

Signal Processing using a Computer Algebra System

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Abstract: This paper is an extension of Introducing Fourier Series with Derive [2]. The aim is to show that a Computer Algebra System is also very suitable in a lot of subjects in Digital Signal Processing like Aliasing, Discrete Fourier Transform and the Sampling Theorem.

The overall philosophy is that we are able, using a Computer Algebra System (CAS), to shift the emphasis from acquiring skills to gaining insight.

1. Introduction

It is impossible to overestimate the importance of signals to modern life, if we think of signals as information carriers, we might imagine TV, mobile phones, radar, sonar, X-ray, music, video, voicemail etc. Clearly, Analog and Digital Signal Processing is a major topic in Electrical Engineering.

However, the theory of signals places a high demand on the calculating skills of the students attending the course. The students involved should be completely familiar with:

- odd/even functions,
- features of convergence,
- understanding of complex numbers,
- basics of trigonometry,
- harmonic functions,
- integration by parts.

Furthermore the combination of these subjects may even prove to be a formidable obstacle to better than average students. Consequently, the courses would be slowed down by lengthy calculating exercises, leaving insufficient time to acquire a thorough understanding of their applications.

2. Fourier Series

The introduction of Fourier Series. In Electrical Engineering functions such as the Square Wave and Saw Tooth, shown below, are frequently used.

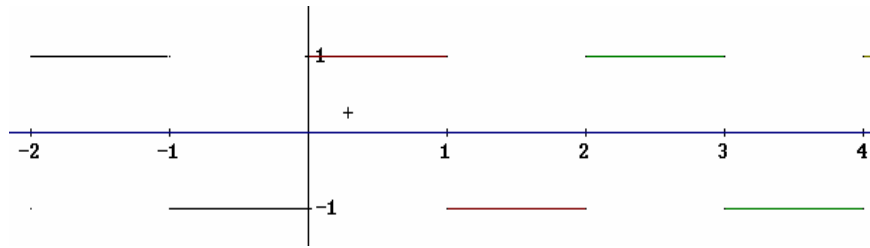


Figure 2.1 The Square Wave

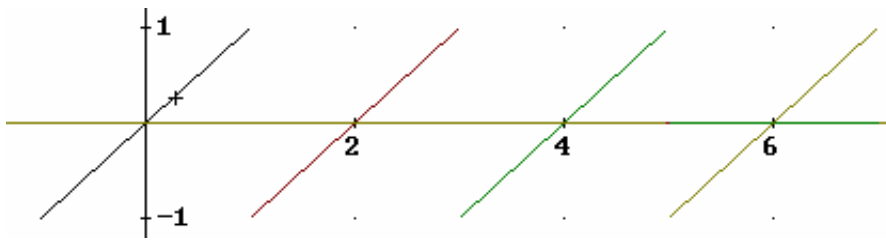


Figure 2.2 The Saw Tooth

These periodic functions may well be approximated by so-called Fourier series.

For example the next series is an approximation of the Square Wave

$$\frac{4\sin(5\pi t)}{5\pi} + \frac{4\sin(3\pi t)}{3\pi} + \frac{4\sin(\pi t)}{\pi} \quad (2.1)$$

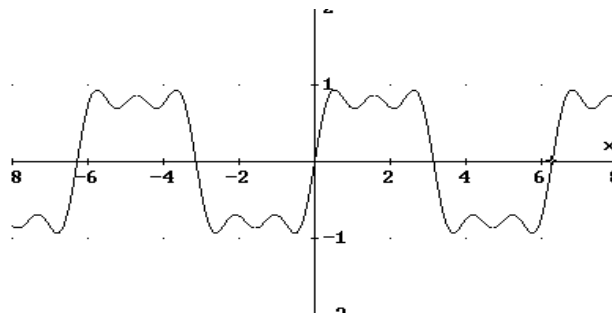


Figure 2.3 The approximation of the Square Wave (2.1)

The traditional way to introduce the Fourier series is to derive the Fourier coefficients (See [1])

$$a_n = \frac{2}{T} \cdot \int_1 f(t) \cdot \cos(n\omega t) dt \quad \text{and} \quad b_n = \frac{2}{T} \cdot \int_1 f(t) \cdot \sin(n\omega t) dt \quad (2.2)$$

from the series:

$$\frac{1}{2} a_0 + \sum_{n=1}^N (a_n \cos(n\omega t) + b_n \sin(n\omega t)) \quad (2.3)$$

For some years a Derive practical exercise (See [2]), and more recently the use of the Ti89 calculator, has been attached to Fourier Courses, enhancing the students' enthusiasm for the subject considerably. The illustrations they can produce definitely contribute to this effect.

The CAS Derive (See [5]) contains the Utility INT_APPS.MTH, which contains the function Fourier(f(t),t,a,b,n) where:

- t is the independent variable.
- f(t) is a periodic function
- [a,b] is the begin and end of one period.
- n is the number of harmonic terms you wish to demand.

Example 2.1

Find the first 5 harmonic terms of Sq(t) as given below.

$$Sq(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \end{cases} \quad \text{period } T=2 \text{ (See Figure 2.1)} \quad (2.4)$$

$$Sq(t) := \chi(0,t,1) - \chi(1,t,2)$$

Fourier(Sq(t),t,0,2,5) Simplify

$$\frac{4 \sin(5\pi t)}{5\pi} + \frac{4 \sin(3\pi t)}{3\pi} + \frac{4 \sin(\pi t)}{\pi} \quad (2.5)$$

Derive allows you to draw the individual harmonic functions of (2.5) and their summation side by side, which gives an excellent insight into the structure of the Fourier-Series.

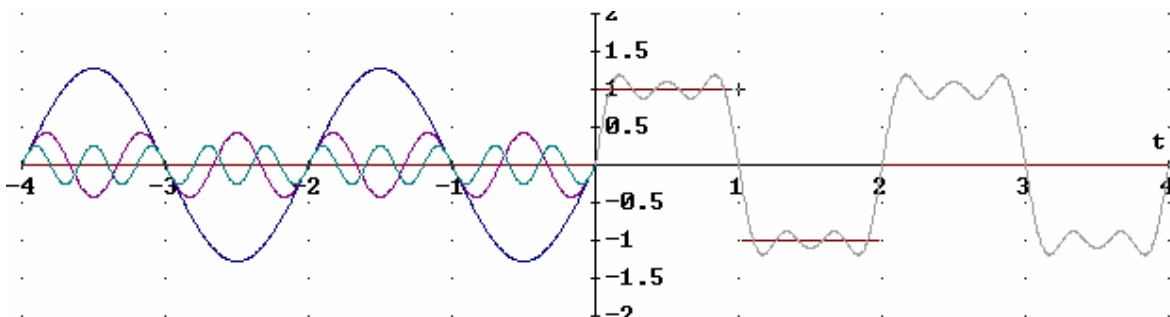


Figure 2.4 Approximation of the Square Wave in detail (2.5)

After this introduction, students are keen to make a thorough study of the underlying theory and eventually are able to calculate the following Fourier coefficients:

$n \in \text{integer}$

$$a(n) := \int_0^2 (\text{chi}(0, t, 1) - \text{chi}(1, t, 2)) \cos(n\pi t) dt = 0$$

$$b(n) := \int_0^2 (\text{chi}(0, t, 1) - \text{chi}(1, t, 2)) \sin(n\pi t) dt = \frac{2}{n\pi} (1 - (-1)^n)$$

The outcome $\sum_{n=1}^N \frac{2}{n\pi} (1 - (-1)^n) \sin(n\pi t)$ can immediately be equated to the series (2.5) found above.

The frequency spectrum can be easily found with:

$$\text{VECTOR} \left(\left[\begin{array}{cc} n\pi & 0 \\ n\pi & |b(n)| \end{array} \right], n, 1, 10 \right) \quad (2.6)$$

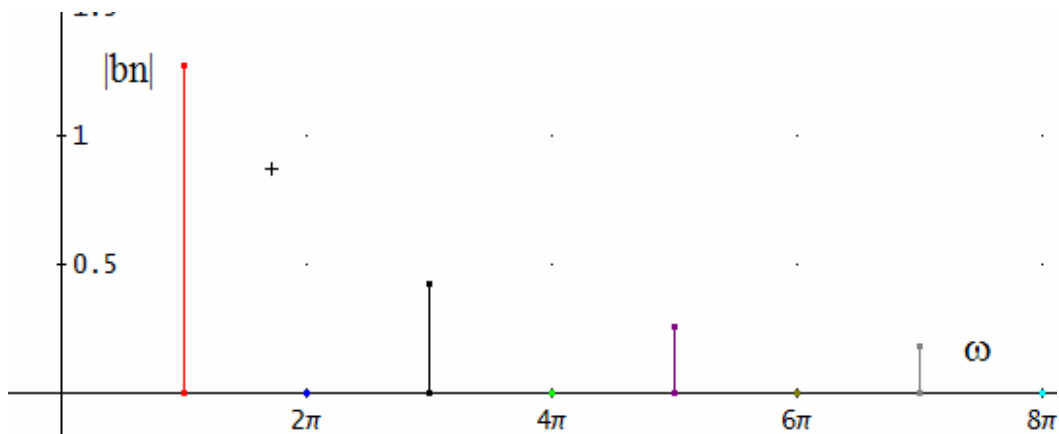


Figure 2.5 Amplitude Spectrum (2.6)

All this does not take long, there are no annoying computing errors and the resulting graphs provide instant feedback.

Until recently, any investigation of the *Gibbs Phenomenon* used to be impossible, whereas in the present situation each student can perform calculations in this field.

Example 2.2

Laboratory: Gibbs Phenomenon.

Use the square wave of $Sq(t)$ (2.4) of example 2.1. Express the Fourier series, with respectively: $N=10$ and $N=100$

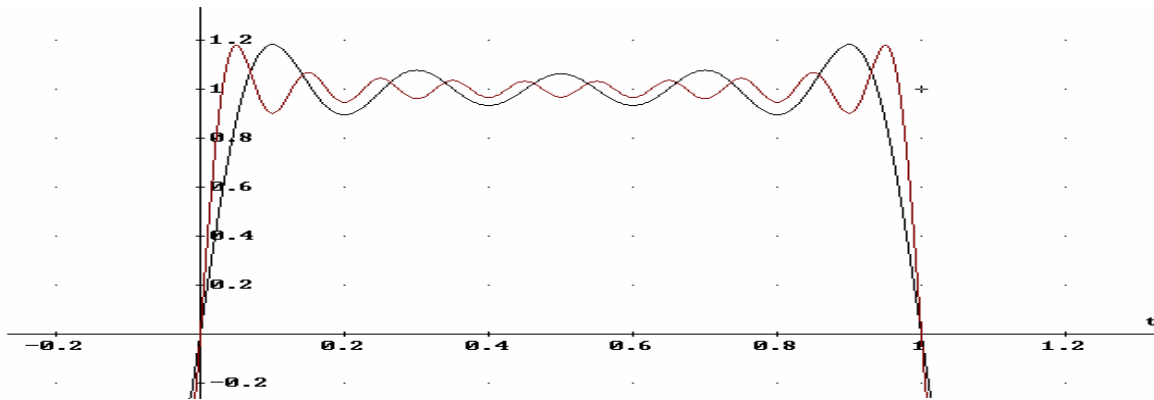


Figure 2.6 Gibbs Phenomenon

Plot the graphs on one screen.

As you see, the series approaches $Sq(t)$. However, the amount of overshoot near to the discontinuities does not tend to zero with increasing N .

Calculate with Derive this overshoot for $N=10$ and $N=100$.

Compare your result with the limit (See [3]):

$$\frac{2}{\pi} \int_0^{\pi} \frac{\sin(x)}{x} dx = 1.17897 \quad (2.7)$$

$$\frac{d}{dt}(\text{FOURIER}(Sq(t), t, 0, 2, 10))$$

$$4 \cos(9\pi t) + 4 \cos(7\pi t) + 4 \cos(5\pi t) + 4 \cos(3\pi t) + 4 \cos(\pi t) = 0$$

$$\text{NSOLVE}(4 \cos(9\pi t) + 4 \cos(7\pi t) + 4 \cos(5\pi t) + 4 \cos(3\pi t) + 4 \cos(\pi t) = 0, t, 0, 0.15)$$

$$t=0.1$$

$$Sq(0.1)=1.18233$$

For 100 terms Derive offers a maximum of 1.179013 which amounts to 8.95 % of the total jump.

I will show, without further comment, how to use a Computer Algebra System in proving the limit (2.7) above!

$$\frac{4\sin((2n-1)\pi t)}{(2n-1)\pi} + \dots + \frac{4\sin(5\pi t)}{5\pi} + \frac{4\sin(3\pi t)}{3\pi} + \frac{4\sin(\pi t)}{\pi} = \sum_{m=1}^n \frac{4\sin((2m-1)\pi t)}{(2m-1)\pi}$$

$$\frac{d}{dt} \sum_{m=1}^n \frac{4\sin((2m-1)\pi t)}{(2m-1)\pi} = \frac{2\sin(2\pi n t)}{\sin(\pi t)}$$

$$\text{SOLVE}\left(\frac{2\sin(2\pi n t)}{\sin(\pi t)}, t\right) \quad t = -\frac{1}{2n} \vee t = \frac{1}{2n} \vee n = 0$$

$$\sum_{m=1}^n \left(\frac{4\sin\left(\frac{(2m-1)\pi}{2n}\right)}{(2m-1)\pi} \right)$$

$$\sum_{m=1}^n \frac{\frac{2}{\pi} \sin\left(\frac{(2m-1)\pi}{2n}\right)}{\frac{(2m-1)\pi}{2n}} \cdot \frac{\pi}{n}; t = (2m-1)\pi \rightarrow \sum_{m=1}^n \frac{\frac{2}{\pi} \sin(t)}{t} \Delta t \xrightarrow{n \rightarrow \infty} \frac{2}{\pi} \int_0^{\pi} \frac{\sin(t)}{t} dt$$

Apart from the Gibbs Phenomenon, there are many other examples in this field, such as, the consequences of shifting a function for the amplitude spectrum and the phase spectrum and filtering (See 2).

3. Discrete Signals

In other parts of signal processing, such as the Discrete Fourier Transform, a CAS is very useful. Simple analog signals or continuous-time signals can be represented mathematically as a function of one variable $x(t)$, where t is normally the time.

For many reasons, like storing and processing, it is desirable to look at a signal at isolated points in time.

For example $t=nT_s$ with $n=0,1,2,3,4,\dots$ and $T_s=50\text{ms}$. We denote: $x[n]=x(nT_s)$.

If $x(t) = 4\cos(8\pi t)$ than $x[n] = 4\cos(8\pi n \frac{50}{1000}) = 4\cos(\frac{2\pi n}{5})$

The continuous-time $x[t]$ is by sampling transposed in a corresponding discrete-time signal $x[n]$.

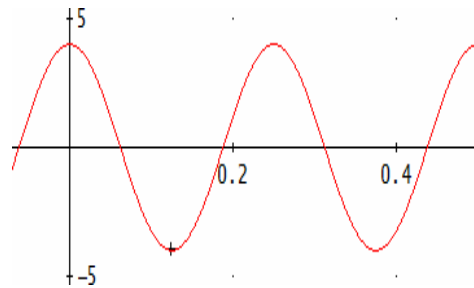


Figure 3.1 $x(t)=4\cos(8\pi t)$

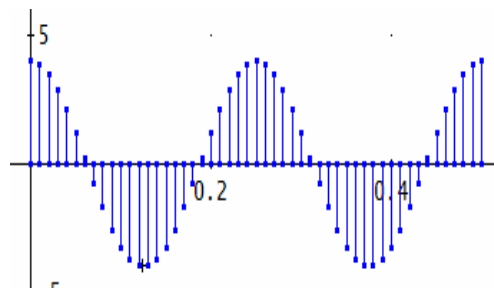


Figure 3.2 $x[n]=4\cos(2\pi n/5)$

Some students say “ Every signal has its own Fourier transform”, so for Discrete-time signals there is a Discrete Fourier Transform (See [4]).

$$\mathbf{X}(k) = \sum_{n=0}^{N-1} x[n] e^{-\frac{2\pi n k}{N} i} \quad \text{with } k=0,1,2,\dots,N-1. \quad (3.1)$$

Calculating $X(k)$ delivers just a string of complex numbers.
But the next example is an eye opener!

Given the discrete-time signal $x[n] = \cos\left(\frac{2\pi \cdot n}{5}\right)$, it then follows for $N=20$:

$$\mathbf{X}(k) = \sum_{n=0}^{19} x[n] e^{-\frac{2\pi n k}{N} i} = \sum_{n=0}^{19} \cos\left(\frac{2\pi \cdot n}{5}\right) e^{-\frac{2\pi n k}{20} i}$$

Using Derive, we make a plot of the so called frequency spectrum $|X(k)|$, with the expression:

$$\mathbf{VECTOR}\left(\left[\begin{array}{c} k \\ k \end{array} \right], k, -10, 10\right)$$

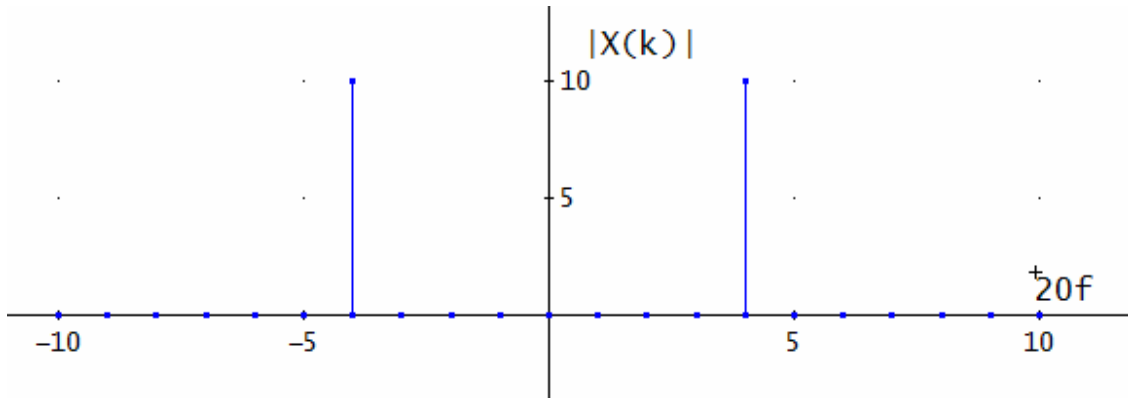


Figure 3.3 Frequency spectrum $|X(k)|$

$x[n] = \cos\left(\frac{2\pi \cdot n}{5}\right)$ has the frequency $f_x = \frac{1}{5} = \frac{4}{20}$ and exactly that frequency is displayed in the frequency spectrum above.

There is even then much to explain, but the first step has been made to analyse signals, one of the most important issues in Signal Processing these days.

We can combine this last example with the lab exercise as follows. Plot the frequency spectrum of

$$y[n] = \cos\left(\frac{8\pi \cdot n}{5}\right)$$

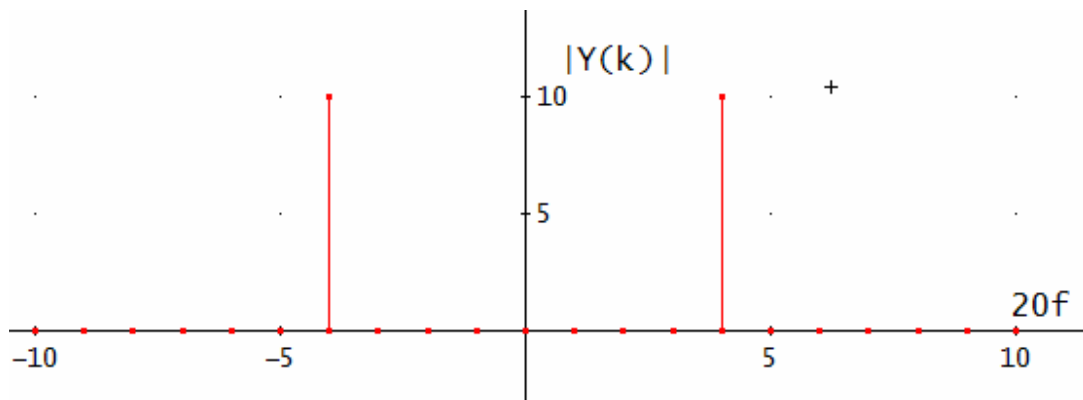


Figure 3.4 Frequency spectrum $|Y(k)|$

Note that the fact that $f_x \neq f_y$ and $|X(k)| = |Y(k)|$ is a good reason to talk about the Sampling Theorem(See [4]).

4. Concluding remarks

There are lots of toolboxes to perform signal processing. Most books are using MATLAB and I must say it is excellent software and a must for students on a higher level in this field. But I think a start using DERIVE, MAPLE or MATHEMATICA is indispensable! Analysing, interpretation and decision-making is possible only with a profound knowledge of theory. Students are forced, during laboratory work, to find their way using books and other sources.

An approach as described above has its consequences. Computer laboratory and software licences should be available. Every student should have the Ti-89 at his disposal. The big challenge is to find examples and laboratory exercises which invite the students to explore and search and also encourage them to note the theory.

I think that the Gibbs Phenomenon is a good example. It even contains some suspense: when the students are surfing the internet, they will be surprised at the amount of sites, containing theory, Java applets etc.

Examination methods have been altered as well.

At the end of the course the students' progress is assessed in three different ways:

- 1) by oral presentation on the outcome of the investigations,
- 2) a written report on the practical coursework,
- 3) a written examination covering mainly theoretical knowledge.

One of the major considerations leading to the revised approach was whether the emphasis in coursework could be shifted from acquiring skills to gaining insight.

In my view, experience gained over recent years working with CAS tells me we are on the right track.

Acknowledgements.

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References.

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