Studying Self-affine Tiles with the Aid of Winfeed

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Abstract: Let A be a 2×2 integral matrix with $|\det A| = N$ and all its eigenvalues having moduli greater than one. Let $D = \{v_1, v_2, \ldots, v_N\}$ be a set of integral vectors in $\mathbb{R}^2$. We call D a *digit set*. It is well-known that there exists a unique compact set $T = T(A,D)$ satisfying $AT = T + D := \bigcup_{k=1}^{N}(T + v_k)$. Hence T is said to be a *self-affine set*. If the Lebesgue measure of T is positive, then T is called a *self-affine tile*. A self-affine tile may be regarded as a two-dimensional analogue of the unit interval on the real line. In other words, it may be thought of as the unit interval of a generalized number system in $\mathbb{R}^2$ (with number base A and digit set D). We concentrate on the case $D = \{0, v, 2v, \ldots, (N-1)v\}$ for some $v \in \mathbb{Z}^2 \setminus \{0\}$. We study the following questions in this paper:

1. When will T be a self-affine tile?
2. When will T be connected?
3. When will T be disklike (i.e. homeomorphic to the closed unit disk)?

Given A and D, we can draw the picture of T easily using the freeware *Winfeed* (a software designed for drawing fractals). These pictures helped formulate some conjectures in regard to the above questions. We can also verify and consolidate the answers to these questions by inspecting the pictures of various T.

1. Introduction

Throughout this paper, A denotes a 2×2 integral matrix with $|\det A| = N$. We assume that A is *expanding* (i.e. $|\lambda| > 1$ for every eigenvalue λ of A). Let $D = \{v_1, v_2, \ldots v_N\} \subset \mathbb{Z}^2$. The set D is called a *digit set* (or more precisely, an N*-digit set*). It is well-known (see e.g. [1], [3]) that there exists a unique compact set $T = T(A,D)$ satisfying:

$$AT = T + D := \bigcup_{k=1}^{N}(T + v_k). \tag{1.1}$$

Hence T is said to be a *self-affine* set. We can also regard T as the *attractor* (or *invariant set*) of the *iterated function system* (IFS) :

$$\varphi_i(x) = A^{-1}(x + v_i) \tag{1.2}$$

for $= 1, 2, \ldots, N$. The set T can also be expressed explicitly as

$$T = \left\{ \sum_{i=1}^{\infty} A^{-i} v'_i : v'_i \in D \right\}. \tag{1.3}$$

If the Lebesgue measure $\mu(T)$ of T is positive, T is called a *self-affine tile*.

Example 1.1 The following figure (Figure 1.1) shows the most famous self-affine tile (studied in details in [5]) called the *twindragon*. It is generated by $A=\begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix}$, $D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right\}$. Observe that it is composed of two identical parts, each is similar to the whole tile.

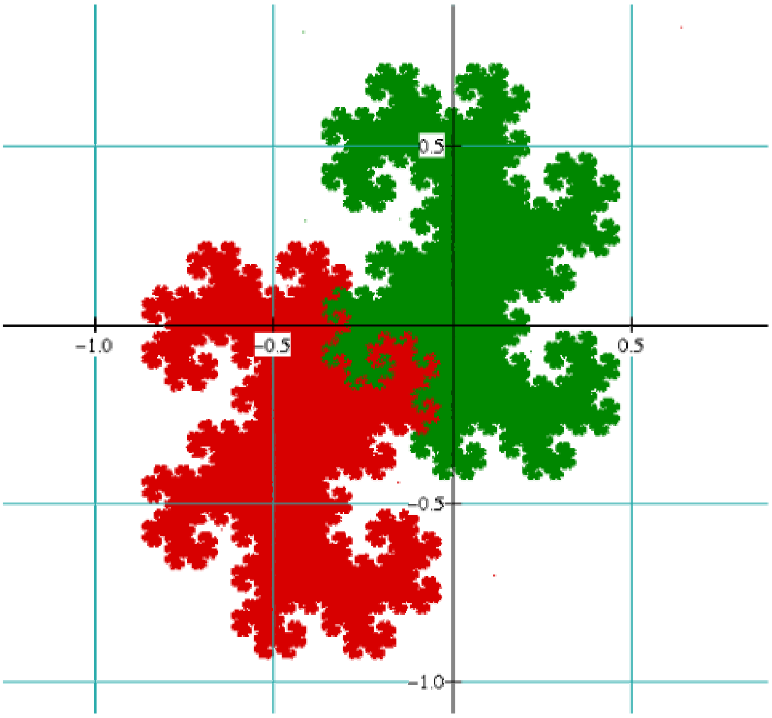

Figure 1.1 The Twindragon

To emphasize A is an integral matrix and $D\subset\mathbb{Z}^2$, T is referred to as an *integral self-affine tile*. We can always assume $0\in D$ when studying the geometric or topological properties of T. It is because if D_1 and D_2 are two N-digit sets related by $D_2=x_0+D_1$, then it can be shown that $T_2=T_1+(A-I)^{-1}x_0$, where $T_1=T(A,D_1)$ and $T_2=T(A,D_2)$. It is known that if T is a self-affine tile, it is a tile in the ordinary sense. More precisely, a self-affine tile T tiles $\mathbb{R}^2$ by a set L (called the *tiling set*) of translations in the sense that $T+L=\mathbb{R}^2$ and $T+g_1$, $T+g_2$ are *essentially disjoint* (i.e. their intersection has Lebesgue measure zero) for any distinct $g_1,g_2\in L$. If L is a lattice (i.e. $L=\mathbb{Z}x+\mathbb{Z}y$ for some linearly independent $x,y\in\mathbb{Z}^2$), $T+L$ is called a *lattice tiling* of $\mathbb{R}^2$ by T with the lattice L. In Section 2, we examine two conditions for a self-affine set to be a tile. We study the connectedness of T in Section 3. In Section 4, we study an equivalent condition for a class of T to be homeomorphic to the closed unit disk. We illustrate this condition with the pictures of different T. These pictures are drawn with the aid of *Winfeed*, a freeware for drawing fractals, which can be downloaded from the Peanut Software homepage http://math.exeter.edu/rparris/default.html .

2. Two Conditions for a Self-affine Set to be a Tile

Before we state the first condition for a self-affine set T to be a tile, we need to give the definition of an A-*invariant* lattice. A lattice L is said to be A-*invariant* if $L \subseteq A(L)$. The following theorem gives a sufficient condition for T to be a tile.

Theorem 2.1 Let A and D be defined as before. Assume further that $D \subset L$ for some A-*invariant* lattice L. A sufficient condition for $T = T(A, D)$ to be a self-affine tile is that D is a complete set of coset representatives of the quotient group $L / A(L)$.

Proof. See [1]. ■

If $D = \{0, v, 2v, \ldots, (N-1)v\}$ for some $v \in \mathbb{Z}^2 \setminus \{0\}$, it is called a *consecutive collinear (CC)* digit set.

Theorem 2.2 Let A be an integral expanding 2×2 matrix with characteristic polynomial $f(x) = x^2 + px + q$. Let $D = \{0, v, 2v, \ldots, (N-1)v\}$, where $N = |\det A| = |q|$, for some $v \in \mathbb{Z}^2 \setminus \{0\}$. Then $T = T(A, D)$ is a self-affine tile if and only if $\{v, Av\}$ is linearly independent.

Proof. Suppose $\{v, Av\}$ is linearly dependent. By this assumption and the fact that

$$0 = f(A)v = A^2 v + pAv + qv, \tag{2.1}$$

we can express $A^{-i}v$, for all $i \in \mathbb{N}$, in the form $A^{-i}v = c_i v$ for some $c_i \in \mathbb{Q}$. Then we can see from (1.3) that T is a subset of the line spanned by v. Hence $\mu(T) = 0$.

To prove the converse, we need to apply Theorem 2.1. Let L be the lattice generated by $\{v, Av\}$, i.e. $L = \mathbb{Z}v + \mathbb{Z}Av$. We see immediately that L is A-invariant and $D \subset L$. We need to show that D is a complete set of coset representatives of $L / A(L)$. We give the proof for the case $f(0) = -N$; the case $f(0) = N$ can be handled in the same manner. From (2.1), we have $Nv = A^2 v + pv$. Consider an arbitrary $u = aAv + bv \in L$. By division algorithm, there exist unique $k, r \in \mathbb{Z}$ such that $b = kN + r$ and $0 \le r \le N-1$. Then $u = A[kAv + (p+a)v] + rv \in A(L) + rv$. Now suppose $r_1 v$ and $r_2 v$ are two distinct elements of D and there exists $w \in [A(L) + r_1 v] \cap [A(L) + r_2 v]$. Then w can be written in two ways: $w = cA^2 v + dAv + r_2 v$ and $w = c' A^2 v + d' Av + r_2 v$. It follows that

$$g(A)v = 0 \tag{2.2}$$

where $g(x) = (c - c')x^2 + (d - d')x + (r_2 - r_1)$. Multiply (2.2) by A, we obtain $g(A)Av = 0$. Since $\{v, Av\}$ is linearly independent, we must have $g(A) = 0$. The linear independence also implies that $f(x)$ is also the minimal polynomial of A. Hence $f(x) | g(x)$. But this implies $N | r_1 - r_2$, which is absurd. ■

3. The Connectedness of Self-affine Tiles

The connectedness of a self-affine tile $T = T(A, D)$ is closely related to the $\mathcal{E}$-*connectedness* (to be defined) of the elements of D. Define the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ of D as follows. The set of vertices $\mathcal{V}$ and the set of edges $\mathcal{E}$ are given by:

$\mathcal{V} := D$ and $\mathcal{E} := \{(v_i, v_j) : (T + v_i) \cap (T + v_j) \neq \phi, v_i, v_j \in D\}$.

Two vertices v_i and v_j are said to be $\mathcal{E}$-*connected* if they are joined by a path, i.e. there exists a finite sequence of vertices $v_{j_1}, v_{j_2}, \ldots, v_{j_k}$ such that $v_{j_1} = v_i, v_{j_k} = v_j$ and $(v_{j_l}, v_{j_{l+1}}) \in \mathcal{E}, 1 \leq l \leq k-1$. Recall that a graph is connected if every pair of distinct vertices can be linked by a path.

Theorem 3.1 T is connected if and only if $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is connected (i.e. any $v_i, v_j \in \mathcal{V}$ are $\mathcal{E}$-connected).

Proof. See [4]. ■

When D is a CC digit set, we have the following corollary.

Corollary 3.2 If $D = \{0, v, 2v, \ldots, (N-1)v\}$ for some $v \in \mathbb{Z}^2 \setminus \{0\}$ and $T \cap (T + v) \neq \phi$, then T is connected.

Proof. See [4]. ■

We can deduce from (1.3) that for a CC digit set D, $T \cap (T + v) \neq \phi$ is equivalent to

$$v = \sum_{i=1}^{\infty} b_i A^{-i} v, \tag{3.1}$$

where $b_i \in \{0, \pm 1, \pm 2, \ldots, \pm(N-1)\}$ for $i = 1, 2, 3 \ldots$.

A matrix is expanding if and only if its characteristic polynomial is *expanding* (i.e. all its roots have moduli greater than one). The lemma below characterizes all integral expanding quadratic polynomials.

Lemma 3.2 Let $f(x) = x^2 + px + q \in \mathbb{Z}[x]$. Then $f(x)$ is expanding if and only if one of the following holds:

(i) $|p| \leq q$ for $q \geq 2$, or

(ii) $|p| \leq |q + 2|$ for $q \leq -2$.

Proof. See [2]. ■

Theorem 3.3 Let A be an integral expanding 2×2 matrix with $|\det A| = N$ and $D = \{0, v, 2v, \ldots, (N-1)v\}$ for some $v \in \mathbb{Z}^2 \setminus \{0\}$. If the set $\{v, Av\}$ is linearly independent, then $T = T(A, D)$ is a connected tile.

Proof. (See also [8], [9]) Theorem 2.2 guarantees that T is a tile. We need to show it is connected. Let the characteristic polynomial of A be $f(x) = x^2 + px + q \ (|q| = N)$. We find an expression of the form (3.1) in each case. Then by Corollary 3.2, T is connected. We consider the cases $q \le -2$ and $q \ge 2$ separately.

(i) $q \le -2$. In this case $q = -N$ and $|p| \le |q+2| = N-2$ (Lemma 3.2). From $f(A) = 0$, we deduce that

$$I = (A^2 - 1)^{-1}[-pA + (N-1)] = [-pA + (N-1)]\sum_{i=1}^{\infty} A^{-2i}$$

$$v = [-pA + (N-1)]\sum_{i=1}^{\infty} A^{-2i} v$$

(ii) $q \ge 2$. In this case $|p| \le q = N$ (Lemma 3.2). We first consider the subcase $|p| < q = N$. From $f(A) = 0$, we deduce that

$$\begin{aligned} I &= (A^2 + I)^{-1}[-pA - (N-1)I] \\ &= [-pA - (N-1)I]\sum_{i=1}^{\infty} (-1)^{i+1} A^{-2i} \end{aligned}$$

$$v = [-pA - (N-1)I]\sum_{i=1}^{\infty} (-1)^{i+1} A^{-2i} v$$

When $|p| = q = N$, $f(x) = x^2 \pm Nx + N$. Let $g(x) = x \mp 1$. From $g(A)f(A) = 0$, we get $I = (A^3 \mp I)^{-1}[\mp(N-1)A^2 \pm (N-1)I]$. Expanding $(A^3 \mp I)^{-1}$ and then multiplying by v, we obtain a series of the form (3.1).

■

4. The Disklikeness of Self-affine Tiles

A self-affine tile $T = T(A, D)$ is said to be *disklike* if it is homeomorphic to the closed unit disk. If D is a CC digit set, then the disklikeness of T is determined by the coefficients of $f(x)$ (the characteristic polynomial of A).

Theorem 4.1 Let A be an integral expanding 2×2 matrix with characteristic polynomial $f(x) = x^2 + px + q \ (|q| = N)$. Let $D = \{0, v, 2v, \ldots, (N-1)v\}$ for some $v \in \mathbb{Z}^2 \setminus \{0\}$ such that the set $\{v, Av\}$ is linearly independent. Then $T = T(A, D)$ is a disklike tile if and only if $2|p| \le |q+2|$.

Proof. See [8], [9]. ■

The pictures of a number of T were drawn and inspected before formulating the conjecture on the criterion $2|p|\leq|q+2|$ for the disklikeness of T in the above theorem. Using the function of drawing the attractor of IFS in Winfeed, we can draw T without doing any programming. The pictures of some T are shown below to illustrate the condition for disklikeness. Figure 4.1 and 4.2 are the tiles generated by the companion matrices of $f(x)=x^2+2x+4$ and $f(x)=x^2+3x+4$ respectively, and the same digit set $D=\{0,v,2v,3v\}$, where $v=\begin{pmatrix}0\\1\end{pmatrix}$. The inequality $2p\leq q+2$ are satisfied in these two cases. It seems quite convincing to judge from these pictures that the two tiles are disklike.

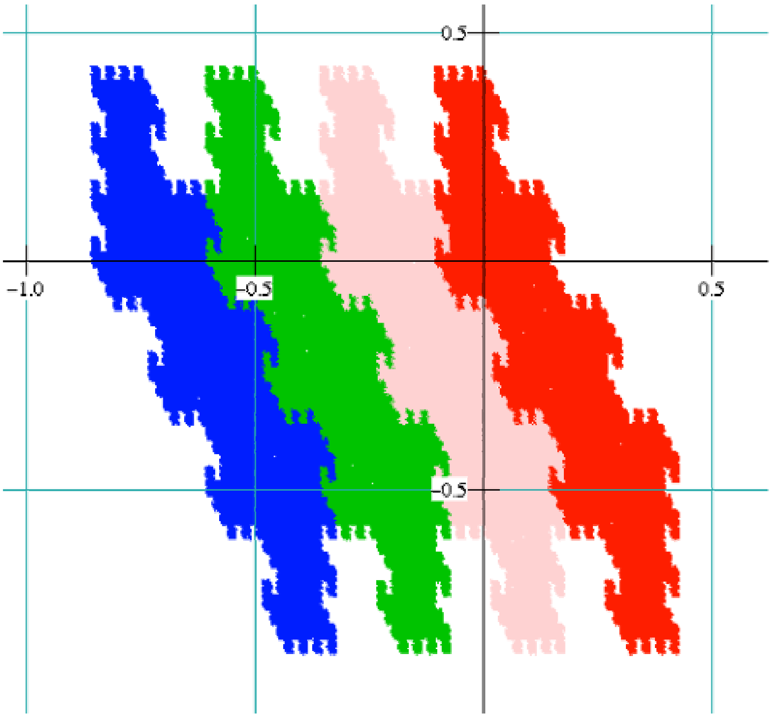

Figure 4.1 The tile generated by $A=\begin{pmatrix}0 & 1\\-4 & -2\end{pmatrix}$, $D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}\right\}$.

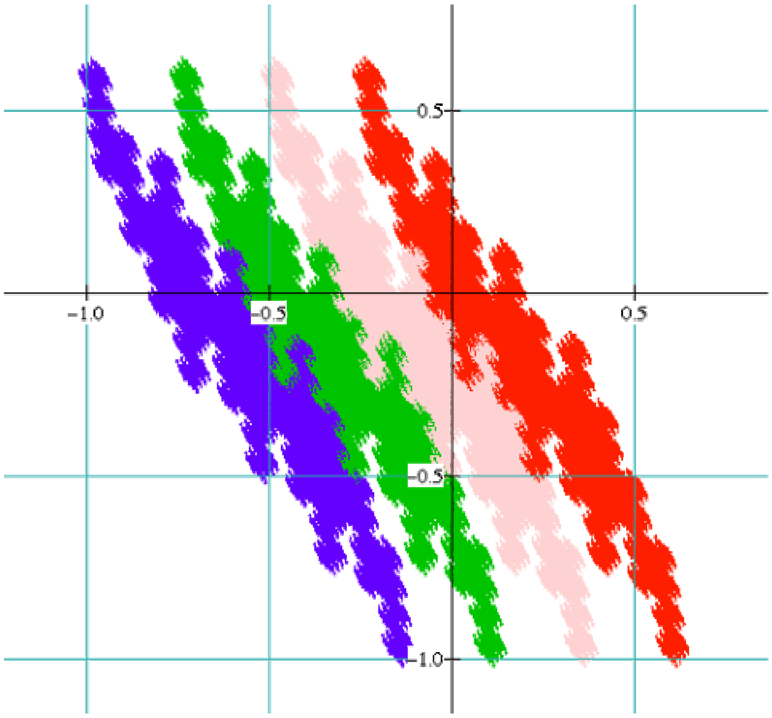

Figure 4.2 The tile generated by $A=\begin{pmatrix}0 & 1\\ -4 & -3\end{pmatrix}$, $D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}\right\}$.

Figure 4.3 is generated by the companion matrix of $f(x)=x^2+4x+4$ and the same digit set. Here $2p>q+2$. However, this picture does not suggest that T is non-disklike. But if we enlarge a part of it (see Figure 4.4), we see that T has great many long and thin slits. Bearing in mind that every point plotted has a size, Figure 4.4 supports the theoretical conclusion drawn from Theorem 4.1 that T is non-disklike.

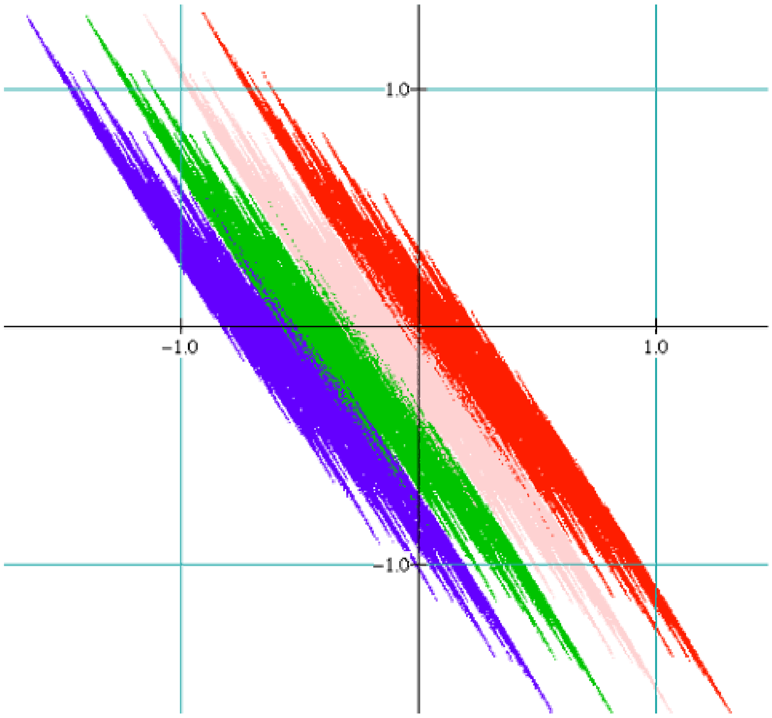

Figure 4.3 The tile generated by $A = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}$, $D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$.

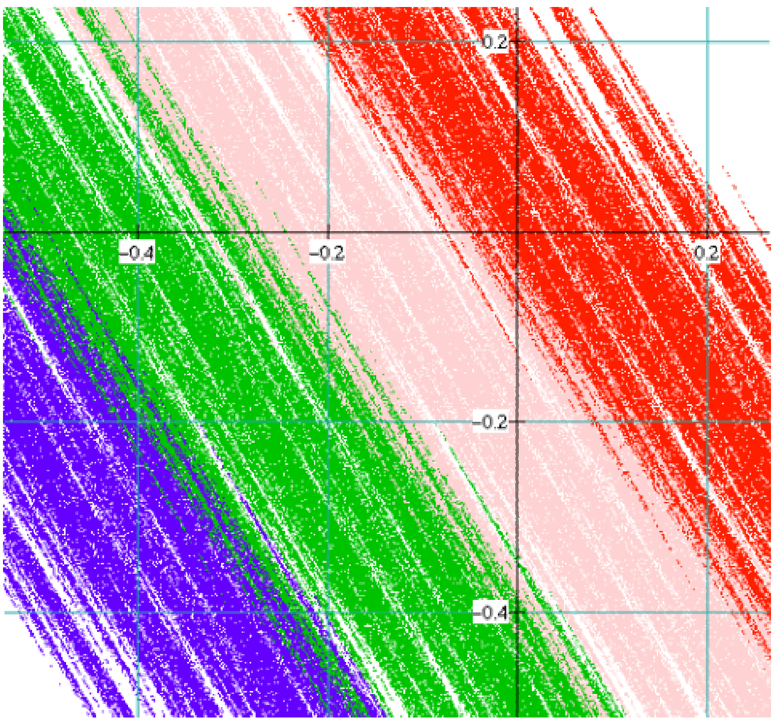

Figure 4.4 A magnified portion of the tile generated by $A = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}$, $D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$.

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