Use of Free Internet Resources to Design Mathematics Problem Solving Activities

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Abstract: Hanoi Tower is a popular mathematical game and traditionally it is made of wood. With the advancement of Internet technology, this game is now available in many websites and students can play this game in an interactive way without being charged. This paper attempts to illustrate how this kind of free Internet resources can be utilized to design mathematics problem solving activities. In particular, three learning activities developed for students with different mathematics background and cognitive levels will be shared in this paper. Moreover, the advantages and disadvantages of working on simulated Hanoi Tower on the Internet and on the traditional wooden one will be discussed.

1. Introduction

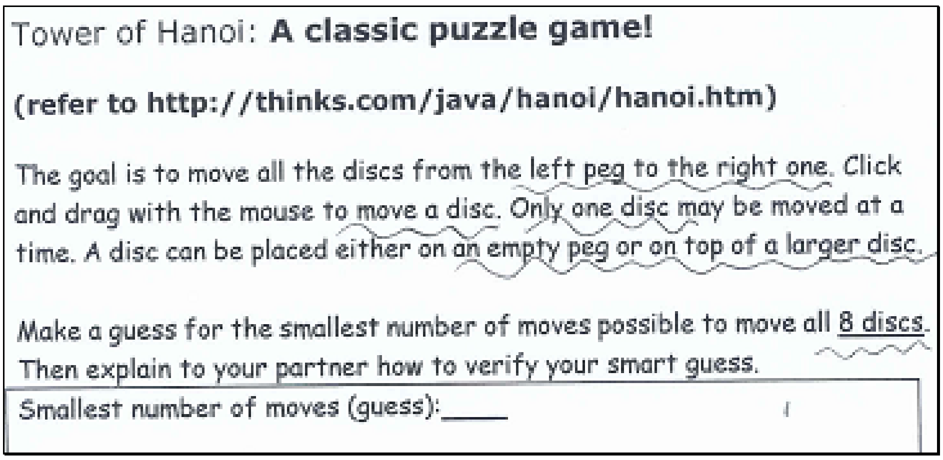
The Hong Kong Special Administrative Government (HKSAR) has been encouraging teachers to use information technology (IT) for effective teaching and learning. However, because of various reasons such as tight working schedule and lack of IT knowledge, many mathematics teachers are unable to develop their own packages for classroom use. And schools can afford purchasing only very few software, due to limited budget. Therefore, free resources on the Internet are valuable to many teachers in designing effective instructions that can engage students in learning.

Problem solving is a process through which a person can use thinking skills to resolve a difficulty and determine a valid course of action. For this reason, problem solving is a very useful process in a knowledge-based society, and the development of problem solving skills is strongly encouraged in the new Hong Kong school curriculum. It is one of the nine generic skills for effective learning stated in *LEARNING TO LEARN* (Curriculum Development Council, 2001). Another document, the *Mathematics Education – Key Learning Area Curriculum Guide* (Curriculum Development Council, 2002), recommends schools and teachers to develop pupils' skills in exploring patterns, acquiring basic strategies for estimating answers and measures, learning to observe, analyze, reason and make judgments, formulating and solving problems, and using modern technology to learn mathematics. The recommendations are in line with the updated standards published by the National Council for Teachers of Mathematics (2000) that identifies problem solving as an essential component of mathematics learning for all grade levels.

In the next section, three problem solving activities that are built on free Internet resources are suggested. These activities look similar but they are thoughtfully designed for students with different mathematical backgrounds and they serve for different purposes.

2. Problem Solving Activities

Activity 1. Being mathematics educators, the competence of student-teachers in mathematics is our concern. So, the first problem solving activity was designed to assess student-teachers' problem solving skills. Figure 1 shows the worksheet prepared for this activity. Student-teachers who took a course in problem solving at The Hong Kong Institute of Education were invited to visit a website. They were first asked to make a guess on the smallest number of moves to relocate eight discs from one peg to another (see Figure 2). Then they were encouraged to explore the interactive Hanoi Tower and to explain their solutions to their partners.


Tower of Hanoi: **A classic puzzle game!**

(refer to http://thinks.com/java/hanoi/hanoi.htm)

The goal is to move all the discs from the left peg to the right one. Click and drag with the mouse to move a disc. Only one disc may be moved at a time. A disc can be placed either on an empty peg or on top of a larger disc.

Make a guess for the smallest number of moves possible to move all 8 discs. Then explain to your partner how to verify your smart guess.

Smallest number of moves (guess):_____

Figure 1: Problem Solving Activity for Assessment

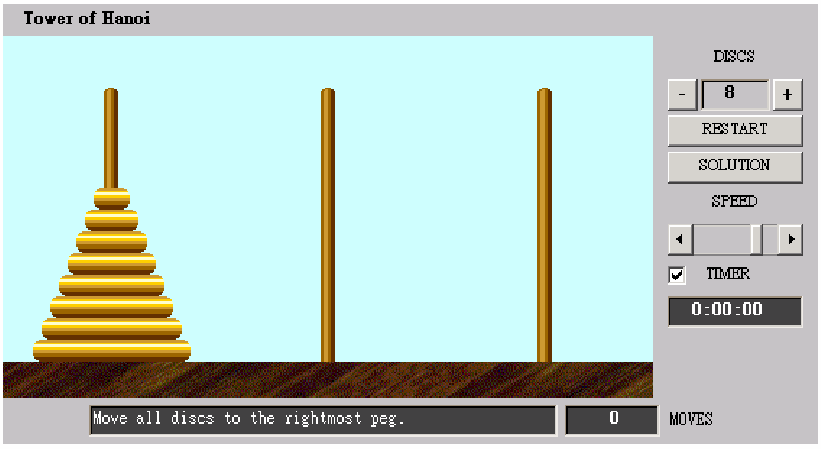


Figure 2: Hanoi Tower with 8 Discs

In fact, there are many websites providing interactive version of this popular mathematical game. Table 1 just contains a few of them. In all these websites, users can select the number of discs according to their needs. The first three websites have some common friendly features. They automatically count the number of moves made by user and record the time since the first move. The fourth and the fifth websites do not have these features. However, the fourth one has another kind of feature that offers suggested move at any stage when users ask for it. The response in the fifth one is a bit slow and the moves are made by "point-and-click" instead of "drag-and-drop".

The last three websites are displayed in Chinese. They are all similar and simply offer a platform for users to select the number of discs and move them around. Neither timer nor move counter are provided in these websites.

Table 1: Hanoi Tower Websites

Websites in English:

1. http://www.mazeworks.com/hanoi/
2. http://thinks.com/java/hanoi/hanoi.htm
3. http://www.geocities.com/mathfair2002/games/hanoi.htm
4. http://www.cut-the-knot.org/recurrence/hanoi.shtml
5. http://www.superkids.com/aweb/tools/logic/towers/

Websites in Chinese:

6. http://math.hhsh.cy.edu.tw/game/hanoi/
7. http://newchess.com.cn/games/tower.htm
8. http://www.mathland.idv.tw/game/hanoi/hanoi.htm

Figures 3 and 4 show the work of two student-teachers, Mary and Jean (made-up names). Both Mary and Jean did not put any number for their guess. It seemed that they did not have prior experience on this game, thus the problem was a genuine one to them.

At the very beginning, Mary tried three discs that is the minimum number of discs offered by the website. Sooner she figured out that "three discs" should not be the simplest condition. So, she estimated the least numbers of moves for two simpler cases, "1 disc" and "2 discs". Then, she continued with "4 discs" and "5 discs". Based on the results, she tried to explore the relationship among the numbers. Eventually, she ended up with a geometric series for the solution. Mary's exploration on this problem solving activity was not very smooth. Indeed, she made several attempts and made a few mistakes. For example, the attempt of using an arithmetic series was not appropriate and the obtained number for moving five discs was not the least. Somehow, her work demonstrated that she had used several problem solving strategies such as simplifying the problem, estimation, looking for patterns, and trial and error.

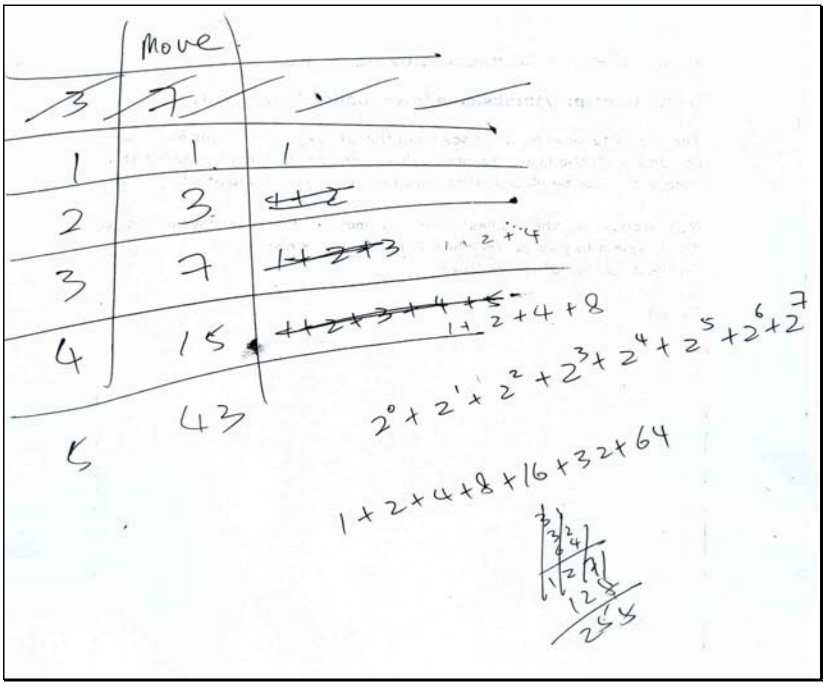


Figure 3: Mary's solution to the problem

Jean's work was not the same as Mary's. Instead of starting with one disc, Jean started with two. In each game, Jean examined the number of moves for each individual disc, from the smallest to the largest. She ended up with three conclusions. First, when the number of disc increases by one, the number of moves for the smallest disc doubles. This pattern applies to the second smallest disc and so on. Second, the number of moves for a certain disc is just half of the one on its top. Third, the total number of moves for all discs is a geometric series. Jean's work demonstrated that she also used several problem solving strategies including simplifying the problem and looking for patterns. Apparently, her observations were in more details.

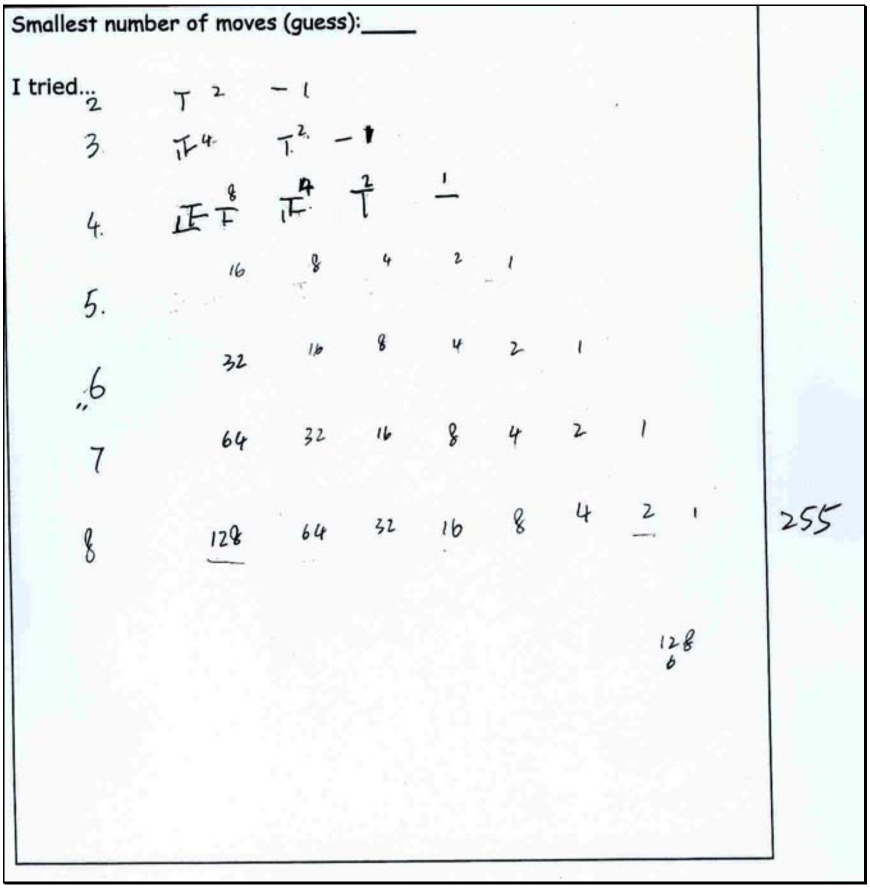


Figure 4: Jean's solution to the problem

Before solving this problem by working on the interactive Hanoi Tower, Mary, Jean and their classmates had already learned in the course many frequently used problem solving strategies such as working backwards, making tables, looking for patterns, simplifying the problem, using deduction, "guess, check and revise", exhausting possible solutions and eliminating, and drawing diagrams. About three quarters of the class (23 student-teachers) got the right answer by various solution methods. The other seven student-teachers needed hints from the instructor to start working with "3 discs" and then to look for patterns. Indeed, this problem has served the purpose of assessing student problem solving skills.

Activity 2. Besides assessing student-teacher's problem solving skills, this mathematical game can also be modified to develop *secondary* students' problem solving skills. Instead of working freely, secondary students are guided by Table 2. They are asked to explore the interactive Hanoi Tower for different numbers of discs and to write down their observations in the table. When the students are engaged in the interactive learning activity, the teacher needs to provide guidance to those who cannot get the least number of moves for the simplest case. After having got the least numbers of moves for "3 discs", "4 discs", and "5 discs" respectively, the students are asked to project the least number of moves for "8 discs". By doing so, the students are forced to analyze the observed data and have an opportunity to develop the ability of looking for patterns and making projection by reasoning.

Table 2: Problem Solving Activity with Little Guidance

Number of Discs	**Least Number of Moves**	**Remarks**
3		*Working on computer*
4		*Working on computer*
5		*Working on computer*
8		*By projection*

Activity 3. The above activity can also be modified to help develop *primary* students' problem solving ability of looking for patterns and making estimation. As the cognitive skills and number sense of primary students are normally weaker than those of secondary students, more guidance is incorporated in the activity. For primary students, looking for a pattern among the numbers of 7, 15, 31,… would be harder than finding a pattern for 2, 4, 8, 16, …. Therefore, a column of (n + 1) is introduced in Table 3 for primary students. In this activity, students may work in pair so that they can provide mutual support to each other during the activity. In order to find the least number of moves for "7 discs", students have to analyze the given data and the data obtained through working on the computer. Through the learning process, students can develop problem solving abilities of looking for patterns and making estimation by reasoning.

Table 3: Problem Solving Activity with More Guidance

Number of Discs	**Least Number of Moves, n**	**n + 1**	**Remarks**
1	1	2	*Given*
2	3	4	*Given*
3			*Working on computer*
4			*Working on computer*
7			*By projection*

3. Comparison between Interactive Tower and Wooden Tower

A wooden Hanoi Tower can serve as solid manipulative that helps small children understand the action of moving the discs. It is appropriate for junior primary students who are young and do not have much experience in working on computer. However, a simulated Hanoi Tower is preferred for secondary students and student-teachers as most of them understand the rules of moving the discs and have sufficient experience in working on computer. Furthermore, a simulated Tower has many advantages over a wooden one as follows:

1. It takes no additional physical place to keep the Tower;
2. It is environmental friendly as no wood is destroyed for making it;
3. It can count the number of moves automatically so that players would not be distracted to keep the number of moves by heart or by writing;
4. It can keep the time of the process so that the activity would be more exciting as players can compete with each other not only for getting a correct answer but also for minimum time; and
5. It guarantees that all counts are eligible as it does not allow any "illegal" move that may happen in wooden tower (e.g., see Figure 5).

Figure 5: An "Illegal" Move in Wooden Tower

4. Discussion

This paper aims to share with teaching professionals an experience of using free Internet resources for teaching and learning. With thoughtful design and planning, an available mathematical game can be transformed into various problem solving activities for students of different mathematical background and cognitive levels.

Genuine problem solving requires a problem that is just beyond students' skill level so that no automatic solution will be recalled. The problem should be non-routine, in that students perceives it as challenging and unfamiliar, yet not undefeatable (Becker & Shimada, 1997). Hanoi Tower is not involved in any parts of the current formal mathematics curricula. So, the majority of secondary and primary students do not have any experience on this mathematical activity. As such, the three activities proposed here provide a good opportunity for them to exercise thinking skills and to develop problem solving abilities of looking for patterns and making projection by reasoning.

Thompson (1989) argues that the art of teaching mathematical problem solving is best mastered over a long period of time. Schoenfeld (1992) also concurs that teaching problem solving is difficult, as teachers must decide when to intervene, and what suggestions will help the students while leaving the solution essentially in their hands, and carry this through for each student. So, although the three problem solving activities are purposefully designed separately for student-teachers, secondary students and primary students, the effectiveness of using them will depend on the skills and experience of the teachers.

Even for the same mathematics topic, instructions and learning activities may vary for different ability groups. So, teachers may make necessary changes on the proposed activities to cater for the needs of their students. For example, if the mathematics background of the students is strong, the teacher may ask students to find the general solution for the least number of moves for n discs and to explain their thinking to others. This change in instruction drives the students to exercise three key practices in mathematics learning: modeling, generalizing, and justifying (Carpenter & Romberg, 2002). As no definite number of discs is provided, the thinking process involves the practices of modeling and generalizing, and the explanation requires the practice of justifying.

The free software provides a platform for students to explore an interactive activity in order to find out the least numbers of moves for a large number of discs. As the software is readily available on the Internet and it counts the number of move automatically, students can concentrate on the echoed responses on the monitor and focus on the ways of getting the least number of moves. Activities 2 and 3 provide students with an opportunity to develop thinking skills while the software engages students in the learning process. The potential factors affecting the effectiveness of these activities and associated worksheets may worth further investigation.

All in all, there are a lot of free resources on the Internet. If these resources are used thoughtfully, many meaningful learning activities can be developed. When students are engaged in interactive and purposefully designed activity on the Internet, their learning interest and competence in the subject will be further enhanced.

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