The Learning of Linear Algebra Concepts: Instrumentation of CAS Calculators

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Abstract: While a relatively small group of researchers internationally has addressed some of the problems in the learning of linear algebra, including the use of technology, there are still many problems for students. Many of them find a number of aspects of linear algebra difficult to learn and often seem to prefer to engage in procedural manipulations rather than a study of the underlying concepts and ideas. At Auckland University computer algebra system (CAS) calculators have, in recent years, been made available to beginning students of linear algebra. This research considered the reactions of a group of these first year university students to the use of the CAS calculator in their learning of linear algebra. This was the first time that most of these students had used the CAS, and so we considered issues associated with their initial instrumentation of the CAS and their attitudes to using it in their learning. We found that the cost of the technology is an issue preventing many from obtaining it, and that those few students who did choose to purchase and use the CAS did not often use it to improve understanding of conceptual ideas. Generally they only used the CAS procedurally, usually employing it to check answers or to perform single step direct calculations to calculate, for example a determinant, or an inverse, of a matrix. Our research supports the view that instrumentation of CAS calculators does not occur naturally or spontaneously, even when students desire to integrate the technology. We infer that it is not enough for lecturers to simply be using the technology but that its use needs explicit and sustained attention.

1. Introduction

In recent years researchers have increasingly been concerned with difficulties related to first year university linear algebra courses (see [10], [11] and [4]). Teaching and learning in this cognitively demanding course can be a frustrating experience for both teachers and students, and some feel it is likely to remain so for most students (see [4]). According to [11] one reason for this is that, unlike calculus, linear algebra is the first course that students encounter that is based on mathematical theory, built systematically from the ground up. Linear algebra has been described as a unifying and generalizing theory which is also a formal theory (see [3]), and while it is thus a very powerful theory for solving a wide range of problems, it is familiarity that breeds usefulness. In contrast, for the learner the subject can be very intense, with rapid introduction of ideas and definitions and an apparent lack of connection to mathematics they already know.

Dorier and Sierpinska ([4], p. 256) declare two major sources of difficulty with the linear algebra course; “the nature of linear algebra itself (conceptual difficulties) and the kind of thinking required for the understanding of linear algebra (cognitive difficulties)”, and claim that in most cases these two sources are inseparable. Another problem area is that a typical linear algebra course comprises many languages and representations. Hillel ([12], p. 232), for example, has proposed three type of languages or levels of description:

- The language of the general theory (vector spaces, subspaces, dimension, operators, kernels, etc.)
- The language of the more specific theory of R^n (n–tuples, matrices, rank, solutions of the system of equations, etc.)
- The geometric language of 2– and 3– space (directed line segments, points, lines, planes and geometric transformations).

He believes that while these languages are interchangeable they are definitely not equivalent. Dorier and Sierpinska [4] add 'graphical', 'tabular' and 'symbolic' modes or languages to the above list, and a typical linear algebra course would include Cartesian and parametric representations of subspaces too. It may be that teachers and textbooks move between these languages and modes naturally and rapidly, not allowing students time to discuss and interpret their validities. But it is a mistake to assume that students will pick up correct understandings along the way, since, as Hillel ([12], p. 233) suggests, "knowing when a particular language is used metaphorically, how the different levels of description are related, and when one is more appropriate than the others is a major source of difficulty for students."

One of the methods that has been trialled to improve learning in linear algebra is the introduction of technology. For example, Sierpinska, Hillel, and Dreyfus [17], attempted to overcome the obstacle of formalism by designing a linear algebra entry based on a model of the two dimensional vector space within the dynamic Cabri Geometry II environment. They found that by using computers the students achieved a level of understanding of linear algebra rarely encountered in ordinary classrooms. One technology that has been of special interest over the past decade is the CAS calculator, whose perceived advantages over computer-based CAS include greater portability and affordability. Hillel ([13], p. 372), believes that CAS may be of value if R^n is used as the exemplar space:

> While CAS afford a great tool for manipulating matrices and solving systems of equations, they do not offer obvious means to help students' understanding of the abstract constructs of the general theory of vector spaces. But, if one stays within the confines of the concrete level of R^n CAS activities can be used in a variety of ways to enhance students' understanding and appreciation of the subject.

In this vein a study by Quesada [16] used a CAS calculator to modify a traditional linear algebra course into a matrix-oriented one and students felt the calculator helped them to understand better and to feel more confident doing linear algebra. The aim was to try to improve students' conceptual understanding of linear algebra by reducing the amount of by-hand calculations required, freeing students to consider concepts and help them think geometrically. A survey showed that students "found the calculator very helpful to solve problems, to investigate, to cover the material more in depth, and to facilitate the study of different applications. They seem to believe that the calculator helped them to better understand the course content and to feel more confident doing linear algebra." (see [16], p. 321).

One of the key issues with CAS use is the extent to which students form a partnership with it in their everyday mathematical work. French researchers, including Artigue [1], Guin and Trouche [8], and Lagrange ([14] and [15]) have applied the distinction between *tool* and *instrument* use to CAS. This *instrumentation,* which includes the adaptation of the CAS to a particular task, deciding what it might be useful for, how it might be applied, and development of the skills needed to use it for the task, turns out to be unexpectedly complex (see [1]). Through their individual interactions students have to learn to decide what CAS is useful for and what will be better done by hand (see [20]). This aspect of integration was addressed by Flynn, Berenson and Stacey [5] who suggest that historically students have been encouraged to develop proficiency in by-hand techniques, and the change to an instrumented approach will require deliberate attention. Herwaaden and Gielen [9] found that students often don't seem to be able to incorporate CAS into their mathematical thinking. They integrated CAS into a traditional course with a special attempt to include pen and paper with computer algebra techniques, and found that this facilitated the process of helping students to include CAS into their mathematical way of thinking.

Of course there can be drawbacks with technology use too. One such problem is described by Tall ([18], p. 35), who mentions how "students learn what they do. If they press buttons, they learn about button-pressing sequences. What is therefore important is to build a sense of meaning through reflection on the underlying mathematics. It is here that a *versatile* approach may prove of real value." Without this emphasis, he argues, in some cases students may even lose some of their basic abilities. Further, an approach that fails to allow students to build such a sense of meaning is in danger of losing students interest. Goos, Galbraith, Geiger and Renshaw [7] found this to be true of some students in their study. They report student comments such as: "I just don't understand what I'm learning here. I mean all I have to do is ask the machine to solve the problem and it's done. What have I learned?" (*ibid*, p. 226).

In this paper we describe the results of a study that considered the reactions of a group of first year university students to the use of a computer algebra system (CAS) calculator in their linear algebra learning. Some issues surrounding instrumentation of the CAS and the students' attitudes to its use are addressed.

2. Method

This case study research involving sixty-five stage one (Maths 151) mathematics and science students from the University of Auckland took place in October 2003. Maths 151 is a core course designed for mathematics majors, covering both calculus and linear algebra, and the majority of students who enrol in this paper have no prior familiarity with linear algebra. TI-89 CAS calculators have recently been introduced as an optional component of the course and are available for all students to purchase. One of the two lecturers integrated TI-89 CAS calculators in the course and used them in teaching the linear algebra, by demonstrating and carrying out various commands during most of the lectures, while the other did not. The linear algebra part of the course, which we were concerned with, begins with Gaussian elimination and its applications, and follows with matrix arithmetic, including the inverse and determinants of matrices, and concludes with a very brief introduction to vector spaces and linear transformations. There is a recommended textbook and students also receive a study guide. In addition, an assistance room is provided, where help from tutors is available on a daily basis, although the majority of the tutors available are not familiar with the CAS calculators. Of the 116 students enrolled in this paper about ninety were attending classes and sixty-five agreed to take part in the study. These students were mostly 17-18 years old, with a majority of Asian students. Very few, if any, had ever used a CAS calculator before this course. During a fifty-minute tutorial towards the end of the course each student was given a linear algebra test and a questionnaire on their attitude toward the use of CAS calculator. Students using a calculator were asked to indicate with a ◊ whenever they used the TI-89, and to state why they decided to use it.

There were 9 questions in the test. The first 7 questions (see Figure 1) were aimed at student understanding of basic linear algebra concepts such as echelon form, types of solutions, linear independence, solution consistency, and matrix invertibility. The last 2 questions were related to linear equations and are not discussed in this paper. For example, in question 1 they were asked to find the reduced row echelon form of the augmented matrix. We wondered if the students using the TI-89 would have more time to think about the solutions, since they did not have to carry out the row reduction procedure. Flexibility in moving between representations was also examined in question 3. The question was presented algebraically but required a graphical understanding of the situation in order to answer it. Question 5 examined the ability to interpret $\det(A)=0$, and question 4 was a challenging investigative type of question where solution method was of key interest, along with whether the CAS would be of assistance.

1. Find the general solution of the following system of equations by finding the reduced row echelon form of the augmented matrix of the system.

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & - & x_3 & = & -5 \\ x_1 & + & x_2 & + & x_3 & = & 2 \\ x_1 & + & 2x_2 & + & 2x_3 & = & 5 \end{array}$$

2. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Is det(5*A*)=5det(*A*)? Explain your answer.
3. Each of the following equations determines a plane in R^3. Do the two planes intersect? If so, describe their intersection.

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & 5x_3 & = & 0 \\ 2x_1 & - & x_2 & + & 8x_3 & = & 9 \end{array}$$

4. If the matrix $M = \begin{bmatrix} a & -2 \\ 8 & 4 \end{bmatrix}$, can you find any values of *a* for which $M^k = 0$ for some *k*?
5. Use the determinant to decide if $v_1,\ v_2,\ v_3$ are linearly independent when

$$v_1 = \begin{pmatrix} 5 \\ -7 \\ 9 \end{pmatrix},\quad v_2 = \begin{pmatrix} -3 \\ 3 \\ -5 \end{pmatrix},\quad v_3 = \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix},$$

6. For what values of *h* and *k* is the following system consistent?

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & h \\ -6x_1 & + & 3x_2 & = & k \end{array}$$

7. Mark each statement True or False. Justify each answer.
   a. If *A* is invertible, then the inverse of A^{-1} is *A* itself.
   b. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and *ad*=*bc*, then *A* is not invertible.
   c. If *A* can be row reduced to the identity matrix, then *A* must be invertible.
   d. If *A* is invertible, then elementary row operations that reduce *A* to the identity I_n also reduce A^{-1} to I_n.

Figure 1. Some of the Test Questions.

The attitude questionnaire (see Figure 2), used a three point Likert scale (agree, neutral, disagree), and comprised 17 statements to elicit student opinions on linear algebra. In addition a few open questions, including a space for additional comments, were included.

3. Results

Question one essentially tested students' procedural ability to find the reduced echelon form of a 3 by 4 matrix representing a linear system in three variables. We found that the 65 students were generally able to perform some kind of Gaussian elimination (not always successfully), however they often failed to give a full answer. Only 34% answered this question correctly, while 26% were able to complete the row reduction but did not give a general solution. Even among those with correct answers very few mentioned the existence or role of a free variable.

Question 2 in the test could be answered by use of the general rule that $\det(kA) = k^n \det(A)$, when *A* is a matrix of size *n* by *n*. However, we found that students did not use this. Instead, 49% took a procedural approach to the question, calculating both det(5*A*) and 5det(*A*).

Some (about 9%) made errors in their calculations and so ended up with a wrong conclusion (i.e. they said ‘yes they are equal’), and 12% of students did not respond to this question at all.

1. The TI-89 graphical calculators do not improve my understanding of mathematics.
2. I waste a lot of time trying to get the TI-89 calculator going.
3. I am glad that I can use the TI-89 calculator during the exam.
4. TI-89 calculators help me to visualise the problems.
5. I can solve problems using TI-89 calculators even though I don’t quite understand the theory.
6. My answers are usually different from the answers that TI-89 calculator gives me.
7. I wish we had a better manual so we could know our TI-89 calculators better.
8. I often check my answers using the TI-89 calculators.
9. I would like to learn more about the TI-89 calculators, so I can use them fully.
10. I believe technology is the way to go to learn mathematics.
11. I hope to use my TI-89 calculator in other courses when applicable.
12. Most tutors are not familiar with the TI-89 graphic calculators.
13. My lecturers are very supportive and encouraging in using the TI-89 calculators.
14. TI-89 calculators make mathematics fun.
15. Since I have been using the TI-89 calculator, I have forgotten how to do the basic skills.
16. I like to use both TI-89 calculator and pen and paper when working on maths problems
17. I only use a TI-89 calculator when I am stuck using pen and paper for maths problems

Figure 2. The Attitude Questions.

It is worth noting that students were more disposed to calculate answers using an algorithmic approach to this problem than they were to employing a conceptual approach. This may be because they did not know the relationship, or were sufficiently unsure to check it, or because they preferred to work this way. However, there was some evidence that students did not know the result, since only 7% of students calculated both determinants and then went on to say that $\det(5A)=5^2\det(A)$. However, none generalised their findings to present the result as $\det(kA) = k^n \det(A)$.

In question 3 we were interested to see if students could make a connection between the algebraic and graphical representations. Although the question explained that the equations determine a plane in R^3, it was not clear that students would see the intersection of the planes represented by the two equations as a straight line. While 35% of students didn't answer this question at all, most students who did try it went as far as doing row reduction (some correct and some not) without mentioning the existence of a line. Only 11% of students obtained a correct answer to this question. For question 4 only 21.5% were correct and 64.6% did not answer at all. Contrary to expectations none of the CAS users even attempted this question. Question 5 was targeted at the concept of linear independence of vectors, and 27.7% correctly found a zero determinant and its meaning. 35.3% did not answer and the others either found the determinant correctly but forgot its meaning, or calculated incorrectly. The answers to the corresponding question in two variables, question 6, followed a similar pattern. Again, many students went through the process of row reduction but said nothing about their findings, 23% did not answer the question at all, and 38% answered the question about the concept of consistency well. In question 7 on invertible matrices only 15.4% were fully correct with explanations, with 9.2% not replying to this question. In summary, a mean facility of 22.1% on questions 1-7 (range 7–38%) indicates that there were some problems with understanding the basic concepts represented in the questions asked. There is also some indication that students were better at following procedures than they were at considering the underlying concepts. A summary of these results is shown in Table 1.

Table 1. Percentages of Students Responding to the Test Questions

Questions	Correct response (%)	No response (%)	Comments
Q1	34	0	Very few mentioned the existence of a free variable
Q2	7	12	No one showed the general solution
Q3	11	35	Most students didn't mention the existence of a line
Q4	21.5	64	None of the CAS users attempted this question
Q5	27.7	35.3	Some found the det correctly but forgot its meaning
Q6	38	23	Students row reduced but said nothing more
Q7	15.4	9.2	Most students didn't explain, just said true or false

3.1 CAS Calculator Use

A survey of the purchase of the TI-89s for the whole of the Maths 151 course showed that penetration was running at around 15%. Hence it was not surprising that of the sample of 65 students in the study only 9 (13.8%) had the calculator. There were 5 male and 4 female students, 5 European and 4 Asian, with 4 aged 19-22, 3 23+ and 2 17–18. While this is a relatively small sample, and no other demographics were collected for these students, some conclusions will be made below about what their data reveals. Since we were still interested in how they had used the technology, and whether there was any evidence of advantage in doing so. Overall there was no significant difference in performance between those who used the TI-89 and those who didn't.

Of the nine students who used the CAS (all of whom were new users), five of them used the calculator primarily to do or check their working, and usually made a comment to that effect, as seen in the work of student 9 in Figure 3. Four of them did not indicate any use of the TI-89 at all, although all of them answered in the questionnaire that they used it to check answers, and one of them commented on the open question of the questionnaire that "Looking at graphs for understanding is helpful. Checking answers is great.". Hence they may simply have failed to make a note where they checked their answers. Such checking was particularly evident for the row reduction in questions 1, 3 and 6, and the determinant calculations of questions 2 and 5. Students wrote, for example, "Use TI-89 to check my solution", "check row reduced form" check my det [determinant] answer", "row reduce check", "used TI-89 to check if solution was right", and "used to check answer (to know where I was going)".

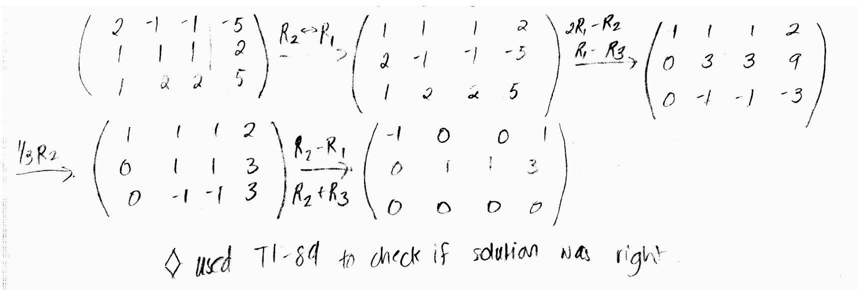

Figure 3. Student 9 Using the TI-89 to Check Working.

In contrast to this, student 3 demonstrated some measure of integration of the CAS into his working. In question 1 (see Figure 4) he used the TI-89 to reduce the matrix to reduced echelon form in one step (no doubt using the single command rref()). He then showed his understanding of what this matrix represented by being able to turn it into an algebraic solution. He also knew about the use of a parameter for the free variable x_3, setting this to t, and was able to write the solution for x_1 and x_2 in terms of t (he has a mistake in finding x_1, copying from the wrong matrix). He liked the CAS and commented that "TI-89 has a good matrix function and is easy to use to row reduce".

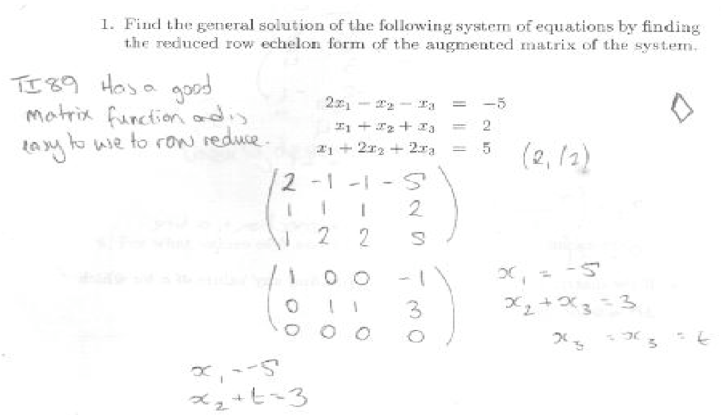

Figure 4. Use of CAS for Working Within a Question.

In question 5 (see Figure 5) he again used the CAS to do the work of finding the determinant of a 3 by 3 matrix, commenting that the "TI-89 very gd [good] for finding determinants". Here too he was able to tie the result of zero to his conceptual knowledge to answer the question and state that the vectors were linearly independent. Student 3 used the TI-89 in all 5 questions he attempted, using the row reduction in question 3 before correctly writing down the intersection and saying that "intersection is a line."

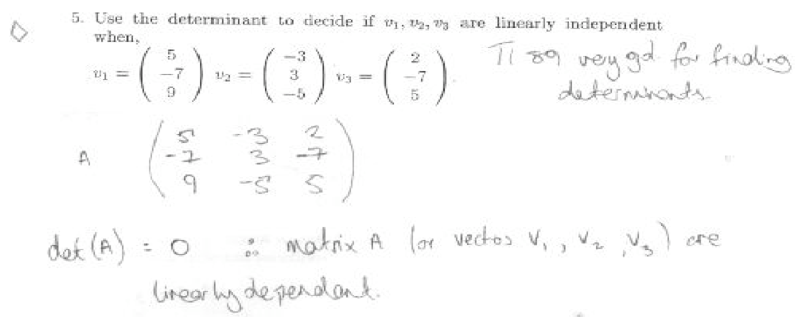

Figure 5. Use of a Direct CAS Command to Find a Determinant.

It appears that the CAS helped him to keep an overview of where the question was heading. It may also have assisted him by eliminating mistakes in the process of row reduction and determinant calculation, saving him time in doing calculations, thus enabling him to concentrate on finding the solutions. Student 9 also decided to use the CAS at least twice in the middle of a question. In question 2, having calculated det(A) she stated she "used TI-89 to calculate 5A (shortcut)". This seems a surprising step to do on the calculator (and certainly to call a shortcut) because it only involved multiplying 4 numbers by 5, and the matrix had to be entered. However, such decisions are part of the instrumentation process, and it may be that the student was unsure how many, and which, of the matrix entries to multiply by 5.

3.2 Questionnaire Analysis

The 17 Likert-style statements in the questionnaire given to all the students addressed a number of different issues. The mean response scores for the CAS students are shown in Table 2, and the results of these students, for whom the questions were relevant, are analysed below. One issue was whether these students believe that the CAS has any influence on their understanding of the mathematics. Statements 1, 5 and 15 addressed this. 5 of the students disagreed with statement 1, and the rest were neutral, and 4 agreed with statement 5. Furthermore, 5 were neutral on question 15, with 3 disagreeing. Taken together these responses indicate a complex picture with regard to understanding. 5 of the students think that the CAS helps their understanding of mathematics, but that on occasions they can get answers right without that understanding, and some feel the CAS can even remove basic by-hand skills. This seems to suggest, as expected, that the manner in which CAS is employed in teaching and learning situations needs careful consideration. It may be easy to get the balance of use wrong, pushing students into button pushing without understanding, and also leading to a loss of skills.

Table 2. The Means of the Questionnaire Responses for the CAS Students

Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Mean	1.4	1.7	2.8	2.6	2	1.5	2.5	3	2.8	2.6	2.8	2.1	2.7	2	1.7	3	2.2

The issue of encouraging instrumentation of CAS is at the heart of this dilemma, and 3 questions, 8, 16 and 17 addressed this. Here, all the students agreed in question 16 that they like to use both TI-89 and pen and paper, but this did not give any information about when they decide to use each. The evidence from statement 17 was that 3 of the students said that they only use the TI-89 when they get stuck, and 5 of the students were neutral (or maybe unsure) on this. However, as noted above, all of them agreed in question 8 that they use it to check answers. This gives a picture of use primarily for checking answers, with further use restricted to occasions when they cannot make progress with a solution by hand. Given that they like to use both, the process of instrumentation of the CAS is clearly at a very early stage, since it is not yet an instrument of first choice.

We believe one primary variable in promoting instrumentation to be the support mechanisms in place, including resource material that encourages CAS use and supportive lecturers and tutors. Questions 7, 9, 12 and 13 addressed these issues. The results revealed that 8 of the students wanted to learn more about the CAS, and 6 wanted a better manual to help them do so (one even wrote “big time” alongside his selection). This is a factor that tertiary staff need to consider closely. 7 of the students were pleased with the amount of support and encouragement from their lecturers. However, the students were less happy with their tutors, with only 3 agreeing with statement 12, with 4 neutral. It seems that most tutors on this course were not familiar with the TI-89 and so were unable to provide much assistance for the students.

A space at the end of the questionnaire was provided for further comment from the students and some interesting observations were made, primarily by those who did not use the CAS, often explaining why. Many of the comments involved some aspect of the high cost of the TI-89 calculators:

> I don't really think a TI-89 is really affordable for me in this situation: So expensive, and I really had spent lot of money for study.
>
> Just too expensive.
>
> Should be made cheaper.

Other reasons given for not using the CAS included being content with a present calculator:

> TI-89 is just a waste of time for some reasons, it is even slower than my old calculator FX-570; and lot more expensive.
>
> I went for the HP 49G because it has more applications to after university life.
>
> Already have a 991w CASIO which is good enough.

Some comments reflected the feeling that the CAS was unfair on those not having one, especially for assessment purposes:

> I don't really suggest that we should use it in exam…It is really not fair to those students that don't want to use a graphic calculator.
>
> Unfair advantages to people who can't afford one. Too easy to check answers. Should be there for assistance only during the year but not be able to have use of them in the exam.

We also found some evidence, in support of Goos, Galbraith, Renshaw, & Geiger [6] that students don't like just pushing buttons, and they often wonder what have they actually learned from the CAS. They commented that:

> I do not feel that a calculator should be given a lot of importance because mental calculations actually helps you to improve your mind and also you know what you are doing.
>
> It is essential for me to first know and understand how to solve the problems than to rely on the calculator. Realistically not everyone carries a calc everywhere they go.
>
> I would only like to use the TI-89 graphic calculators as a last option or an alternative way to confirm or solve mathematical problems.

In conclusion, while the sample in this study was small, and so the results are not generalisable, they seem to indicate that our students had a tendency to prefer procedural working in linear algebra, and were not so good at understanding basic concepts, or applying the results of the procedures. They also had problems relating the working to relevant geometry. We found no evidence of benefits of CAS use to understanding. Instead our results support the findings of Thomas and Hong [19], and others, who have found that instrumental genesis of CAS is a relatively slow process that needs careful monitoring. Thomas and Hong describe a number of qualitatively different ways students interacted with CAS, including: performing a direct, straightforward procedure, checking of procedural by-hand work; performing a direct complex procedure; performing a procedure within a more complex process; and investigating a conceptual idea. In their instrumentation, our students primarily using the CAS to check by-hand work, although there was a little evidence of one or two students starting integration of CAS, with the use of single direct CAS commands within an overall method. It has been argued that CAS can assist students with getting an overview of problems (see [20]), but this study provided very little support for this position. One concern here is that raised by Ball [2], who argues that students need to develop good algebraic strategies for solving equations, since in some cases CAS may not be able to give answers immediately in one step. They need to be able to rewrite the equations in a way that can be recognised by CAS system, in order to find a solution. Clearly instrumentation of CAS is complex and does not occur naturally or spontaneously. Our research supports the view of Flynn, Berenson and Stacey [5] that it is not enough for lecturers simply to be using the technology but that its use needs careful, specific and sustained attention. Furthermore, this is true even when students desire to integrate the technology into their learning, since our students wanted to instrument CAS but lacked the necessary techniques to make good choices about doing so. Currently the mathematics department is producing a draft technology policy aimed at greatly increasing technology participation in its courses. Thus this issue of instrumentation of CAS in learning linear algebra is one that we will be addressing in future studies, where we hope to have more participants.

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