pmatrix} \overset{C'}{15} \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} \overset{x}{\mathbf{3}} & \overset{y}{0} \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} \overset{D}{2} \\ 2 \end{pmatrix} = 2\begin{pmatrix} \mathbf{3} \\ \mathbf{0} \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\times 2 + 0\times 2 \\ 0\times 2 + 1\times 2 \end{pmatrix} = \begin{pmatrix} 6 + 0 \\ 0 + 2 \end{pmatrix} = \begin{pmatrix} \overset{D'}{6} \\ 2 \end{pmatrix}$$

The result of the *matrix transformation* on each vertex of the figure ABCD is shown in figure 4

(By using this *visual* approach students should gain insight into an understanding of the manner by which two matrices are multiplied, without the need for the more formal definition).

(2.3) Horizontal Stretch (along the x-axis) by scaling factor "b". applied to functions

Consider the function $y = f(x)$, defined by $\boldsymbol{y = x}$, under the transformation**: *x*-scale factor 3.**

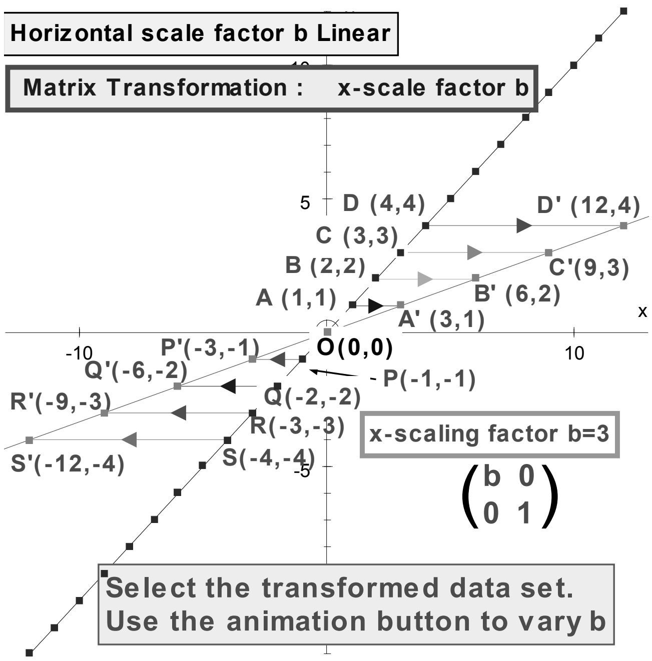


Figure 5

Investigate the effect of the *x*-scale factor by varying the value of **b** in the matrix

Transforming point A	$\begin{pmatrix} \mathbf{3} & 0 \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} \mathbf{3} \\ \mathbf{0} \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\times1 \\ 0\times1 \end{pmatrix} + \begin{pmatrix} 0\times1 \\ 1\times1 \end{pmatrix} = \begin{pmatrix} 3+0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
Transforming point B	$\begin{pmatrix} \mathbf{3} & 0 \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} \mathbf{3} \\ \mathbf{0} \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\times2 \\ 0\times2 \end{pmatrix} + \begin{pmatrix} 0\times2 \\ 1\times2 \end{pmatrix} = \begin{pmatrix} 6+0 \\ 0+2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$
Transforming point C	$\begin{pmatrix} \mathbf{3} & 0 \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} \mathbf{3} \\ \mathbf{0} \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\times3 \\ 0\times3 \end{pmatrix} + \begin{pmatrix} 0\times3 \\ 1\times3 \end{pmatrix} = \begin{pmatrix} 9+0 \\ 0+3 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$
Transforming point D	$\begin{pmatrix} \mathbf{3} & 0 \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4\begin{pmatrix} \mathbf{3} \\ \mathbf{0} \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\times4 \\ 0\times4 \end{pmatrix} + \begin{pmatrix} 0\times4 \\ 1\times4 \end{pmatrix} = \begin{pmatrix} 12+0 \\ 0+4 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \end{pmatrix}$

You could now show the corresponding transformations for the points P, Q, R, and S.
The transformation carried out on the points A, B, C, D could be written in abbreviated form as :

$$\overset{\begin{matrix} x & y \end{matrix}}{\begin{pmatrix} \mathbf{3} & 0 \\ \mathbf{0} & 1 \end{pmatrix}} \overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}} = \overset{\begin{matrix} A' & B' & C' & D' \end{matrix}}{\begin{pmatrix} 3 & 6 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{pmatrix}} \quad \text{Notice that } y = \frac{x}{3}$$

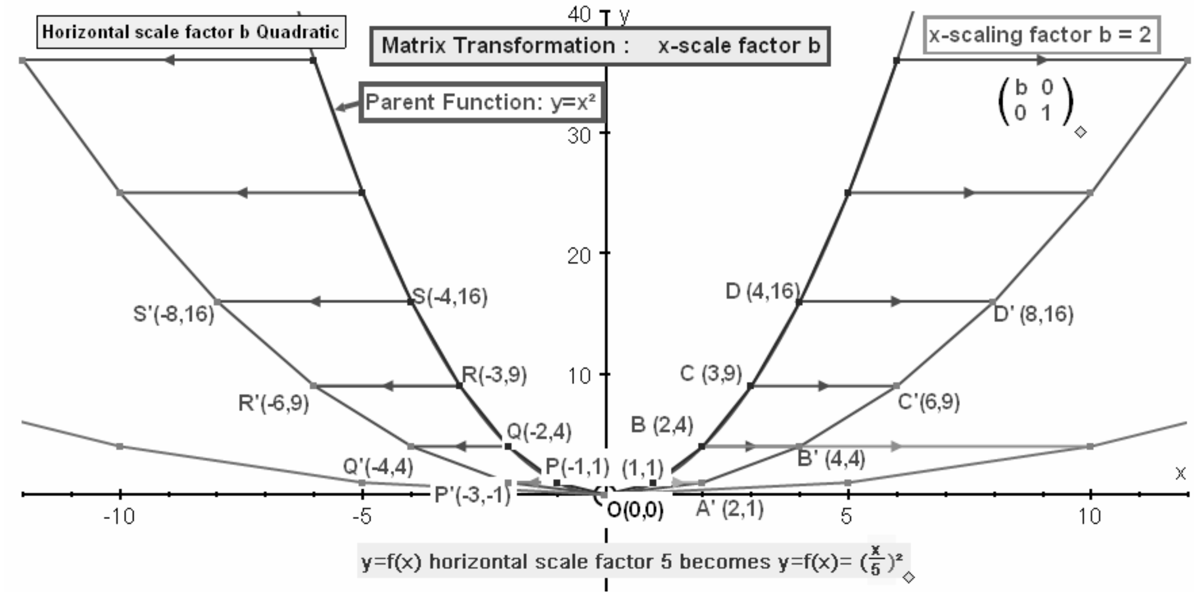
If the function $y = x$ is scaled horizontally ***x*-scale factor 3** the transformed function is defined by

$\boxed{y = f(\frac{x}{3})}$ or $y = \frac{x}{3}$ Note that this is not the same as saying $y = \frac{1}{3}f(x)$

Since <u>the scaling is done on *x*</u> then the stretch factor must be seen as acting on *x* alone.

Given: $y = \mathrm{f}(x)$ then the transformation $\overset{\begin{matrix} x & y \end{matrix}}{\begin{pmatrix} \mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}}$ maps the function $\boldsymbol{y = \mathrm{f}(x)}$ to $y = f(\frac{x}{b})$

Now consider the function $\boldsymbol{y = \mathrm{f}(x) = x^2}$ under the transformation: ***x*-scale factor 2**



<u>figure 6</u>

In figure 6 *investigate the effect* of the *x*-scale factor by animating the value of **b.**
You should first create a data set associated with the given function $y = \mathrm{f}(x)$

Transforming point A	$\begin{pmatrix} \mathbf{2} & 0 \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} \mathbf{2} \\ \mathbf{0} \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\times1 \\ 0\times1 \end{pmatrix} + \begin{pmatrix} 0\times1 \\ 1\times1 \end{pmatrix} = \begin{pmatrix} 2+0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (A → A')
Transforming point B	$\begin{pmatrix} \mathbf{2} & 0 \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} \mathbf{2} \\ \mathbf{0} \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\times2 \\ 0\times2 \end{pmatrix} + \begin{pmatrix} 0\times4 \\ 1\times4 \end{pmatrix} = \begin{pmatrix} 4+0 \\ 0+4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ (B → B')
Transforming point C	$\begin{pmatrix} \mathbf{2} & 0 \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 9 \end{pmatrix} = 3\begin{pmatrix} \mathbf{2} \\ \mathbf{0} \end{pmatrix} + 9\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\times3 \\ 0\times3 \end{pmatrix} + \begin{pmatrix} 0\times9 \\ 1\times9 \end{pmatrix} = \begin{pmatrix} 6+0 \\ 0+9 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$ (C → C')
Transforming point D	$\begin{pmatrix} \mathbf{2} & 0 \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 16 \end{pmatrix} = 4\begin{pmatrix} \mathbf{2} \\ \mathbf{0} \end{pmatrix} + 16\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\times4 \\ 0\times4 \end{pmatrix} + \begin{pmatrix} 0\times16 \\ 1\times16 \end{pmatrix} = \begin{pmatrix} 8+0 \\ 0+16 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \end{pmatrix}$ (D → D')

The transformation carried out on the points A, B, C, D could be written in abbreviated form as :

$$\begin{pmatrix} \mathbf{2} & 0 \\ \mathbf{0} & 1 \end{pmatrix}\overset{\begin{matrix} A & B & C & D \end{matrix}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{pmatrix}} = \overset{\begin{matrix} A' & B' & C' & D' \end{matrix}}{\begin{pmatrix} 2 & 4 & 6 & 8 \\ 1 & 4 & 9 & 16 \end{pmatrix}}$$

Observe the relationship between x' and y'

You could now show the corresponding transformations for the points P, Q, R, and S

The equation of the transformed function with *x*-scale factor = 2, containing **A' B' C' D'** is $y = \left(\frac{x}{2}\right)^2$

Note again the stretch factor, **b**, acts upon the x only and not the y. The defining rule of the function $y = f(x) = x^2$ should accordingly show this. When the x-stretch factor is 2 the transformed function is written as $y = f\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2$. Note that $y = \frac{1}{4}x^2$, although equivalent, does not provide the ready information about the x-stretch transformation.. In fact this is shown below to be a y-stretch factor. We could claim that $y = \frac{1}{4}x^2$ when written this way is really indicating a vertical stretch factor of $\frac{1}{4}$.

(2.4) Vertical Stretch by a scaling factor "a" Geometric approach

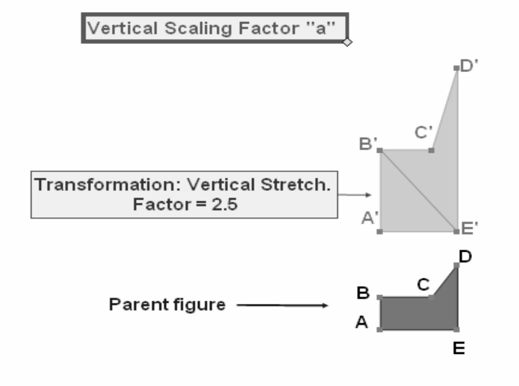


Figure 7

Experiment with different vertical stretch factors **a**.

Investigate what happens when the stretch factor is:

(a) A value greater than 1?

(b) A fraction between 0 and 1? (a shrink)

(c) A negative number?

(d) What occurs when the stretch factor is:

(i) 1?0 (ii) 0? (iii) 1 ⩽ a ⩽ 0?

In figure 7 both the *geometric stretch* and the corresponding *transformation matrix* can be compared, and the connection between the two processes noted.

(2.5) Vertical Stretch by scaling factor a (where a > 0) Algebraic Analysis

A ***y*-stretch** (with scaling factor 2) on any point P$\begin{pmatrix}x\\y\end{pmatrix}$ can be described algebraically as follows:

$$\begin{pmatrix}1 & \mathbf{0}\\ 0 & \mathbf{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = x\begin{pmatrix}1\\ 0\end{pmatrix} + y\begin{pmatrix}\mathbf{0}\\ \mathbf{2}\end{pmatrix} = \begin{pmatrix}x\times 1\\ x\times 0\end{pmatrix} + \begin{pmatrix}y\times\mathbf{0}\\ y\times\mathbf{2}\end{pmatrix} = \begin{pmatrix}x\\ 0\end{pmatrix} + \begin{pmatrix}\mathbf{0}\\ \mathbf{2y}\end{pmatrix} = \begin{pmatrix}x\\ 2y\end{pmatrix}$$

If the vertical or "y-stretch" factor is **2** then:

1. Each y unit $\begin{pmatrix}\mathbf{0}\\ \mathbf{1}\end{pmatrix}$ has been scaled by a factor of 2 and thus $\begin{pmatrix}\mathbf{0}\\ \mathbf{2}\end{pmatrix}$ becomes the new y-unit , while the (x) unit $\begin{pmatrix}\mathbf{1}\\ \mathbf{0}\end{pmatrix}$ stays fixed.
2. The coordinates of any points $\begin{pmatrix}x\\ y\end{pmatrix}$ of the figure to be stretched is determined by applying the transformation $\begin{pmatrix}\mathbf{1} & 0\\ \mathbf{0} & 2\end{pmatrix}$ to the point $\begin{pmatrix}x\\ y\end{pmatrix}$.

The point D$\begin{pmatrix}4\\ 3\end{pmatrix}$ under a vertical stretch of factor 2 is transformed to $\begin{pmatrix}4\\ 6\end{pmatrix}$

(2.6) Vertical Stretch (along y-axis) by scaling factor "a": As applied to functions

Consider the base function $\boldsymbol{y = x}$ with the transformation**:** ***y*-scale factor 2**

To help describe what is actually taking place plot several data points belonging to the given function. Observe the image of each of these points under the given s*caling transformation*

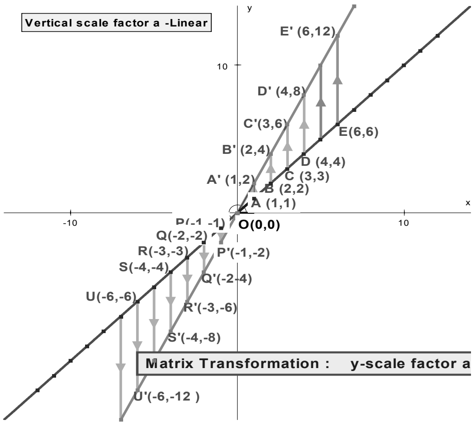


Figure 8

$$\begin{pmatrix}1&\mathbf{0}\\0&\mathbf{2}\end{pmatrix}\overset{B}{\begin{pmatrix}2\\2\end{pmatrix}}=2\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}\mathbf{0}\\\mathbf{2}\end{pmatrix}=\begin{pmatrix}1\times2+\mathbf{0}\times2\\0\times2+\mathbf{2}\times2\end{pmatrix}=\begin{pmatrix}2+0\\0+4\end{pmatrix}=\overset{B'}{\begin{pmatrix}2\\4\end{pmatrix}}$$

$$\begin{pmatrix}1&\mathbf{0}\\0&\mathbf{2}\end{pmatrix}\overset{A}{\begin{pmatrix}1\\1\end{pmatrix}}=1\begin{pmatrix}1\\0\end{pmatrix}+1\begin{pmatrix}\mathbf{0}\\\mathbf{2}\end{pmatrix}=\begin{pmatrix}1\times1\\0\times1\end{pmatrix}+\begin{pmatrix}\mathbf{0}\times1\\\mathbf{2}\times1\end{pmatrix}=\begin{pmatrix}1+0\\0+2\end{pmatrix}=\overset{A'}{\begin{pmatrix}1\\2\end{pmatrix}}$$

$$\begin{pmatrix}1&\mathbf{0}\\0&\mathbf{2}\end{pmatrix}\overset{C}{\begin{pmatrix}3\\3\end{pmatrix}}=3\begin{pmatrix}1\\0\end{pmatrix}+3\begin{pmatrix}\mathbf{0}\\\mathbf{2}\end{pmatrix}=\begin{pmatrix}1\times3+\mathbf{0}\times3\\0\times3+\mathbf{2}\times3\end{pmatrix}=\begin{pmatrix}3+0\\0+6\end{pmatrix}=\overset{C'}{\begin{pmatrix}3\\6\end{pmatrix}}$$

$$\begin{pmatrix}1&\mathbf{0}\\0&\mathbf{2}\end{pmatrix}\overset{D}{\begin{pmatrix}4\\4\end{pmatrix}}=4\begin{pmatrix}1\\0\end{pmatrix}+4\begin{pmatrix}\mathbf{0}\\\mathbf{2}\end{pmatrix}=\begin{pmatrix}1\times4+0\times4\\0\times4+\mathbf{2}\times4\end{pmatrix}=\begin{pmatrix}4\\0\end{pmatrix}+\begin{pmatrix}0\\8\end{pmatrix}=\overset{D'}{\begin{pmatrix}4\\8\end{pmatrix}}$$

The transformation carried out on the points A, B, C, D could be written in abbreviated form as :

$$\begin{pmatrix}1&\mathbf{0}\\0&\mathbf{2}\end{pmatrix}\begin{pmatrix}\overset{A}{1}&\overset{B}{2}&\overset{C}{3}&\overset{D}{4}\\1&2&3&4\end{pmatrix}=\begin{pmatrix}\overset{A'}{1}&\overset{B'}{2}&\overset{C'}{3}&\overset{D'}{4}\\2&4&6&8\end{pmatrix}$$

The equation of the transformed function containing **A' B' C' D'** is $\frac{\boldsymbol{y}}{\mathbf{2}}=\boldsymbol{x}$ Compare this with $\boldsymbol{y}=\mathbf{2}\boldsymbol{x}$ or $y=\frac{\boldsymbol{x}}{\frac{1}{2}}$

The form $\frac{y}{2}=x$ is more informative as it indicates that the scaling is a **y-scaling** and not an *x*-scaling. Note how $\boldsymbol{y}=\mathbf{m}\boldsymbol{x}$ is normally approached in the secondary mathematics classroom. You should investigate the distinction between Vertical and Horizontal scaling transformations. Now consider the function $\boldsymbol{y}=\mathbf{f}(\boldsymbol{x})=\boldsymbol{x}^2$ under the transformation: "***y*-scale factor 2"!**

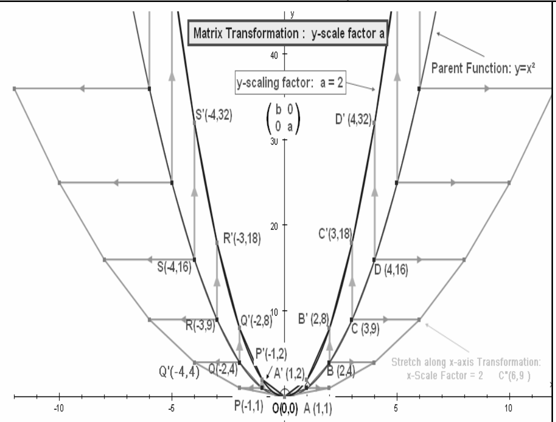


figure 9

$$\begin{pmatrix}1&\mathbf{0}\\0&\mathbf{2}\end{pmatrix}\overset{A}{\begin{pmatrix}1\\1\end{pmatrix}}=1\begin{pmatrix}1\\0\end{pmatrix}+1\begin{pmatrix}\mathbf{0}\\\mathbf{2}\end{pmatrix}=\begin{pmatrix}1\times1\\1\times0\end{pmatrix}+\begin{pmatrix}1\times\mathbf{0}\\1\times\mathbf{2}\end{pmatrix}=\begin{pmatrix}1+0\\0+2\end{pmatrix}=\overset{A'}{\begin{pmatrix}1\\2\end{pmatrix}}$$

$$\begin{pmatrix}1&\mathbf{0}\\0&\mathbf{2}\end{pmatrix}\overset{B}{\begin{pmatrix}2\\4\end{pmatrix}}=2\begin{pmatrix}1\\0\end{pmatrix}+4\begin{pmatrix}\mathbf{0}\\\mathbf{2}\end{pmatrix}=\begin{pmatrix}1\times2\\0\times2\end{pmatrix}+\begin{pmatrix}\mathbf{0}\times4\\\mathbf{2}\times4\end{pmatrix}=\begin{pmatrix}2+0\\0+8\end{pmatrix}=\overset{B'}{\begin{pmatrix}2\\8\end{pmatrix}}$$

$$\begin{pmatrix}1&\mathbf{0}\\0&\mathbf{2}\end{pmatrix}\overset{C}{\begin{pmatrix}3\\9\end{pmatrix}}=3\begin{pmatrix}1\\0\end{pmatrix}+9\begin{pmatrix}\mathbf{0}\\\mathbf{2}\end{pmatrix}=\begin{pmatrix}1\times3+9\times0\\0\times3+9\times2\end{pmatrix}=\begin{pmatrix}3+0\\0+18\end{pmatrix}=\overset{C'}{\begin{pmatrix}3\\18\end{pmatrix}}$$

$$\begin{pmatrix}1&\mathbf{0}\\0&\mathbf{2}\end{pmatrix}\overset{D}{\begin{pmatrix}4\\16\end{pmatrix}}=4\begin{pmatrix}1\\0\end{pmatrix}+16\begin{pmatrix}\mathbf{0}\\\mathbf{2}\end{pmatrix}=\begin{pmatrix}1\times4+\mathbf{0}\times16\\0\times4+\mathbf{2}\times16\end{pmatrix}=\begin{pmatrix}4+0\\0+32\end{pmatrix}=\overset{D'}{\begin{pmatrix}4\\32\end{pmatrix}}$$

The transformation carried out on the points A, B, C, D could be written in abbreviated form as:

$$\begin{pmatrix}1&\mathbf{0}\\0&\mathbf{2}\end{pmatrix}\begin{pmatrix}\overset{A}{1}&\overset{B}{2}&\overset{C}{3}&\overset{D}{4}\\1&4&9&16\end{pmatrix}=\begin{pmatrix}\overset{A'}{1}&\overset{B'}{2}&\overset{C'}{3}&\overset{D'}{4}\\2&8&18&32\end{pmatrix}$$

The equation of the transformed function with *y*-scale factor = 2, containing **A' B' C' D'** is $\frac{y}{2}=(x)^2$

Given $\boldsymbol{y}=\mathbf{f}(\boldsymbol{x})=\boldsymbol{x}^2$ the traditional way of writing the scaled function is $y=2x^2$ but for many students the coefficient 2 in this position is counter-intuitive! Consider how you teach $y=\mathbf{a}\,x^2$.

3. TRANSLATIONS

In the translation shown, figure ABCDEF is translated horizontally to A'B'C'D'E'F' by a distance equal to the length of the vector $\overrightarrow{\mathbf{PQ}}$ and in the direction of the translation vector $\overrightarrow{\mathbf{PQ}}$

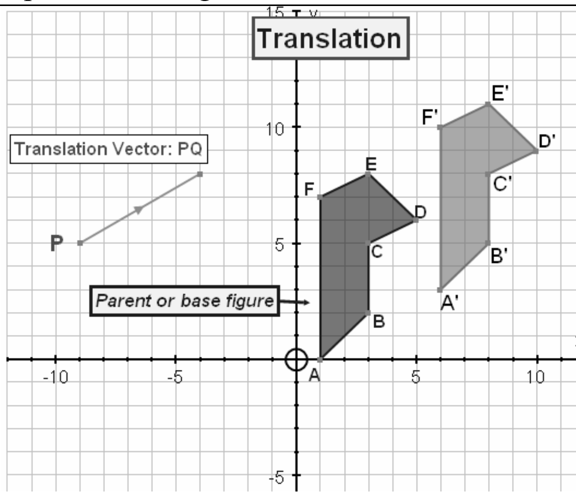


Figure 10

To ***move*** any given figure ***h*** units horizontally and ***v*** units vertically we describe the transformation using a **translation vector** expressed in the form **T** = $\begin{pmatrix} h \\ v \end{pmatrix}$

In figure 10 the figure ABCDEF is to be translated. The translation vector in this case is $\overrightarrow{\mathbf{PQ}} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

All points of the parent object ABCDEF will be moved +5 units horizontally, then + 3 units vertically The translation of the vertices A, B, C, D, E, F, of the parent object in figure 10 to their corresponding *image points* A', B', C', D', E', F', can be described or analysed algebraically as follows

A$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is moved to or *Translated to* A' $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{5} \\ \mathbf{3} \end{pmatrix} = \begin{pmatrix} 1+\mathbf{5} \\ 0+\mathbf{3} \end{pmatrix}$; ie $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow[T]{} \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ or T(A) = A'

B$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is moved to or *Translated to* B' $\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} \mathbf{5} \\ \mathbf{3} \end{pmatrix} = \begin{pmatrix} 3+\mathbf{5} \\ 2+\mathbf{3} \end{pmatrix}$; ie $\begin{pmatrix} 3 \\ 2 \end{pmatrix} \xrightarrow[T]{} \begin{pmatrix} 8 \\ 5 \end{pmatrix}$ or T(B) = B'

C$\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is moved to or *Translated to* C' $\begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} \mathbf{5} \\ \mathbf{3} \end{pmatrix} = \begin{pmatrix} 3+\mathbf{5} \\ 5+\mathbf{3} \end{pmatrix}$; ie $\begin{pmatrix} 3 \\ 5 \end{pmatrix} \xrightarrow[T]{} \begin{pmatrix} 8 \\ 8 \end{pmatrix}$ or T(C) = C'

D$\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ is moved to or *Translated to* D' $\begin{pmatrix} 5 \\ 6 \end{pmatrix} + \begin{pmatrix} \mathbf{5} \\ \mathbf{3} \end{pmatrix} = \begin{pmatrix} 5+\mathbf{5} \\ 6+\mathbf{3} \end{pmatrix}$; ie $\begin{pmatrix} 5 \\ 6 \end{pmatrix} \xrightarrow[T]{} \begin{pmatrix} 10 \\ 9 \end{pmatrix}$ or T(D) = D'

E$\begin{pmatrix} 3 \\ 8 \end{pmatrix}$ is moved to or *Translated to* E' $\begin{pmatrix} 3 \\ 8 \end{pmatrix} + \begin{pmatrix} \mathbf{5} \\ \mathbf{3} \end{pmatrix} = \begin{pmatrix} 3+\mathbf{5} \\ 8+\mathbf{3} \end{pmatrix}$; ie $\begin{pmatrix} 3 \\ 8 \end{pmatrix} \xrightarrow[T]{} \begin{pmatrix} 8 \\ 11 \end{pmatrix}$ or T(E) = E'

F$\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ is moved to or *Translated to* F' $\begin{pmatrix} 1 \\ 7 \end{pmatrix} + \begin{pmatrix} \mathbf{5} \\ \mathbf{3} \end{pmatrix} = \begin{pmatrix} 1+\mathbf{5} \\ 7+\mathbf{3} \end{pmatrix}$; ie $\begin{pmatrix} 1 \\ 7 \end{pmatrix} \xrightarrow[T]{} \begin{pmatrix} 6 \\ 10 \end{pmatrix}$ or T(F) = F'

If ***y* = f(*x*)** is translated by the vector $\begin{pmatrix} h \\ v \end{pmatrix}$ then the rule of the translated function is ***y* − v = f(*x* − h).**

Each point of the parent function ***y* = f(*x*)** is moved **horizontally h** units and **vertically v** units.
For any general point Q$\begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{s} \end{pmatrix}$ of the function **f**, then **s = f(r):** If P(*x,y*) be the image of point Q(r,s) then the translation is described by:

$$T(Q) = P \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{Q}{\begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{s} \end{pmatrix}} + \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{v} \end{pmatrix} = \overset{P}{\begin{pmatrix} x \\ y \end{pmatrix}}$$

$$\Rightarrow \begin{pmatrix} r \\ s \end{pmatrix} \xrightarrow[T]{} \begin{pmatrix} r+h \\ s+v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{where} \quad \begin{cases} r+h=x \\ s+v=y \end{cases}$$

From which we get $r = x - h$ and $s = y - v$

Since $r = \mathrm{f}(s)$ then substituting for r and s $\Rightarrow$ $y - v = \mathrm{f}\left(x - h\right)$

$$ie \quad \overset{Q}{\begin{pmatrix} r \\ s \end{pmatrix}} \xrightarrow[T]{} \overset{P}{\begin{pmatrix} x \\ y \end{pmatrix}} \Rightarrow r = \mathrm{f}(s) \xrightarrow[T]{} \textbf{\textit{image function}} \quad \{(x,y) \mid y - v = \mathrm{f}\left(x-h\right)\}$$

4. SUMMARY AND RATIONALE

(i) HORIZONTAL stretch (along the x-axis)**:**

$$\begin{pmatrix} \mathbf{b} & 0 \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow[T]{} \begin{pmatrix} X \\ Y \end{pmatrix}$$ represents the horizontal **stretch** of $\begin{pmatrix} x \\ y \end{pmatrix}$ to its image $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \boldsymbol{bx} \\ \boldsymbol{y} \end{pmatrix}$

A **Horizontal stretch** along the x-axis by a stretch factor **b** satisfies the following conditions:

- The horizontal dimensions of the parent figure are stretched horizontally by a factor **b**.
- The vertical dimensions of the figure remain the same. (ie y values are invariant)
- The ***area*** of the stretched figure is increased by a factor of **b** (ie by the stretch factor)
- If $\mathbf{f} = \{(x,y) \mid y = \mathbf{f}(x)\}$ then an x-stretch factor of **b** transforms the function **f** to $\boldsymbol{y} = \mathbf{f}\left(\dfrac{\boldsymbol{x}}{\mathbf{b}}\right)$

(ii) VERTICAL stretch (along the y-axis)**:**

$$\begin{pmatrix} 1 & \mathbf{0} \\ 0 & \mathbf{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow[T]{} \begin{pmatrix} X \\ Y \end{pmatrix}$$ represents the vertical **stretch** of $\begin{pmatrix} x \\ y \end{pmatrix}$ to its image $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \boldsymbol{x} \\ \mathbf{a}\boldsymbol{y} \end{pmatrix}$

A **stretch** along the y-axis by a scaling factor **a** satisfies the following conditions:

- The vertical dimensions of the parent figure are stretched vertically by the factor **a**.
- The horizontal dimensions of the figure remain the same. (ie x values are invariant)
- The ***area*** of the stretched figure is increased by a factor of **a** (ie by the stretch factor)
- If $\mathbf{f} = \{(x,y) \mid y = \mathbf{f}(x)\}$ then a y-stretch factor of **a** transforms the function **f** to $\dfrac{y}{\mathbf{a}} = \mathrm{f}(x)$

A TRANSLATION:

- Each point (x,y) of the initial figure is moved **horizontally h** units and **vertically v** units
- The transformation changes the ***position*** of the parent figure by a distance and in a directio equivalent to magnitude and direction of the translation vector.
- The ***dimensions*** and ***orientation*** of the figure stay the same as the parent figure.

Algebraically a figure is translated by vector $\begin{pmatrix} \mathbf{h} \\ \mathbf{v} \end{pmatrix}$ if it satisfies the following condition:

- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} + \begin{pmatrix} \mathbf{h} \\ \mathbf{v} \end{pmatrix} \xrightarrow[T]{} \begin{pmatrix} x \\ y \end{pmatrix}$ This represents the **translation** of any point $\begin{pmatrix} r \\ s \end{pmatrix}$ of the parent figure by the vector $\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{v} \end{pmatrix}$**.** From this $\begin{pmatrix} r + \mathbf{h} \\ s + \mathbf{v} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ ie $\boldsymbol{x} = \mathbf{r} + \mathbf{h}$ and $\boldsymbol{y} = \mathbf{s} + \mathbf{v}$

  OR $\boldsymbol{x} - \mathbf{h} = \mathbf{r}$ and $\boldsymbol{y} - \mathbf{v} = \mathbf{s}$

> Students should understand that every (affine)transformation of a function $\boldsymbol{y} = \mathbf{f}(\boldsymbol{x})$ can be
>
> expressed in the form : $\dfrac{y - v}{\boldsymbol{a}} = \mathrm{f}\left(\dfrac{x - h}{\boldsymbol{b}}\right)$
>
> where the x-axis scaling (dilation) factor is **b** ; the y-axis scaling (dilation) factor is **a**
>
> and the function is then **translated** by the vector $\begin{pmatrix} \mathbf{h} \\ \mathbf{v} \end{pmatrix}$

This approach can be applied to the teaching of all functions studied in secondary mathematics courses. Below is a representative list of SOME functions developed in the course-work of most secondary schools.

Traditional Approach	Suggested approach

• **Linear functions**: $y = \mathrm{m}x + \mathrm{c}$	• If y = f(x) is a linear function; Consider **y = m(f(x)) + c** $\frac{y-c}{\boldsymbol{m}} = \mathrm{f}\left(\frac{x-h}{\boldsymbol{b}}\right)$ where $y = \mathrm{f}(x) = x$
• **Quadratic: (i)** $y = \mathrm{a}x^2 + \mathrm{b}x + \mathrm{c}$ • **Quadratic: (ii)** $y = \mathrm{a}(x - \mathrm{h})^2 + \mathrm{c}$	• Quadratics and general polynomial functions $\boldsymbol{y} = \mathbf{f}(\boldsymbol{x})$ viewed in the form $\frac{y-v}{\boldsymbol{a}} = \mathrm{f}\left(\frac{x-h}{\boldsymbol{b}}\right)$ where $y = \mathrm{f}(x) = x^2$
The reciprocal function $y = \frac{k}{\boldsymbol{x}} + c$	• viewed in the form $\frac{y-v}{\boldsymbol{a}} = \mathrm{f}\left(\frac{x-h}{\boldsymbol{b}}\right)$ where $y = \mathrm{f}(x) = \frac{1}{\boldsymbol{x}}$
• **The Exponential function** $y = ke^x + c$ or $y = ke^{g(x)} + c$	• Viewed in the form $\frac{y-v}{\boldsymbol{a}} = \mathrm{f}\left(\frac{x-h}{\boldsymbol{b}}\right)$ where $y = \mathrm{f}(x) = \mathrm{e}^x$
• **The Logarithmic Function** $y = \log_r x + c$	• Viewed in the form $\frac{y-v}{\boldsymbol{a}} = \mathrm{f}\left(\frac{x-h}{\boldsymbol{b}}\right)$ where f: $y = \log_{\mathrm{r}} x$
• **The trig functions:** $y = \mathrm{f}(x)$ where f(x) = sin$x$; f(x) = cos$x$; or f(x)=tan$x$	• Viewed in the form $\frac{y-v}{\boldsymbol{a}} = \mathrm{f}\left(\frac{x-h}{\boldsymbol{b}}\right)$ where f(x) = sin$x$ eg $y = 5\sin(2x-6)+4$ viewed as $\frac{y-4}{5} = \sin\left(\frac{x-3}{\frac{1}{2}}\right)$

1. What approach would you use when analysing the graph of the function $\boldsymbol{y = 2x^2 - 24x + 81}$?
One approach in common use is to first complete the square $y = 2(x-6)^2 + 9$ then outline the translation. This is usually stated as translating $\boldsymbol{y = 2x^2}$ **6units to the right** then **9 units upwards**.
In one instance (− 6) moves the graph to the **right** (ie + 6 units); while (+9) moves the graph **down** (ie − 9 units). Is this approach logically consistent? Students find this approach confusing!
Students will gain a clearer understanding by using the form $\frac{y-9}{2} = f(x-6) = \frac{(x-6)^2}{1}$?

2. How does this approach relate to work in Statistics or Correlation? eg Consider change of scale and change of origin; or standardized scores. How is the concept applied in other areas of mathematics? The approach suggested in this paper is consistent in all these areas.	$z = \frac{x-\mu}{\sigma}$	$\left(\frac{x-\bar{x}}{s_x}\right)\left(\frac{y-\bar{y}}{s_y}\right)$

3. Consider $x^2 + y^2 = 64$: How is this related to $(x-5)^2 + (y+7)^2 = 64$?

4. Which of the forms $y = 5\sin(2x-6)+4$ or $\frac{y-4}{5} = \sin\left(\frac{x-3}{\frac{1}{2}}\right)$ is the more informative?

5. A study of the basic transformation of functions involves a study of transformations of the form:
$\begin{pmatrix} \boldsymbol{b} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{a} \end{pmatrix}\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} + \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{pmatrix}$ where $\boldsymbol{a}$ and $\boldsymbol{b} \neq \mathbf{0}$ From which $y = f(x) \rightarrow \frac{Y-v}{a} = f\left(\frac{X-h}{b}\right)$
The effect of each parameter
a (The vertical scaling factor); **b**(The horizontal scaling factor);
h(The horizontal translation component); **v** (The vertical translation component)
should be considered in turn and their relation to the coefficients in the function $y = \mathrm{f}(x)$ noted.

Teaching using this approach needs, and is greatly enhanced by, the use of appropriate technology.