TEACHING SIGNAL ANALYSIS USING SCIENTIFIC NOTEBOOK

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Abstract

At Murdoch University we offer a signal analysis unit to undergraduate students in mathematics and physics. In the teaching of signal analysis at this level it is essential that students see and generate the signals that are being presented. For this reason it is beneficial to both the student and the teacher to have available a program that is easy to implement and that gives immediate on-screen plots. This is where Scientific Notebook comes into its own. With this software package we can easily generate standard sinusoidal functions and see them on the screen, making it straightforward to explain and understand terms such as period and frequency. However, it isn't immediately clear how to automatically generate line graphs for the corresponding discrete versions, especially as Scientific Notebook has no programming facility. We also have to consider the different behaviour of the MAPLE (Scientific Notebook versions 4 and earlier) and MuPAD (Scientific Notebook versions 4 and later) computation engines. In this paper we will outline, in the teaching context, our approach to these questions and present some algorithms for generating a wide range of discrete signals.

1 Introduction

There are many references to teaching mathematics with Scientific Notebook; see [1], [2], [3] and the examples presented in http://forum.mackichan.com:81/%7Emackichan. In this paper we investigate how Scientific Notebook can be used in the teaching of signal analysis at the undergraduate level. While there are higher level packages to carry out the calculations typically encountered in such a unit, Scientific Notebook has the advantage that students can easily produce plots interactively. Indeed, Scientific Notebook is designed for on-screen display of both the text and calculations.

In my teaching of signal analysis I have not only presented the "straightforward" aspects that can be done easily with Scientific Notebook, but I have also developed techniques to generate more complicated discrete signals in a reasonably automatic way. This has the distinct advantage in teaching the subject in such a way that the student sees the immediate outcome of her/his calculations, and furthermore the plots are really quite attractive.

We commence with the introduction of some fundamental terms used in signal analysis and consider how Scientific Notebook deals with the continuous sinusoidal examples. We then examine the discrete case, which arises when the sinusoidal signals are sampled at regular intervals. With these techniques we can cover most of a standard undergraduate unit in signal analysis. In fact the only aspect that will need other packages is when incorporating noisy signals, in which case MATLAB, MAPLE or even BASIC could be used.

Just a word about the different versions of Scientific Notebook. Up to and including v.4 the computation engine used was MAPLE. From v.4 on it was MuPAD, and for v.4.1 that I have both of these are available. Of course MuPAD has developed considerably since then. You will need to check how the techniques work with your version.

2 Some signal analysis terms

A function f is periodic if its values repeat themselves at regular intervals. Mathematically, we say that f is λ-periodic if

$$f(t) = f(t+\lambda) \text{ for all real numbers } t \tag{1}$$

Equation (1) can hold for more than one value of λ. In particular, if f is λ-periodic then f is $k\lambda$-periodic for every integer k. Furthermore, constant functions are λ-periodic for every real number λ. The period of a non-constant periodic function f is the smallest of all positive values of λ for which (1) holds.

A periodic function is completely determined by its behaviour on any interval of length λ, where λ is its period. If we know its values on any such interval then its values at all other points along the real line can be found by repeated use of (1).

3 Sinusoids

Sinusoids are functions having graphs the same shape as the familiar cosine and sine curves. There are various ways of expressing such functions mathematically, each with its own special advantages. But first, observe in Figure 1 the usual way that a plot of the sine function is presented in textbooks which is (with a change in the View Intervals in View, Plot Properties)

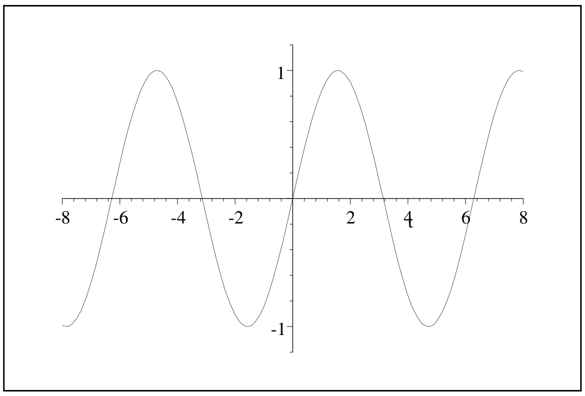

Figure 1: Sine function $\sin t$

the default for Scientific Notebook. In fact the sine curve is somewhat flatter, as can be seen using Scientific Notebook and choosing Equal Scaling Along Each Axis under Axes, Plot Properties; see Figures 2 and 3. This looks much more like a wave at the sea shore!

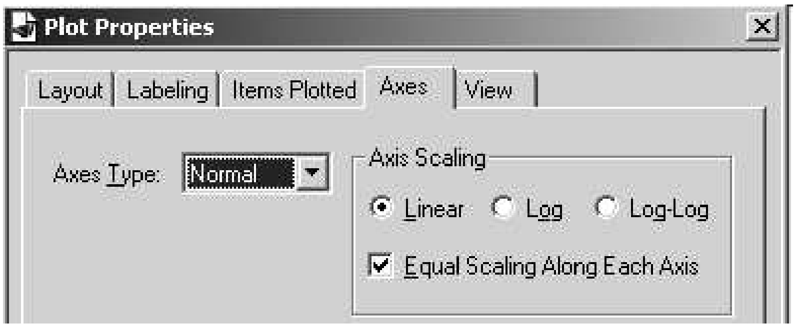

Figure 2: Scientific Notebook window

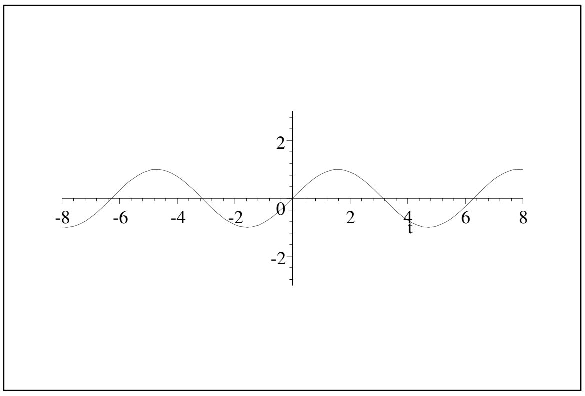

Figure 3: Sine function $\sin t$

The period of the sine function, which is 2π, can be seen using either plot.

3.1 Amplitude and phase form

A sinusoid may be defined by an equation of the form

$$f(t) = r\cos(\omega t - \varphi) \text{ or } f(t) = r\sin(\omega t - \varphi') \tag{2}$$

The constant r, which we normally assume to be positive, is called the amplitude, ω is the angular frequency and φ and φ' are the phase angles. The cosine and sine forms are equivalent; we can pass from one to the other simply by changing the phase angle. Figure 4 shows the graph of a typical sinusoid. It can be obtained from the elementary cosine or sine graphs by dilating or stretching by the factor r in the y-direction, compressing by a factor ω in the t-direction, and then translating or shifting the graph φ or φ' to the right.

Setting $r = \omega = 1$ and $\varphi = \varphi' = 0$ in (2) gives the standard cosine and sine functions. Constant functions are obtained by setting $\omega = 0$.

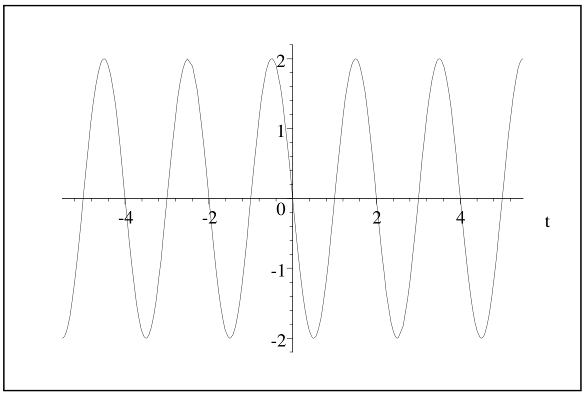

Figure 4: Sinusoid $2\cos\left(\pi t - \frac{3\pi}{2}\right)$

Since $\cos t$ and $\sin t$ both have period 2π, the period of either sinusoid given in (2) is $2\pi/\omega$. If we think of the independent variable t as time (measured in seconds), a common situation in practice, then the sinusoid completes $\omega/2\pi$ full cycles every second. The number $\omega/2\pi$ is called the frequency and the basic unit of frequency is 1 cycle per second or 1 Hz.

3.2 Cosine and sine form

A sinusoid can also be defined as a sum of a sine and a cosine function with phase angles both 0. Thus a sinusoid is any function of the form

$$f(t) = a\cos\omega t + b\sin\omega t \tag{3}$$

The equivalence of (2) and (3) can be established using standard trigonometric identities.

3.3 Linear combinations

The periodic waveforms encountered in nature seldom have the simple shape of a sinusoid. More commonly they look something like the one shown in Figure 5, which is the graph of $2\cos t + \sin 2t + 0.5\cos 10t$. Waveforms such as these can be obtained by combining sinusoids of different frequencies.

Suppose that f is a linear combination of sinusoids, that is, $f(t)$ is a sum of the form

$$f(t) = r_0 + \sum_{n=1}^{N} r_n \cos(\omega_n t - \varphi_n) \tag{4}$$

where the r_n, ω_n and φ_n are constants. The term r_0 represents the constant part of f, and is known in electrical engineering as the DC or direct current component of f. The remaining terms in the sum (4) constitute the AC or alternating current in the waveform f.

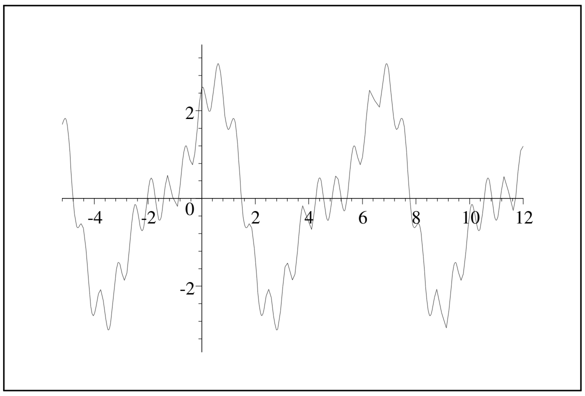

Figure 5: Linear combinations of sinusoids

3.4 Fourier series

The Fourier series of a continuous $2\pi-$periodic function is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \tag{5}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt \text{ and } b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt \tag{6}$$

The numbers a_n and b_n are called the real Fourier coefficients of f. For example, the Fourier coefficients for the square wave

$$h(t) = \begin{cases} 1 & \text{if } \; 2k\pi \le t < (2k+1)\pi \text{ for some integer } k \\ -1 & \text{otherwise} \end{cases}$$

are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -\cos nt\,dt + \frac{1}{\pi}\int_0^{\pi}\cos nt\,dt = 0 \text{ for all } n$$

and

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin nt\,dt = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

The square wave can be easily plotted in Scientific Notebook as a piecewise-defined function in the usual way giving Figure 6. Alternatively, the square wave can be written as

$$h(t) = 2\lfloor \sin t \rfloor + 1$$

where Scientific Notebook uses $\lfloor x \rfloor$ to denote the floor function (also called the integer part) of x, defined as the greatest integer less than or equal to x. In this case, when we plot the graph of h we don't have the option of inserting the vertical dashed lines as these are produced automatically, and in fact as solid lines as shown in Figure 7.

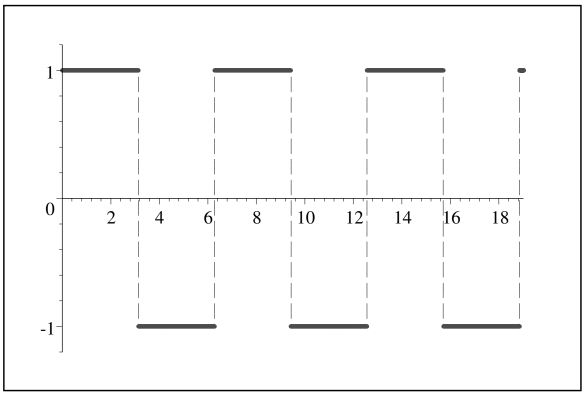

Figure 6: Square wave $h\left(t\right)$

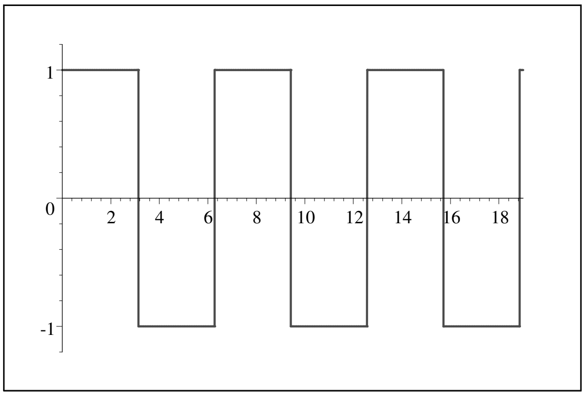

Figure 7: Square wave $h\left(t\right)$

The Fourier series of h is then

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(2n+1\right)t}{2n+1} = \frac{4}{\pi}(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \frac{1}{7}\sin 7t + \ldots)$$

We write

$$s_N = \frac{4}{\pi}\sum_{n=0}^{N}\frac{\sin\left(2n+1\right)t}{2n+1}$$

for the N^{th} partial sum of the Fourier series. The graphs of h and of s_1 (dots), s_5 (dashes), and s_{15} (solid) are shown in Figure 8. It is clear that $s_N(t) \to h(t)$ as $N \to \infty$ for all values of t except where h is discontinuous, namely $t = k\pi$ for some integer k. Incidentally, the graph gives a nice picture of the overshoot near the points of discontinuity (Gibbs' phenomenon).

4 Discrete signals

A discrete signal $x = x[n]$ is just an ordered sequence where $-\infty \leq n \leq \infty$. If $x[n]$ is obtained by sampling from the analogue signal $x[t]$ then $n = 0$ is assumed to correspond to $t = 0$, in

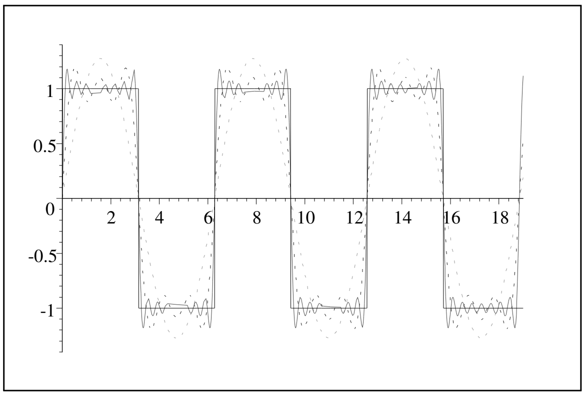

Figure 8: Fourier approximants to the square wave

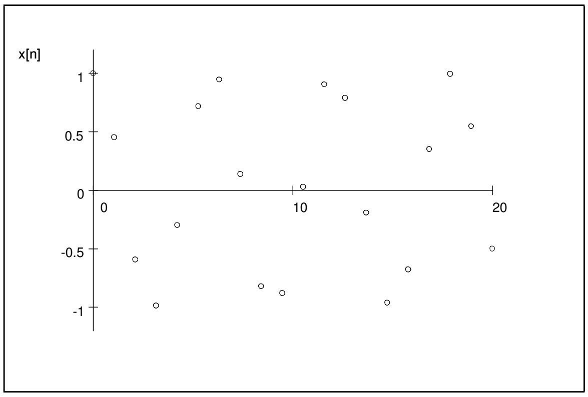

Figure 9: 20 sample plot of $\cos\frac{n\pi}{3}$ (MuPAD)

which case a marker $\Downarrow$ indicates the origin ($n = 0$). $x[n]$ is also called a sampled signal and, for each n, $x[n]$ is called a sample and n is just the sampling instant.

Any N satisfying $x[n \pm kN] = x[n]$ for $k = 0, 1, ...$ is called a period of $x[n]$ in which case $x[n]$ is termed periodic. The smallest such N is called its (fundamental) period. Note that the period of a periodic discrete signal is always a positive integer. The fundamental period of a linear combination of two discrete signals is the least common multiple of their fundamental periods.

Plotting discrete signals is certainly possible using Scientific Notebook. For example, to plot 20 samples of $x[n] = \cos\frac{n\pi}{3}$ just use the 2−dimensional plot as usual and then click on the picture and select Items Plotted, Point (with box or one of the other choices) then Variables and Intervals and choose 20. If the Computation Engine is set as MuPAD then the sequence of values will appear exactly as desired; the resulting plot is shown in Figure 9. Unfortunately, if MAPLE is selected then the program interpolates some points where there are significant changes in the values and this defeats the purpose of plotting discrete signals; see Figure 10.

In fact, discrete signal plots are usually presented as line graphs. For example, the plot of $x[n] = \cos\frac{n\pi}{3}$ with 20 samples commencing with $n = 1$ gives Figure 11.

We now describe a procedure that produces line plots by considering this example but simplifying everything by taking only 4 sample points, starting with $n = 1$. We let $\lceil x \rceil$ denote

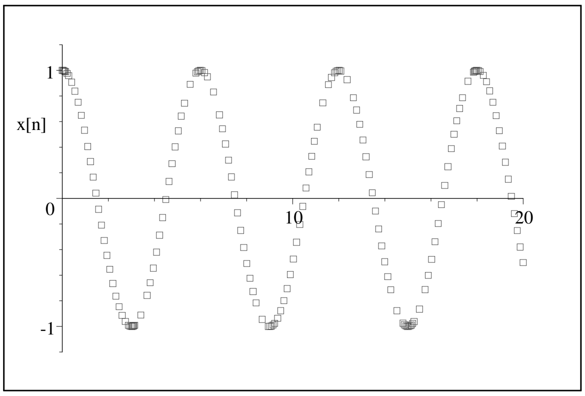

Figure 10: 20 sample point plot of $\cos\frac{\pi n}{3}$ (MAPLE)

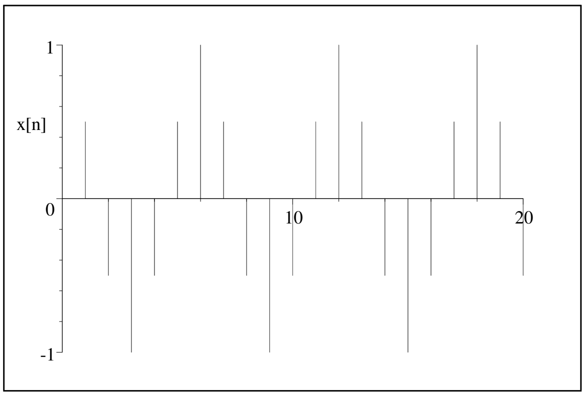

Figure 11: 20 sample line plot of $\cos\frac{\pi n}{3}$ (MAPLE)

the ceiling function of x, defined as the smallest integer greater than or equal to x. First form two row matrices using the FILL command with the defining functions h, k given by

$$h\,(i,j) = \left\lceil \frac{j}{3} \right\rceil \quad \text{and} \quad k(i,j) = (((j-1) \bmod 3) \bmod 2) \cos \frac{\left\lceil \frac{j}{3} \right\rceil \pi}{3}$$

which give

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & -1 & 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix}$$

respectively, and then reshape them into column matrices (or we could form the column matrices from the outset by replacing j by i in the right-hand side of each of these equations; here we are just saving space by using rows).

The first matrix serves as the index matrix and the second as the matrix of signal values separated by pairs of zeros. Now concatenate these column matrices to give:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 3 \\ 3 \\ 3 \\ 4 \\ 4 \\ 4 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix}, \text{concatenate}: \begin{bmatrix} 1 & 0 \\ 1 & \frac{1}{2} \\ 1 & 0 \\ 2 & 0 \\ 2 & -\frac{1}{2} \\ 2 & 0 \\ 3 & 0 \\ 3 & -1 \\ 3 & 0 \\ 4 & 0 \\ 4 & -\frac{1}{2} \\ 4 & 0 \end{bmatrix}$$

All three operations (FILL, FILL, CONCATENATE) should be performed in sequence without altering the cursor position. Finally, again without moving the cursor, choose PLOT, 2D and this gives the line plot in Figure 12.

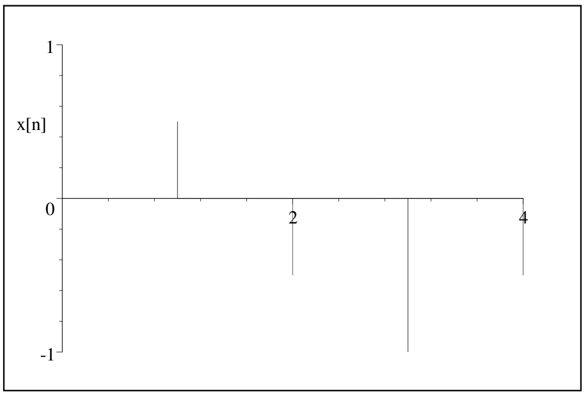

Figure 12: 4 sample line plot of $\cos\frac{\pi n}{3}$ (MAPLE)

What is happening here is that the polygonal plot feature of Scientific Notebook is being used; you will notice that the lines are being retraced as the points joined go in order

$$\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \cos\frac{\pi}{3} \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & \cos\frac{2\pi}{3} \end{pmatrix}, \begin{pmatrix} 2 & 0 \end{pmatrix},$$
$$, \begin{pmatrix} 3 & 0 \end{pmatrix}, \begin{pmatrix} 3 & \cos\frac{3\pi}{3} \end{pmatrix}, \begin{pmatrix} 3 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \end{pmatrix}, \begin{pmatrix} 4 & \cos\frac{4\pi}{3} \end{pmatrix}, \begin{pmatrix} 4 & 0 \end{pmatrix}$$

Now this process will work for any sequence $x[n]$ (replacing $\cos\frac{n\pi}{3}$). However, note the restriction set by Scientific Notebook with the default setting for the matrix size being 30×30. While the number of rows or columns can be changed under TOOLS, USER SETUP there is an overall restriction of 1000 cells. Since we don't require any more than two columns (if filling in columns) we can go up to 500 rows. This will need to be done in TOOLS, USER SETUP. Furthermore, by the way the matrices are chosen there are three entries (rows) corresponding to each sample point, which means that the largest number of sample points that can be handled by this method is 166. This is of course quite sufficient for any standard example.

5 Triangular waves

Using Scientific Notebook to plot piecewise-defined functions can be quite tedious, and for this reason it pays to devote the time to working out alternative forms. For example, for

$$2\left(\arccos\left(-\sin 6t\right)-\frac{\pi}{2}\right)=\begin{cases} 12t & \text{if} \quad 0<t<\frac{\pi}{12} \\ -12t+2\pi & \text{if} \quad \frac{\pi}{12}<t<\frac{3\pi}{12} \\ 12t-4\pi & \text{if} \quad \frac{3\pi}{12}<t<\frac{5\pi}{12} \\ -12t+6\pi & \text{if} \quad \frac{5\pi}{12}<t<\frac{7\pi}{12} \\ 12t-8\pi & \text{if} \quad \frac{7\pi}{12}<t<\frac{9\pi}{12} \\ -12t+10\pi & \text{if} \quad \frac{9\pi}{12}<t<\frac{11\pi}{12} \\ 12t-12\pi & \text{if} \quad \frac{11\pi}{12}<t<\frac{13\pi}{12} \\ -12t+14\pi & \text{if} \quad \frac{13\pi}{12}<t<\frac{15\pi}{12} \\ 12t-16\pi & \text{if} \quad \frac{15\pi}{12}<t<\frac{17\pi}{12} \\ -12t+18\pi & \text{if} \quad \frac{17\pi}{12}<t<\frac{19\pi}{12} \\ 12t-20\pi & \text{if} \quad \frac{19\pi}{12}<t<\frac{21\pi}{12} \\ -12t+22\pi & \text{if} \quad \frac{21\pi}{12}<t<\frac{23\pi}{12} \\ 12t-24\pi & \text{if} \quad \frac{23\pi}{12}<t<2\pi \end{cases}$$

it is easier to simply plot the trigonometric expression on the left-hand side rather than constructing the 13−row column matrix on the right-hand side. Furthermore, Scientific Notebook has problems with such piece-wise defined expressions when the intervals defining the function have irrational end-points.

6 Conclusion

There are many approaches to generating the one signal as, for example, the two given for the square wave. We have concentrated on producing some of the more straightforward ones using Scientific Notebook rather than using a higher level package with programing facility.

References

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[3] Majewski M. (1998), Scientific Notebook - an Interactive Approach in Teaching Mathematics, in Proc. Int. Conf. Teach. Math., John Wiley & Sons.