# Encouraging 'Learning by Discovery' Through Lab Activities Via Computer Algebra.

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**Abstract:** This paper describes a research study that examines the impact of integrating the use of computer algebra with activities in a mathematics lab. The study has been conducted in a senior secondary school in New Delhi that follows the curriculum prescribed by one of the central boards responsible for education in India. In the prescribed curriculum the emphasis is on developing 'by hand' skills and teaching with the help of technology is not a practice. However, at the school the author is in charge of a mathematics lab, which encourages 'learning by discovery' by engaging students in investigative activities, some of which integrate technology. This paper relates the experiences of the author in conducting such activities in the lab using Mathematica. The research focused on enabling the students to explore the problems via Mathematica, discover patterns, make conjectures and then generalize results through carefully designed lab modules. The subjects of this study were 24 students of year 11 and 25 students of year 12. This paper describes two modules. The first one deals with the famous Birthday Paradox (year 11) and the second with the Buffon's Needle Problem (year 12). A structured feedback from the students at the end of the lab sessions revealed that Mathematica was instrumental in enabling them to obtain a deeper insight into the problem. It also revealed that students were comfortable with using Mathematica and were eager to explore other problems in a similar manner. This provided encouragement for larger scale integration of CAS into the activities of the mathematics lab.

## Introduction.

The use of computer algebra for teaching and learning mathematics has been the subject of research in recent years ([8],[9]). Researchers and educators have to deal with various issues related to the use of computer algebra systems (CAS) in mathematics instruction such as: How can CAS be integrated into the secondary school curriculum and how can it be used to improve conceptual understanding in mathematics? What is the role of paper-pencil skills in a computer algebra environment? What kind of examination questions is relevant in a CAS environment? ([1] – [7])

The use of CAS has added a new dimension to mathematics learning namely that of visualization and exploration. This has been made possible because of their powerful graphing and symbolic manipulation features. CAS has indeed redefined mathematics instruction since it enables the student to learn on his own by allowing him to investigate and make observations.

This paper reports my experiences in conducting activities in a mathematics lab with the help of computer algebra. At the school where I teach, I am in-charge of a mathematics laboratory. This is an experimental project funded by the Department of Education and one of its objectives is to explore the use of technology for teaching mathematics. The mathematics lab is an additional course offered to mathematically motivated students of classes 6 to 12. Activities conducted in the lab supplement the mathematics taught in the classroom and encourage 'learning by discovery' by engaging students in investigative activities, some of which require the use of technology. The lab is equipped with computers, CAS (only Mathematica), graphing calculators (HP38G, Casio CFX 9850 GB PLUS) and other software packages such as Autograph. It has been my attempt to augment my regular classroom lessons (where using technology is not a practice) with sessions in the mathematics laboratory where Mathematica is used as a vehicle for exploration, visualization, and demonstration. Carefully designed lab modules are developed which guide the students in exploring the problem at hand. This paper describes two lab modules. The first one based on the birthday problem and was conducted with 24 students of year 11. Here Mathematica was used to simulate the problem by randomly generating birthdays. The second one, based on the Buffon's needle problem was conducted with 24 students of year 12. Here students used Mathematica to estimate the value of $\pi$ by simulating needle throws. In both modules the focus was on enabling the students to explore the problems, discover patterns, make conjectures and then generalize results through carefully designed exercises and activities. This study reveals that through the use of CAS students can learn by discovering on their own.

## Lab Module 1: The Birthday Problem

**Educational Setting and Background Knowledge of Students.**
This module was conducted with 24 students of year 11 who were selected from various sections of class 11 in the school and had opted for mathematics lab as an additional course. At the beginning of the academic session, all 24 students were made to go through five 30-minute sessions on Mathematica where they were familiarized with commands such as **`Plot`**, **`Solve`**, **`Random`** and **`Table`**. During the lab each student was given a worksheet and students worked in pairs, each pair sharing one computer on which they had access to Mathematica. They had acquired the basic concepts of probability theory through regular traditional classroom teaching. This included

(i) Concept of randomness, sample space and random experiment.
(ii) Difference between the theoretical and empirical probability of an event.
(iii) Computation of the probability of events using the basic definition of probability.

**Aim of Lab Module**
The primary objective of this module was to enable the students to explore the birthday problem through simulation. In particular the aims were to:

- Reinforce the concept of randomness.
- Introduce the concept of simulation and highlight its importance.
- Simulate experiments such as the throw of a die and the birthday problem.

**Method and Teaching Sequence.**
Each student was given a worksheet, which posed exercises and problems leading the students to explore the problem in a step-by-step manner.

**(a) Concept of Randomness**: Here students generated random numbers using the **`Random`** command. **`Random[]`** produced a pseudo-random number between 0 and 1. Similarly **`Random[Integer,{1,10}]`** randomly produced an integer between 1 and 10. An exercise required students to generate 1000 integers between 1 and 10 randomly and repeat this experiment 10 times, each time counting the number of integers greater than 5. This was achieved using the following code:

```
data=Table[Random[Integer,{1,10}],{i,1,1000}];
Sum[Count[data,k],{k,6,10}]
```

Here the first line of the code uses the **`Random`** and **`Table`** commands to generate 1000 integers between 1 and 10, which is assigned to **`data.`** The second line uses the **`Sum`** and **`Count`** commands to count the number of integers from 6 to 10. Different students obtained different answers. One student's results have been tabulated in Table 1.

Table 1. Randomly generating integers between 1 and 10 using Mathematica.

| **Simulation** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Number of integers > 5** | 490 | 526 | 512 | 502 | 510 | 496 | 483 | 472 | 483 | 519 |
| Fraction of integers > 5 = 4993/10000 | | | | | | | | | | |

The fractions of integers obtained by the students were successively combined to obtain larger and larger number of integers. This exercise offered a good way of highlighting the fact that as the number of integers increased the fraction of integers greater than 5 approached ½.

**(b) Concept of Simulation**.

In the next part of the module, the concept of simulation was introduced as the process of estimating the probability of an event by running an experiment a large number of times. It was pointed out that an experiment can be simulated by a random device such as a pseudo-random number generator in a computer or a graphic calculator. In a simulation experiment a large number of trials are generated and the relative frequencies of the required event are used to estimate the probability of that event. To estimate the probability of obtaining a 1 in the throw of a die, for example, one could actually throw a die a large number of times and use the relative frequency of 1's (number of 1's/total number of throws) to estimate the probability or simulate the throws by randomly generating the numbers 1 to 6.

**(c) Simulating the Throw of a Die**

Students simulated 100 throws of a die by randomly generating 100 integers from 1 to 6 using the `Random` and `Table` commands as follows:

```
diethrows=Table[Random[Integer,{1,6}],{i,1,100}]
```

An exercise posed the following question:

*Compute the theoretical probability of obtaining a sum of 7 in the simultaneous throw of a pair of dice. Simulate 1000 throws using Mathematica and compute the empirical probability.*

Students concluded that the theoretical probability of obtaining a sum of 7 is 1/6 since 7 can result from the six combinations (1,6), (2,5), (3,4), (4,3), (5,2) and (6,1) out of 36 possible combinations. The following program was used to simulate 1000 throws of a pair of die, sum the numbers and count the number of 7's.

```
die1=Table[Random[Integer,{1,6}],{i,1,1000}];
die2=Table[Random[Integer,{1,6}],{i,1,1000}];
total=die1+die2;
Count[total,7]
```

Students executed this program 10 times and then computed the empirical probability of obtaining a sum of 7.

**(d) The Birthday Problem**

In the subsequent lab session students were asked to consider the following question:

*How many people do you need in a group to ensure that the probability of at least two of them having the same birthday is about half? Can you explain your answer?*

The estimates provided an interesting range of group sizes. Some were as low as 15 whereas others were as high as 365 or 366! Only two students gave the correct answer 23 but no explanations were provided. The above question was then rephrased as follows:

*Given a group of n people what is the probability that at least two of them will have the same birthday?*

Students were asked to begin by considering a group of three people. The probability that all three have different birthdays is $\frac{365}{365}\times\frac{364}{365}\times\frac{363}{365}$. The probability of the complement of this event, that is *at least two of three persons have the same birthday* is $1-\frac{365}{365}\times\frac{364}{365}\times\frac{363}{365}$ = 0.0082. This was worked out in Mathematica by typing the following:

```
N[1-(364/365)*(363/365),4]
```

In the next exercise the students were asked to compute the probabilities of at least one repeated birthday for groups of 4 and 5 people and generalize this to find a formula for *n* people. 17 out of 24 students were able to extend the calculations using Mathematica and obtained the probabilities as $1-\frac{365\times 364\times 363\times 362}{365^4} = 0.0164$ and $1-\frac{365\times 364\times 363\times 362\times 361}{365^5} = 0.0271$ for 4 people and 5 people respectively. They generalized the answer and obtained $1-\frac{365\times 364\times 363\times ..........\times(365-(n-1))}{365^n}$ or $1-\frac{365!}{365^n(365-n)!}$

In the next exercise students were asked to generate a table of probabilities for *n* varying from 10 to 50 and also obtain a plot of *n* versus the probabilities. This was achieved by executing the following program

```
prob[n_]:=1-((365!/(365-n)!)*(1/365^n))
Table[{n,N[prob[n]]},{n,10,50}]//TableForm
ListPlot[prob,PlotStyle->PointSize[0.03]]
```

In the above program the function **prob[n_]** is the probability of at least one repeated birthday for a group of *n* people. In the second line **Table** computes the probabilities for group sizes varying from 10 to 50 and presents the data in the form of a table. The third line uses **ListPlot** to plot *n* versus the probabilities. The output indicates that in a group of 23 people the probability of at least one repeated birthday is little more than half (0.507). The plot is shown in figure 1 below.

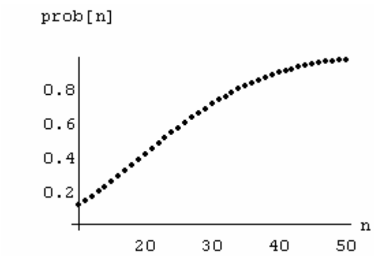


Figure 1 Mathematica plot indicating the probabilities of at least one repeated birthday for *n* (group size) varying from 10 to 50.

In order to verify that the answer is indeed 23, each student was asked to write down ten different birthdays (of friends or relatives) on different slips of paper, which were folded and put in a box. Ten sets of 23 birthdays each were created from the contents of the box. Students working in threes were assigned one set each and had to check for a repeated birthday. Five groups out of 10 had at least one repeat (2 out of these had two repeats) the first time. This experiment was repeated several times to convince the students that the probability of finding at least one repeated birthday in a group of 23 people is indeed about half!

**(e) Simulating the Birthday Problem**

To simulate the problem it was required to randomly generate sets of 23 birthdays. This was done using the following code

```
month=Table[Random[Integer,{1,12}],{i,1,23}];
day=Table[Random[Integer,{1,31}],{i,1,23}];
bday=100*month+day;
Sort[bday]//TableForm
```

In the above program **`month`** is a list of 23 randomly generated integers between 1 and 12 inclusive which denotes the month while **`day`** is a list of 23 randomly generated integers between 1 and 31 inclusive which denotes the day of the month. The third line converts the randomly generated dates to three or four digit numbers whose first one or two digits indicate the month and the last two digits indicate the day of that month. For example, 127 indicates 27th of January and 1120 indicates 20th of November. Thus **`bday`** is a list of 23 randomly generated birthdays. In case the list contains an impossible date such as 31st April, the entire list may be rejected and a new one may be generated. The last line of the program sorts **`bday`** so that if there is a repeated birthday the numbers (representing the birthday) appear one after the other. Each student was asked to run the simulation 10 times and check for a match each time. In case of a match the repeated birthday was recorded. One students' data has been entered in Table 2.

Table 2. Outcomes for 10 simulations of 23 random birthdays

| Simulation | Repeated birthday/No match |
|---|---|
| 1 | No match |
| 2 | No match |
| 3 | No match |
| 4 | 524 (24th May), 1210 (10th December) |
| 5 | 530 (30th May) |
| 6 | 903 (3rd September),<br>1201 (1st December) |
| 7 | 716 (16th July) |
| 8 | 621 (21st June) |
| 9 | No match |
| 10 | No match |

Most students obtained a match 5 out of 10 times! However this algorithm does not take care of impossible birthdays. An efficient way of avoiding this is to identify the days of the year with numbers 1 to 365, then randomly generate 23 integers from this set and check for a repeated integer by typing the following

```
bdaylist=Sort[Table[Random[Integer,{1,365}],{23}]];
```

In both methods mentioned above one has to scan the list of numbers to check for repeated birthdays. This can be avoided by using the **`Union`** command and the **`If`** condition as follows

```
If[bdaylist=!=Union[bdaylist],Print["match"],Print["no match"]]
```

The **`Union`** command outputs the elements of a set without any repetitions. In the above program a list of 23 integers from 1 to 365 inclusive are randomly generated. If this list and its union are the same it implies that there is no birthday match and the program outputs 'no match'. In case of a match there will be repeated numbers in the list, which **`Union`** will output without repetition. The output of the program will then be 'match'. After running this program most students obtained a 'match' 5 out of 10 times which further confirmed the fact that the chances of at least one repeated birthday in a group of 23 people is about half.

**Student Response and Comments.**
This module took three one-hour classes. At the end of the module, students were asked to fill a questionnaire where they responded to the statements given in Table 3 by indicating one of the following: Strongly agree (SA), Agree (A), Not Sure (NS), Disagree (D) and Strongly Disagree (SD).

Table 3. Student response to the questionnaire for feedback on the Birthday problem.

| Item No. | Item | SA | A | NS | D | SD |
|---|---|---|---|---|---|---|
| 1. | Before going through this module you were familiar with the birthday problem. | -- | 2 | -- | -- | 22 |
| 2. | Prior to these lessons you had a good idea of the concept of randomness. | 3 | 2 | 5 | 5 | 9 |
| 3. | Simulating the problem using Mathematica helped you to explore the problem and verify the results on your own. | 15 | 4 | 3 | 2 | -- |
| 4. | Mathematica should be integrated into other activities in a similar manner. | 14 | 7 | 2 | 1 | -- |
| 5. | The lab module was particularly interesting because of Mathematica. | 20 | 2 | 2 | -- | -- |

Some general comments on the lessons were

- *These lessons helped me to explore and understand the basic principles behind the birthday problem. Although I had read about the problem earlier, I had never really understood it until now.*
- *It was real fun trying to verify the answer (probability of at least one repeated bday in a group of 23 is about half) by taking actual birthdays and then simulating it on Mathematica by generating fictitious birthdays.*

## Lab Module 2: The Buffon's Needle Problem

**Educational Setting and Background Knowledge of Students.**
This module was conducted with 24 students of year 12 who had opted for mathematics lab as an additional course. These students had attended the same course in class 11 and were conversant with the basic features of Mathematica such as plotting graphs, solving equations, generating random numbers etc. As in the previous module, two students shared one computer and each student was given a worksheet, which was designed to guide their exploration.
The prerequisite knowledge required for this activity was

(i) Elementary trigonometry.
(ii) Basic probability theory. In particular the concept of theoretical and empirical probability of an event and concept of simulation was required.

(iii) Basic laws of definite integration.

Students were already familiar with (i) and (ii) from their regular mathematics course in class 11. (iii) had been taught just before this lab.

**Aim of Lab Module**

The primary objective of this module was

- To introduce the student to the Buffon's needle problem which is an interesting application of elementary calculus, probability and trigonometry.
- To enable the students to explore the needle problem through simulation.
- To use the needle problem to estimate the value of $\pi$.

**Method and Teaching Sequence.**

The worksheet given to the students posed exercises and problems leading the students to explore the problem in a step-by-step manner.

**(a) The Needle Problem.**

The worksheet began by introducing the needle problem as follows

*Suppose you are given a sheet of paper with equidistant parallel lines drawn horizontally. Let the distance between the lines be l. You are also given a large number of needles of length l. What is the probability that a randomly thrown needle will either touch or cross a line? The fascinating thing about this problem is that it can be used to estimate the value of* $\pi$ *!*

**(b) Exploring the Needle Problem**. As a preliminary exercise students were asked to make rough sketches of the various ways in which a needle can fall on the sheet of paper. Some of the sketches are shown in figure 2.

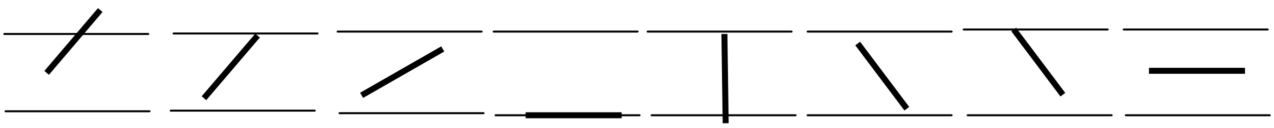

Figure 2. Sketches of possible needle throws.

In order to describe each needle throw, two random variables, $d$ and $\theta$ were defined, where

$d$ = distance of the center of the needle from the nearest line,

$\theta$ = the acute angle made by the needle with the horizontal.

Students easily figured out that $0 \le d \le \frac{l}{2}$ and $0 \le \theta \le \frac{\pi}{2}$. In the exercise that followed students were asked to consider three cases of needle throws keeping $\theta$ constant.

(i) A needle touches a line and makes an angle $\theta$ with the horizontal.

(ii) A needle crosses a line and makes an angle $\theta$ with the horizontal.

(iii) A needle falls between two lines making and angle $\theta$ with the horizontal.

They were required to find a relationship between $d$, $l$ and θ using trigonometry for each case. For (i) they easily concluded that $\sin\theta = \frac{d}{l/2}$ or $d = \frac{l}{2}\sin\theta$ (Figure 3(a)). They used this relation to further conclude that $d < \frac{l}{2}\sin\theta$ for (ii) (Figure 3(b)) and $d > \frac{l}{2}\sin\theta$ for (iii) (Figure 3(c)). Thus keeping $\theta$ constant the condition for the needle to touch or cross (that is 'hit') a line is $d \le \frac{l}{2}\sin\theta$.

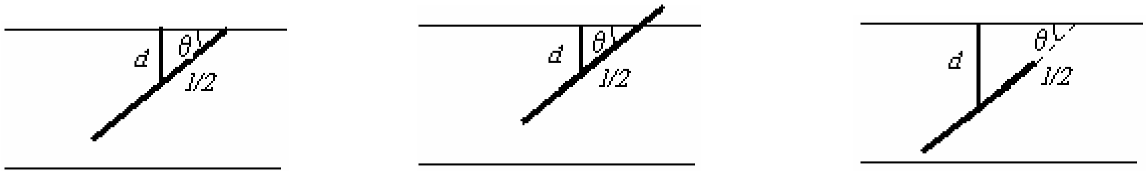


Figure3. (a) (b) (c)

The next section of the worksheet provided the following explanation.

*Corresponding to every needle throw we have an ordered pair* ($d$, $\theta$) *where* $0 \le d \le \frac{l}{2}$ *and* $0 \le \theta \le \frac{\pi}{2}$. Representing $\theta$ along the $x$-axis and $d$ along the y-axis, *all possible values of d and* $\theta$ *can be represented by a rectangle with sides* $\frac{l}{2}$ *and* $\frac{\pi}{2}$. *For every needle throw we will obtain a point on or inside this rectangle. Plotting the graph of* $d = \frac{l}{2}\sin\theta$ *within the rectangle we get the following figure.*

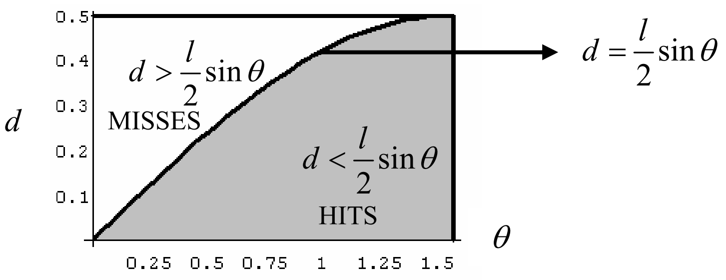


Figure 4 Rectangle representing all possible needle throws. The shaded portion represents the 'hits'.

It was further explained that points which satisfy the inequality $d \le (l/2)\sin\theta$ (representing needles which hit a line) will lie on or below the curve representing $d = (l/2)\sin\theta$. Thus if we throw the needle a large number of times, the fraction of 'hits' **= area under** $d = (l/2)\sin\theta$ **/total area of the rectangle.** This can also be interpreted as the probability of a needle hitting a line.

In the next exercise students computed the area under $d = (l/2)\sin\theta$ as follows

$$\int_0^{\pi/2} d\ d\theta = \frac{l}{2}\int_0^{\pi/2} \sin\theta\, d\theta = \frac{l}{2}\left|-\cos\theta\right|_0^{\pi/2} = \frac{l}{2}\left|-\cos\frac{\pi}{2} + \cos 0\right| = \frac{l}{2}.$$

Also if the needle is thrown a large number of times the (empirical) probability of a hit can be estimated as **number of needles, which hit a line / total number of throws**. This is equal to **area under** $d = (l/2)\sin\theta$ **/area of the rectangle** $= \frac{l/2}{\pi l/4} = \frac{2}{\pi}$

Thus, ***p*** **= probability of a hit** = **number of needles which hit a line / total number of throws = 2/π.**

A large number of needle throws can therefore be used to estimate the value of π.

**Simulating the Needle Problem Using Mathematica.**

Students agreed that it is impractical to throw needles on a sheet of paper, count the number of hits and then estimate π. They decided to resort to simulation using the **Random** command. For simplicity $l$ was taken as 1. In order to simulate each needle throw two random numbers were generated one belonging to the interval [0,1/2] (for $d$) and the other belonging to [0,π/2] (for $\theta$). The following Mathematica code simulates 100 needle throws and plots them within the rectangle.

```
needlethrows=
Table[{Random[Real,{0,N[Pi/2]}],Random[Real,{0,0.5}]},{i,1,100}]
ListPlot[needlethrows,PlotStyle->PointSize[0.03]]
```

Subsequently students decided to simulate 1000 throws of the needle and plot the points along with the graph of $d = \frac{1}{2}\sin\theta$. The program and its output are as follows.

```
moreneedlethrows=Table[{Random[Real,{0,N[Pi/2]}],
Random[Real,{0,0.05}]},{i,1,1000}]
a=ListPlot[moreneedlethrows,PlotStyle->PointSize[0.03]]
b=Plot[(1/2)Sin[x],{x,0,Pi/2},PlotStyle->RGBColor[1,0,0]]
Show[a,b]
```

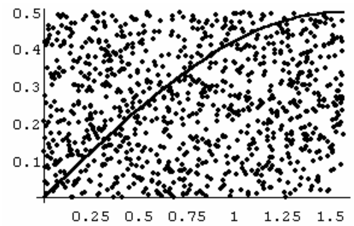


Figure 5 Mathematica plot showing 1000 simulated needle throws along with the graph of $\frac{1}{2}\sin\theta$.

Although the above program gives a visually interesting output it does not help in finding the probability of a randomly thrown needle hitting a line. For this it is necessary to count the number of hits (points which lie on or below the graph of $d = \frac{1}{2}\sin\theta$). In the following program, 1000 pairs of random numbers, **r1** and **r2** are generated, where **r1** represents $\theta$ and **r2** represents $d$. The **If** condition is used to check whether each randomly generated ordered pair $(d, \theta)$ satisfies the inequality $d \le \frac{1}{2}\sin\theta$. A counter **hit** counts the number of hits and gets incremented every time an ordered pair satisfies the inequality.

```
r1:=Random[Real,{0,N[Pi/2]}];
r2:=Random[Real,{0,0.5}];
hit=0;
m=Table[{random1=r1,random2=r2,
If[random2<=N[0.5*Sin[random1]],hit+=1,miss]},
{i,1,1000}]//TableForm
```

Students were asked to run the above program 10 times so as to obtain 10000 throws. The total number of hits was then used to estimate the value of $\pi$ by using the **NSolve** command.

```
NSolve[hit/10000==2/pi,pi]
```

One student's results have been given in table 4.

Table 4. Number of 'hits' for 10 simulations of 10000 needle throws

| **Simulation** | **Number of hits** |
|---|---|
| 1 | 6446 |
| 2 | 6393 |
| 3 | 6404 |
| 4 | 6385 |
| 5 | 6339 |
| 6 | 6367 |
| 7 | 6386 |
| 8 | 6390 |
| 9 | 6336 |
| 10 | 6317 |
| Total hits | 63763 |
| **Estimated value of $\pi$** | **3.13662** |

**Student Response and Comments.**
This module took three one-hour classes. At the end of the module, students were asked to fill a questionnaire similar to that of the birthday problem. The responses are given in Table 5.

Table 5. Student response to the questionnaire for feedback on the Buffon's Needle problem.

| Item No. | Item | SA | A | NS | D | SD |
|---|---|---|---|---|---|---|
| 1. | Before going through this module you were familiar with the needle problem. | -- | -- | 1 | -- | 23 |
| 2. | Prior to these lessons you had a good idea of the concept of randomness. | 7 | 8 | 4 | 2 | 3 |
| 3. | Simulating the problem using Mathematica helped you to explore the problem on your own. | 17 | 7 | -- | -- | -- |
| 4. | Mathematica should be integrated into other activities in a similar manner. | 16 | 5 | 2 | 1 | -- |
| 5. | The lab module was particularly interesting because of Mathematica. | 22 | 1 | 1 | -- | -- |

Some general comments on the lessons were

- *Exploring the needle problem via Mathematica has been a fascinating experience. In fact without Mathematica it would not be possible to explore this problem.*
- *Mathematica helped to simulate the needle problem. A good estimate of $\pi$ can be obtained only after simulating a large number of throws which is possible only because of Mathematica.*

**Discussion.**

This paper describes two lab modules conducted with students of years 11 and 12 in a mathematics lab. These modules are based on the Birthday problem and the Buffon's needle problem both of which lend themselves to investigation and form interesting exploratory activities for students. Since simulation is essential for exploring both problems Mathematica proved to be a great asset. Worksheets were designed to enable students to explore the problems in a step-by-step manner leading to the final answer. Each exercise required the student to write a code or program in Mathematica and make a conjecture or a generalization from the output leading to 'learning by discovery'. In the birthday problem, students had to generalize a formula for the probability of at least one repeated birthday in a group of $n$ people and use Mathematica to compute the probability for group sizes 10 to 50. The conclusion that the probability of at least one repeated birthday in a group of 23 people is about half was drawn from the output of the program. The answer was verified by collecting actual data and then was simulated using Mathematica by randomly generating birthdays. In the needle problem, students were first required to formulate the problem in terms of the random variables $d$ and $\theta$ and then investigate the problem by simulating needle throws. A good estimate of $\pi$ was obtained only after simulating a large number of throws. Throughout the modules, students worked in pairs sharing their ideas. My role (as their instructor) was more of a facilitator guiding them in the course of their investigations and 'discovery'. The encouraging feedback from students led to the further integration of Mathematica into the activities of the lab.

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