Dynamic Visualization of Complex Integrals with Cabri II Plus

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Abstract: Dynamic visualization helps us understand the concepts of mathematics. This paper shows that of complex integrals with Cabri II Plus. The geometrical interpretation of complex numbers was introduced by Gauss using the complex plane, but in general this geometrical idea has not been sufficiently made much of in the scene of education. This is only taught in the beginning, and is rarely used in teaching complex integration. Furthermore its dynamic visualization is hardly found out. And this is one of the most difficult fields for students, while it often appears in studying science. This paper highlights not only the visualization of mathematical concepts but also the effectuality of moving figures. Observing their movements makes it easier to find out mathematical properties. Cabri Geometry enables us to make moving figures without the knowledge of programming. Complex integrals are visualized with the idea of the quadrature mensuration by parts, using the geometrical interpretation of the product of complex numbers; rotation and dilation. This dynamic visualization helps students understand complex integration deeply.

1. Introduction

This paper gives the dynamic visualization of complex integrals with Cabri II Plus, which is a dynamic geometry software ([1]). The integral of a real variable function $y=f(x)$ on the xy-plane is easily visualized as an area. This idea is often used in the scene of the education, and helps students understand the meaning of the integration. As for the complex integration, such visualization is hardly shown, and it may drive students to think that complex integration is difficult. There are only some works ([2], [3], and [4]) for the visualization of complex integrals. I'd like to expand their works. In this paper we can get results of integrals <u>without calculation</u>. <u>We only need to draw a circle (or an arc) with a certain length of a string</u>. Furthermore, in this way, we can find the symmetric property of complex integrals. Thus we easily understand the special property of $1/z$, and it leads us to open the door to the concept of residues, which explains us the necessity of Laurent expansion. Cabri II Plus provides us opportunities to create moving figures only with basic knowledge of geometry.

2. Visualizing Four Operations of Complex Numbers

We will get the values of integrals by repetition of multiplication and addition, which are geometrically visualized on Gauss plane. Addition and subtraction of complex numbers are shown as those of vectors. We can use tools of Cabri II Plus to draw the sum of vectors. As for multiplication and division, they are shown as rotation and dilation;

$$|z_k| = r_k,\ \arg z_k = \theta_k,\ i.e.,\ z_k = r_k e^{i\theta_k}\ (k = 1,2,3),\quad z_3 = z_1 z_2. \Rightarrow \quad r_3 = r_1 r_2,\ \ \theta_3 = \theta_1 + \theta_2 .$$

The macro in Cabri II Plus can remember a flow of constructions and is useful for multiplication and quotient of vectors which need repeated constructions.

In this paper we regard the arguments of complex numbers to be the positive least.

3. Visualizing Complex Integrals

3.1 Quadrature Mensuration by Parts

Def.1 $$\int_C f(z)dz = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(z_k)\cdot \Delta z_k \qquad (\Delta z_k = z_{k+1} - z_k .)$$

$$= \lim_{n\to\infty} \{ f(z_0)\cdot\Delta z_0 + f(z_1)\cdot\Delta z_1 + \cdots + f(z_{n-1})\cdot\Delta z_{n-1} \} .$$

The integral is defined as the quadrature mensuration by parts, which is the sum of products. For simplification, we divide the contour C equally by n; all the Δz_k 's have the same length.

As for the real integration, it can be explained as the area;

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\cdot\Delta x_k \quad (= S : area) .$$

3.2 Complex Integrals of z^α along the Circle centered at the origin and radius *r* (α : integer)

The following examples visualize complex integrals by multiplication and addition as we have seen in the previous chapter.

Th.1 $$\oint_C z^\alpha dz = \begin{cases} 0 & (\alpha \neq -1) \\ 2\pi i & (\alpha = -1) \end{cases}.$$

C : the circle centered at the origin and radius r, $\oint$: the loop integral

Ex.1

$$\oint_C z dz = 0.\ (\alpha = 1.)$$

$$\int_C zdz \overset{def}{=} \lim_{n\to\infty}\left(z_0\cdot\Delta z_0 + z_1\cdot\Delta z_1 + \cdots + z_{n-1}\cdot\Delta z_{n-1}\right). \qquad (\Delta z_k = z_{k+1} - z_k.)$$

With n large enough, we can regard $\overrightarrow{OP_k}$ and $\overrightarrow{P_kP_{k+1}}$ to be perpendicular and the total sum of $|\Delta z_k|$'s to be the whole circumference of the circle C.

Now let's think about the result of the product $z_k\cdot\Delta z_k$. See Fig.1. (This figure is constructed with the tools of Cabri II Plus such as "Circle", "Perpendicular Line", "Measurement Transfer", etc. We can make a macro to construct $z_k\Delta z_k$ directly from z_k and Δz_k.) Let n be 20, for example.

$$|z_k| = r,\quad |\Delta z_k| = \frac{2\pi r}{20},\quad \arg z_k = k\left(\frac{2\pi}{20}\right) = \theta_k,\quad \arg(\Delta z_k) = \theta_k + \frac{\pi}{2}.$$

$$\Rightarrow\quad |z_k\Delta z_k| = \frac{2\pi r^2}{20},\quad \arg(z_k\Delta z_k) = 2\theta_k + \frac{\pi}{2}.\quad (k = 0,1,\cdots,n-1.)$$

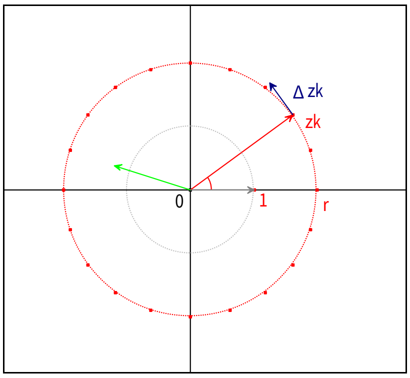


Fig.1 The Result of the Product $z_k\cdot\Delta z_k$

Then we take the sum for $k = 0,1,\cdots,19$. See Fig.2. (Macros made in Cabri II Plus help us repeat the constructions.)

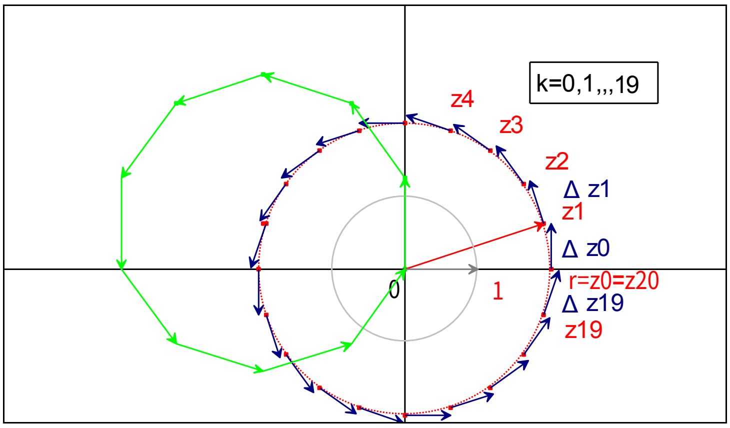


Fig.2 Taking the Sum of $z_k\cdot\Delta z_k$'s for $k = 0,1,\cdots,19$

Imagine a string winded from the origin around a circle. This string has the length $2\pi r^2$ $\left(=\sum_{k=0}^{19} z_k \cdot \Delta z_k\right)$, and goes around anticlockwise along the circle twice until the end of the whole circle, because the sum of the exterior angles equals to 4π $\left(=\sum_{k=0}^{19}\left\{2\cdot\left(\theta_{k+1}-\theta_k\right)\right\}=20\cdot 2\cdot\frac{2\pi}{20}\right)$. The result of this calculation appears as the end point of the string; $\underline{0}$. Now we get

$$\sum_{k=0}^{19} z_k \cdot \Delta z_k = 0. \ (n = 20.)$$

Therefore,

$$\oint_C z dz = 0. \ (\alpha = 1.)$$

Ex.2

$$\boxed{\oint_C \frac{dz}{z} = 2\pi i \ (\alpha = -1)}$$

Now let's think about the results of the product $\frac{1}{z_k}\Delta z_k$ .See Fig.3.

$$\left|\frac{1}{z_k}\right| = \frac{1}{r}, \ \left|\Delta z_k\right| = \frac{2\pi r}{n}, \ \arg\left(\frac{1}{z_k}\right) = -k\cdot\left(\frac{2\pi}{n}\right) = -\theta_k, \ \arg\left(\Delta z_k\right) = \theta_k + \frac{\pi}{2}$$

$$\Rightarrow \left|\frac{1}{z_k}\Delta z_k\right| = \frac{2\pi}{n}, \ \arg(\frac{1}{z_k}\Delta z_k) = -\theta_k + \left(\theta_k + \frac{\pi}{2}\right) = \frac{\pi}{2} \quad (k = 0, 1, \cdots, n-1)$$

Note that every $\left|\frac{1}{z_k}\cdot\Delta z_k\right|$ is independent of r, and $\arg(\frac{1}{z_k}\cdot\Delta z_k)$ always equals to $\frac{\pi}{2}$.

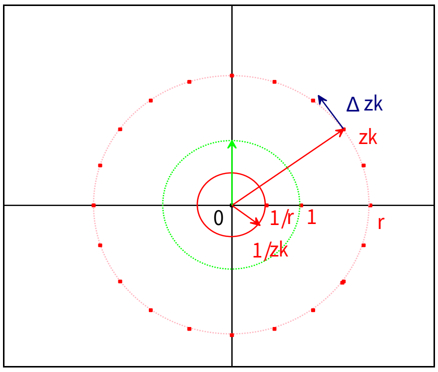


Fig.3 The Result of the Product $\frac{1}{z_k}\Delta z_k$

Let n be 20, and we shall observe the sum of $\frac{1}{z_k}\cdot\Delta z_k$'s $(k=0,1,\cdots,n-1)$. See Fig.4.

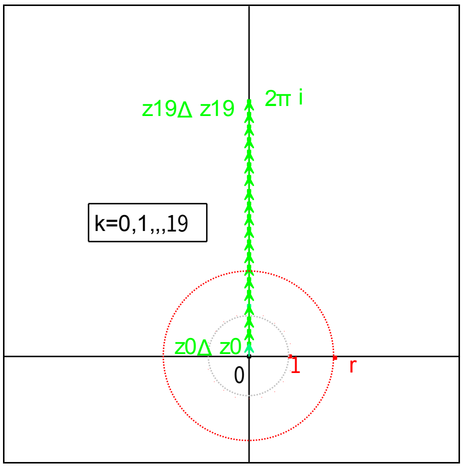


Fig.4 Taking the Sum of $\frac{1}{z_k}\cdot\Delta z_k$'s for $k=0,1,\cdots,19$

This time the string has the length 2π, and goes upright. The result of this calculation appears as the end point of the string; $\underline{2\pi i}$. Now we get

$$\sum_{k=0}^{19}\frac{1}{z_k}\cdot\Delta z_k=2\pi i \ \ (n=20)$$

Therefore,

$$\oint_C\frac{dz}{z}=2\pi i. \ \ (\alpha=-1.)$$

In addition, this example has also visualized the following integral;

$$\underline{\int\frac{dz}{z}=\log z+C.}$$

Ex.3

$$\boxed{\oint_C\frac{dz}{z^2}=0 \ \ (\alpha=-2)}$$

Now let's think about the result of the product $\frac{1}{{z_k}^2}\cdot\Delta z_k$. See Fig.5.

$$\left|\frac{1}{{z_k}^2}\right|=\frac{1}{r^2},\ \ |\Delta z_k|=\frac{2\pi r}{n},\ \ \arg\left(\frac{1}{{z_k}^2}\right)=-2k\left(\frac{2\pi}{n}\right)=-2\theta_k,\ \ \arg(\Delta z_k)=\theta_k+\frac{\pi}{2}$$

$$\Rightarrow \left|\frac{1}{{z_k}^2}\Delta z_k\right|=\frac{1}{r}\cdot\frac{2\pi}{n},\ \ \arg(\frac{1}{{z_k}^2}\Delta z_k)=\frac{\pi}{2}-\theta_k \ \ \ (k=0,1,\cdots,n-1).$$

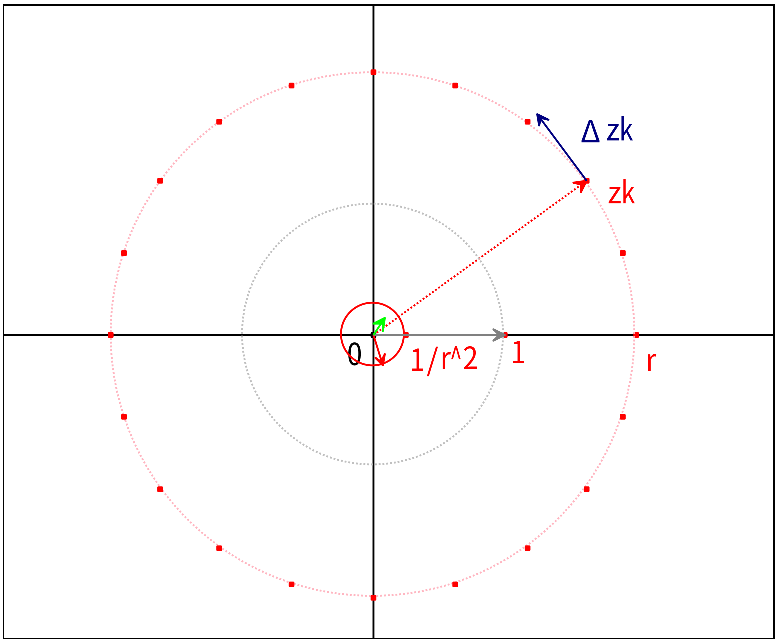


Fig.5 The Result of the Product $\frac{1}{{z_k}^2} \cdot \Delta z_k$

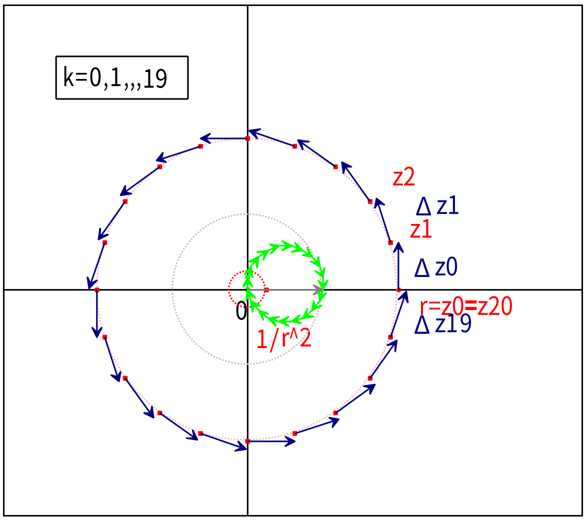


Fig.6 Taking the Sum of $\frac{1}{{z_k}^2} \cdot \Delta z_k$'s for $k = 0,1,\cdots,19$

This time the string has the length $\frac{2\pi}{r}$, and goes around clockwise along the circle once until the end of the whole circle. The result of this calculation appears as the end point of the string; $\underline{0}$. See Fig.6. Now we get

$$\sum_{k=0}^{19} \frac{1}{z_k^{\ 2}} \cdot \Delta z_k = 0. \ (n = 20.)$$

Therefore,

$$\oint_C \frac{1}{z^2} dz = 0 \ (\alpha = -2).$$

Summary of 3.2

Substituting $\alpha = \beta - 1$ into *Th.1*, we obtain the following theorem.

Th.2	$$\oint_C z^\beta \frac{1}{z} dz = \begin{cases} 0 \ (\beta \neq 0) \\ 2\pi i (\beta = 0) \end{cases}$$
	C : the circle centered at the origin and radius r, $\oint$: the loop integral

As well as Ex.1 to 3,

$$\int_C z^\beta \frac{1}{z} dz \overset{def}{=} \lim_{n\to\infty} \left(z_0^{\ \beta} \frac{\Delta z_0}{z_0} + z_1^{\ \beta} \frac{\Delta z_1}{z_1} + \cdots + z_{n-1}^{\ \beta} \frac{\Delta z_{n-1}}{z_{n-1}} \right).$$

$$\left| z_k^{\ \beta} \right| = r^\beta, \quad \left| \frac{\Delta z_k}{z_k} \right| = \frac{2\pi}{n}, \quad \arg\left(z_k^{\ \beta} \right) = \beta\theta_k, \quad \arg\left(\frac{\Delta z_k}{z_k} \right) = \frac{\pi}{2}$$

$$\Rightarrow \left| z_k^{\ \beta} \cdot \frac{\Delta z_k}{z_k} \right| = r^\beta \cdot \frac{2\pi}{n}, \quad \arg(z_k^{\ \beta} \frac{\Delta z_k}{z_k}) = \beta\theta_k + \frac{\pi}{2}.$$

The string has the length $2\pi r^\beta$, and goes around anticlockwise along the circle β times (if $\beta < 0$ then clockwise, and if $\beta = 0$ then goes upright) until the end of the circle. The result of this calculation appears as the end point of the string; $\underline{0(\beta \neq 0)}$ and $\underline{2\pi i(\beta = 0)}$. See Table1.

Table1 Properties of “the String” Appearing in $\oint_C z^{\beta} \frac{1}{z} dz$

The Length of the String $= 2\pi r^{\beta}$. $\left(; \text{ the length of } \sum_{k=0}^{n-1} {z_k}^{\beta} \frac{\Delta z_k}{z_k} = \sum_{k=0}^{n-1} \left\| {z_k}^{\beta} \frac{\Delta z_k}{z_k} \right\| = \sum_{k=0}^{n-1} r^{\beta} \cdot \frac{2\pi}{n} = n \cdot \left(r^{\beta} \cdot \frac{2\pi}{n} \right) = 2\pi r^{\beta}. \right)$
How Many Times the String Turns Around = β *times anticlockwise*. $\left(; \text{ the sum of the exterior angles} = \sum_{k=0}^{n-1} \left({z_{k+1}}^{\beta} \frac{\Delta z_{k+1}}{z_{k+1}} - {z_k}^{\beta} \frac{\Delta z_k}{z_k} \right) = \sum_{k=0}^{n-1} \left\{ \beta \left(\theta_{k+1} - \theta_k \right) \right\} = n \cdot \left(\beta \frac{2\pi}{n} \right) = 2\pi\beta. \right)$
The End Point of the String = $0 \left(\beta \neq 0 \right)$ and $2\pi i \left(\beta = 0 \right)$

The results led by this method are justified by the following calculations;

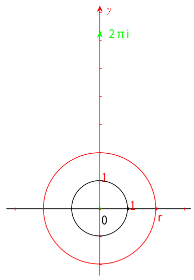
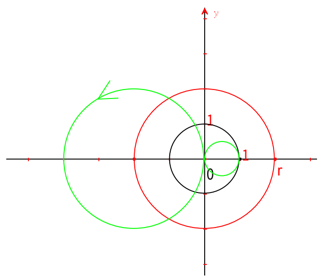
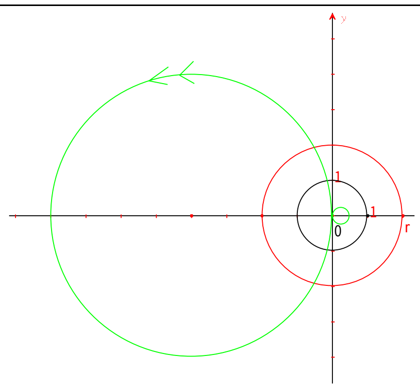
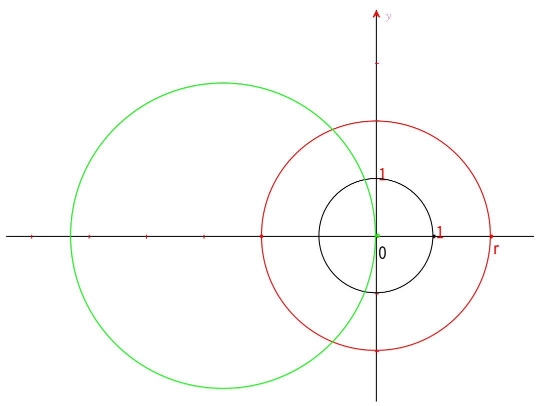
$$z = re^{i\theta},\ \theta : 0 \to 2\pi,\ dz = ire^{i\theta} d\theta.$$

$$\int_C z^{\beta} \frac{1}{z} dz = \int_{\theta=0}^{2\pi} r^{\beta} e^{i\beta\theta} \frac{1}{re^{i\theta}} \cdot ire^{i\theta} d\theta = ir^{\beta} \int_{\theta=0}^{2\pi} e^{i\beta\theta} d\theta$$

$$= \begin{cases} ir^{\beta} \left[\dfrac{1}{i\beta} e^{i\beta\theta} \right]_{\theta=0}^{2\pi} = \dfrac{r^{\beta}}{\beta} \left(e^{2\pi\beta i} - e^{0} \right) = 0. & \left(\beta \neq 0. \right) \\ i \left[\theta \right]_{\theta=0}^{2\pi} = 2\pi i. & \left(\beta = 0. \right) \end{cases}$$

These results are symmetric with respect to β. Now we get “the symmetric property of integrals”. See Table2.

Table2 The Symmetric Property of Integrals

$\beta = 0$ $\oint_C \frac{dz}{z} = 2\pi i$ $length = 2\pi$ $rotarion = 0\,times$		
$\beta = 1$ $\oint_C dz = 0$ $length = 2\pi r$ $rotarion = 1\,time$		$\beta = -1$ $\oint_C \frac{dz}{z^2} = 0$ $length = 2\pi r^{-1}$ $rotarion = -1\,times$
$\beta = 2$ $\oint_C zdz = 0$ $length = 2\pi r^2$ $rotarion = 2\,times$		$\beta = -2$ $\oint_C \frac{dz}{z^3} = 0$ $length = 2\pi r^{-2}$ $rotarion = -2\,times$
$\beta = 3$ $\oint_C z^2 dz = 0$ $length = 2\pi r^3$ $rotarion = 3\,times$		$\beta = -3$ $\oint_C \frac{dz}{z^4} = 0$ $length = 2\pi r^{-3}$ $rotarion = -3\,times$
. . .	. . .	. . .

3.3 Laurent series and Residue

The discussion in 3.2 shows the following theorem.

Th.3 $\displaystyle\oint_C f(z)dz = 2\pi i \cdot a_{-1}$

where $\displaystyle f(z) = \cdots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots$ (*)

C : the circle centered at the origin and radius r

(*) is called Laurent series, and this is the theorem of residue, which shows the importance of Laurent expansion.

4. Conclusion

Dynamic visualization is so effective. Thanks to this idea, we can understand the complex integrals and find the various properties. In the process of visualization I found the symmetry of integrals. Integrals along the other contours are also visualized in this way, and other theorems, for example Rouche's theorem, could be visualized with this idea. I hope students are taught the complex integrals with this idea and easily understand the concepts with images. For dynamic visualization Cabri II Plus is very useful.

References

[1] Cabri Geometry II plus, http://www.cabri.com.

[2] KOBAYASHI Ichiro, *Grasping the concept of integrals of complex-valued functions using Cabri*, ICME9 Cabri project, 2000, http://www.ies.co.jp/math/ICME9/.

[3] KOBAYASHI Ichiro, *Understanding Complex Analysis concepts with the help of Cabri*, CabriWorld2001 in Montreal, 2001, http://www.ies.co.jp/math/cabri/CabriW2001_HP/.

[4] Tristan Needham, *Visual Complex Analysis*, Clarendon press, Oxford, 1997.