Conceptualising Motion Through Dynamic Graphing

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Abstract

In this paper, I review the use of dynamic graphs for developing understanding of motion in one- and two- dimensional space, as reported in the literature. In the main part of the paper, I report my findings on the use of dynamic graphs in a Year 12 class while they were studying vector calculus. The technology used was the graphics calculator. The graphs showed the trajectories of position vectors $\mathbf{r}(t)$, and the graphing was performed in the parametric graphing facility. Straight line, parabolic, circular and figure eight trajectories were included. Analysis of classroom conversation indicated that the dynamic graphing supported conceptualization of position in multiple ways, but offered few opportunities for defining velocity and acceleration. Reasons for the differing outcomes were that position was displayed in relation to Cartesian axes but velocity and acceleration were not represented in any lasting material way on the dynamic graphs.

Introduction

One benefit that computer technologies offer teaching and learning mathematics is the quick generation of graphs. Static graphs that appear in their entirety and can not be directly manipulated, dynamic graphs that are plotted point-by-point, and graphs that are dynamic because they can be manipulated using the mouse, have been found to support mathematical investigation in the classroom. This paper is mainly about dynamic graphs, which have been found to support learning particularly well. For example, Lapp and Cyrus (2000) report that investigation involving the collection of data with computer-based laboratory (CBL) devices and dynamic graphing of the data assisted the learning of function and rate of change relationships, in a variety of different school settings. Learning seemed to hinge on real-time, dynamic graphing, in conjunction with data generation. Plotting the graphs after collecting the data did not seem so effective.

Standard facilities on a graphics or CAS (computer algebra system) calculator may be used for the graphs. As well, dynamic graphing is a feature of specialised programs and software. For instance, Stroup (2002) describes a graphics calculator program that allows the user to draw a displacement-time graph by pressing arrow keys in fast or slow succession. A point is plotted at each arrow press and a cross moves across the bottom of the screen so that its displacement from the left of the screen corresponds to the vertical height of the point being plotted on the graph. The display allows student investigation of links between the graph and the displacement of the cross, and of links between the graph and velocity of the cross. Stroup also describes software which allows students to graph velocity against time for an elevator. When a simulation is run according to the graph, the height of the elevator (the definite integral of velocity) is displayed dynamically at the side of the screen.

Schnepp and Nemirovsky (2001) report on computer software where students draw displacement-, velocity- or acceleration- time graphs on the screen and the computer activates a toy mini-car according to the graph being drawn. The reverse CBL functionality is also available where motion of the mini-car is recorded with a data-logger and the computer displays displacement and rate of change graphs. The software supports investigation of rate of change and integral relationships.

The topic of interest in the above applications is motion in one dimension. Motion in two-dimensional space can be investigated using parametric graphing capabilities of a graphics calculator or graphing software. Drijvers and Doorman (1996) give the example of students using a parametric graph to model the position of a point on the edge of a coin that was rotating around the edge of another coin. Student-investigation centred on speed of the moving point. Fast and slow speeds were apparent as the graph was plotted.

There are also several specialised software packages available for supporting investigation of motion in two-dimensional space. Yerushalmy and Shternberg (2001) describe a program that allows the user to draw a trajectory on the computer screen and *x-t* and *y-t* graphs appear alongside the drawing. The trajectory can be retraced at different rates and the effects on the steepness of the *x-t* and *y-t* graphs noticed.

Roth, Woszczyna and Smith (1996) report on software that displays motion of an object in two-dimensional space, with constant force acting on the object. The user attaches arrows to an object on the screen, to represent constant force and initial velocity. The 'experiment' is run and the trajectory of the object is displayed as a trace. The position of the object at fixed time intervals is shown on the trace. Force and velocity arrows, with appropriate length and direction, are displayed at each position.

Advantages for learning of these applications are that students can simultaneously observe real phenomena (e.g., motion of the minicar) or virtual phenomena (e.g., motion of the elevator) and graphical or geometric (arrow) representations of the phenomena. Sometimes the user controls the motion of the objects and at other times the representations. Roth et al (1996) identify that such applications allow "coordination of the phenomenal and conceptual" (p. 1002). Students see motion in the software set-ups and as well they physically experience motion in their everyday life. As such, motion belongs to the phenomenal domain. The graphs and vector arrows that are used to portray motion belong to the conceptual domain. They are aspects of conceptualising motion in mathematical terms. Hence, the applications achieve "a bridge between the two domains by enabling what is impossible in the world of our everyday experience: the co-presence of the phenomenal . . . and the conceptual" (p. 1002). Roth et al. also observe that 'static displays can present the conceptual, though not as well as a micro-world, for they lack the dynamic aspects; they cannot present the phenomenal because they display only snapshots. Real experiments on their own can present the phenomenal very well, but they do not present the conceptual' (p. 1003, paraphrased).

Advantages of dynamic graphing for learning can also be defined in mathematical terms and here I draw on Sfard's (1991) theory of mathematical development. She proposes that new understandings emerge from performing mathematical operations on known mathematical objects (viz constructs). The end products are gradually seen as new mathematical objects, when operational understanding of the new objects is said to exist. Eventually new objects may be understood as an entities in their own right, separate from their constituting processes, when structural understanding is said to exist. For instance, with the Roth et al. (1996) software, one intention is that students will infer mathematical understanding of acceleration from vector arrows for velocity. Inference would involve noticing the differences in length and direction of the velocity vectors. Acceleration would be understood operationally in terms of difference (subtraction) operations. Subsequently, students might discern mathematical properties of acceleration and realise that acceleration can be added, subtracted etc., so that acceleration becomes understood as a mathematical entity or structure that can be operated on.

Operational and structural understanding are also known as procedural and conceptual understanding (Hiebert & Carpenter, 1992). They are complementary and inseparable, and while development is usually in the operational-structural order, it is sometimes in the reverse order, particularly for geometric/ graphical representations (Sfard, 1991; Forster & Taylor, 2000). Understanding emerges through use of all types of mathematical representation (Roth et al., 1996; Sfard, 1991) and operational and structural understanding can be brought to bear in interpreting all types of representation (Sfard, 1991).

Inquiry in the Year 12 Class

The main purpose of this paper is to report dynamic graphing activities that I observed being implemented in a Year 12 calculus class. The activities addressed motion in two-dimensional space and vector calculus relationships. They involved students in producing dynamic graphs on their graphics calculators and copying one of the graphs onto graph paper. Vector arrows for displacement, velocity, and acceleration were added to graph-paper graph.

I discuss details of dynamic graphing in the first lesson on vector calculus in the class, and outline the other dynamic graphing and by-hand graphing that followed. Drawing on the theoretical views discussed above (Roth et al., 1996; Sfard, 1991), I propose that the activities allowed students multiple opportunities to conceptualise position and motion. It is not feasible in a short paper to report conversation and other evidence that indicated learning in the class, and this is a limitation of the paper. However, relevant detail is reported in Davis and Forster (2003) and Forster and Taylor (2003).

I attended 21 50-minute lessons in the Year 12 class of which 12 lessons were on vector calculus. My study was non-interventionary in that I did not provide or suggest activities. I observed whole-class work, was an assistant teacher during individual and small-group work, and made field notes in both these roles. The one-to-one conversations of the 13 students in the class were audio-recorded. Tapes were transcribed immediately after the lessons, and I discussed issues with the teacher and students during subsequent lessons. The lessons were video-recorded, with the display from the overhead projector in the field of view. The teacher frequently connected students' calculators or his own calculator to a view-screen and the displays were captured on the video. The teacher provided me with photocopies of assessment tasks and students’ solutions to them.

The unit of analysis in this paper is the class. Therefore, I do not claim understanding for individuals in the activities that I describe, but the various data did evidence very satisfactory progress in the vector topic for all but one student.

The Year 12 students had used graphics calculators for nearly two years before the study and had completed a unit of work on vectors in Year 11. It was limited to straight-line relationships, included parametric graphing on the calculators, but did not include calculus methods. The class had studied velocity and acceleration relationships in Year 12 units on rectilinear and simple harmonic motion.

Results and Discussion

First lesson on vector calculus

In the first lesson on vector calculus, the teacher asked the class what they thought the topic would involve. Vectors in two and three dimensions and differentiation to find velocity and acceleration were mentioned. As a first activity, the teacher asked students to: graph $\mathbf{r}(t) = 2t\,\mathbf{i} + (t-1)\,\mathbf{j}$ for $0 \le t \le 12$ in the parametric aplet of their calculators, watch the graph as it was plotting, and say whether it was "slowing down, speeding up, or anything else that was going on". Similar instructions were given for $\mathbf{r}(t) = 2t\,\mathbf{i} + (t-t^2)\,\mathbf{j}$ and $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$. Students performed the graphing individually, talked about it together and one student's calculator was connected to the view-screen. The projected display was the subject of class discussion.

From an analytical viewpoint, the graphing activity allowed students to experience visually position and motion on three trajectories on their calculator screens. Hence, in Roth et al.'s (1996) terms, the activity addressed the phenomenal domain. The activity also offered opportunities for conceptualising position in terms of position vectors as follows.

With the first graph, the teacher asked students to calculate $\mathbf{r}(0)$ and $\mathbf{r}(12)$ in order to select scales prior to plotting the graph. So, he required them to perform substitution, multiplication and subtraction operations to obtain $\mathbf{r}(0)$ and $\mathbf{r}(12)$ for $\mathbf{r}(t) = 2t\,\mathbf{i} + (t-1)\,\mathbf{j}$. The dynamic graphing showed the point-by-point creation of the straight-line $\mathbf{r}(t)$ graph, starting with $\mathbf{r}(0)$ and $\mathbf{r}(12)$, which were commented on. After students obtained the graph, the teacher asked them to retrieve $\mathbf{r}(t)$ components from the table of values.

All phases of the activity involved procedures that were directed at producing position vectors in numerical or graphical forms, so they called for and encouraged operational (or procedural) understanding of position vectors. In regards to the motion, students said that speed was constant but there was no elaboration of the mathematics of constant speed.

With the second graph (for $\mathbf{r}(t) = 2t\,\mathbf{i} + (t-t^2)\,\mathbf{j}$), the jagged appearance of some students' parabolas (see Figure 1a) provoked discussion. The teacher explained the problem was a large (TSTEP) which is the interval between the times for which points are plotted. The TSTEP is chosen when setting the scales (see Figure 1b) and the teacher suggested using a TSTEP of 0.1. When he asked the class to explain the parabolic shape, students made the link it was due to the presence of t^2 in the equation. In regards to the motion, students commented of the point plotting: "It's going down", "Hey, it goes quick", and "It's getting quicker", etc.

Discussion on the production of points on graph targeted operational understanding of position vectors (judging by the definitions that Sfard (1991) offers). Connecting the algebraic structure of the $\mathbf{r}(t)$ expression to shape of the graph encouraged structural understanding. Regarding the motion, direction ("down") was noticed, judgments were made about speed ("quick") and changing speed ("getting quicker"), but the terms direction and speed were not used explicitly and were not defined precisely. Mathematical understanding of direction and speed were not pursued.

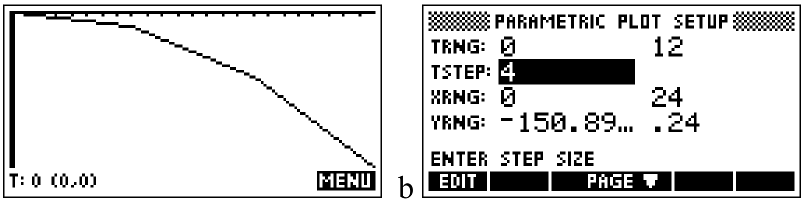

Figure 1. (a) Graph of $\mathbf{r}(t) = 2t\,\mathbf{i} + (t - t^2)\,\mathbf{j}$ for $0 \le t \le 12$. (b) The plot set up.

Two issues arose with the function $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$. Some students used an autoscale facility when plotting the graph and an ellipse appeared (see Figure 2a). With a default decimal scale the graph was a circle (Figure 2b). The teacher led the class in revising how to set symmetric scales (where distances between units on each scale are equal and spatial relationships are preserved), and the conventionally accepted shape for $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ (a circle) was produced. The other issue was students across the class continued to use the range $0 \le t \le 12$ and TSTEP (0.1) that they had used previously, so their calculator graphs of the circle showed nearly two rotations, and second circle did not exactly coincide with the first. When the teacher asked why, a student offered that time for one rotation (the period) was not a multiple of 0.1, the time interval that students were using to plot the graph. The class decided the solution to the problem was plotting one rotation only ($0 \le t \le 2\pi$). Thus, the period, a property common to all position vectors that involve periodic functions was addressed, so the discussion promoted structural understanding of the vectors.

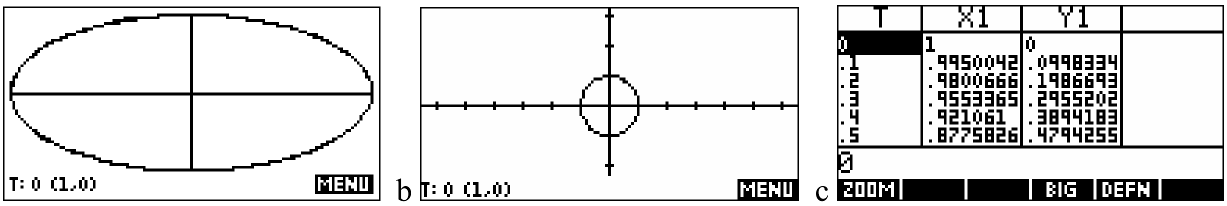

Figure 2. The graph of $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ with (a) asymmetric and (b) symmetric scales. (c) A the table of values for $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$.

The teacher also asked students to access the table of values for the circle (Figure 2c) and asked them for relationships that linked the x and y values. Answers included: "x is decreasing and y is increasing", which was correct for the values that were on display but was not generally true. It was a case where the calculator display was misleading. Eventually a student suggested: "the modulus is one", which the teacher restated as the Cartesian relationship $\sqrt{(x^2 + y^2)} = 1$. Thus, the numerical structure in the $\mathbf{r}(t)$ co-ordinates was generalised. Generalisation and making connections between different mathematical forms are other aspects of structural understanding (Sfard, 1991).

Some students recognised that $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ matched the unit circle given by $(\cos\theta, \sin\theta)$, with which they were familiar. Hence, they linked the algebraic structure of the $\mathbf{r}(t)$ expression to the unit circle, so articulated other structural understanding.

Speed on the circle was not discussed. One reason was the class was distracted by the problems with the graph. It was noticeable though that speed on the ellipse appeared faster along the top and bottom than along the sides because of the point-by-point plotting at equal time intervals. Speed on the circle was constant. Hence, with periodic functions it is imperative that spatial relationships are preserved if relative speeds (faster/slower) are going to be inferred from the graph.

In summary, the dynamic, point-by-point plotting allowed students to experience position and motion visually, and see position represented mathematically as a graph. The simultaneous achievement of the phenomenal and the conceptual potentially encouraged mathematical understanding (Roth et al., 1996). Actions that may have helped develop operational (procedural) understanding of position vectors were: choosing x and y scales, the range for t, and the time interval between points on the graphs; and retrieving co-ordinate values for $\mathbf{r}(t)$ in calculator table of values. Linking algebraic and graphical forms, and identifying the period, modulus and the Cartesian relationship for the circle, targeted structural (conceptual) understanding.

The mathematics of motion was treated to a much lesser extent than the mathematics of position. There was no formal treatment of speed and velocity: the properties that were mentioned were not given numeric, algebraic or graphical definition and no methods of calculation were discussed. In the absence of any comment, it cannot be said that students noticed the varying speed on the ellipse and constant speed on the circle. Thus, the dynamic graphing mainly served the mathematical definition of position and provided background for future work on motion.

Subsequent lessons
In the second lesson on vector calculus, the teacher implemented four dynamic graphing activities. He asked students to:

1. graph $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ using different TSTEP values, and explain the outputs;
2. predict the starting point and direction of the motion (clockwise or anticlockwise) on $\mathbf{r}(t) = \pm\cos t\,\mathbf{i} \pm \sin t\,\mathbf{j}$, for different choices of sign, and check by graphing on the calculator;
3. plot the graph of the motion of a stone given by $\mathbf{r}(t) = 2t\mathbf{i} + (10 + 12t - 4.9t^2)\mathbf{j}$, state the meaning of the components, and state questions that could be asked and provide the solutions;
4. plot and describe the graph of $\mathbf{r}(t) = 10\sin t\,\mathbf{i} + 5\sin 2t\,\mathbf{j}$ (see Figure 3).

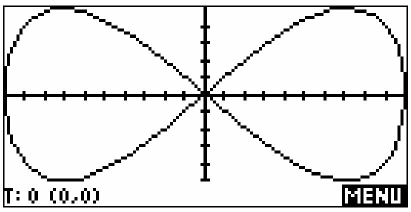

Figure 3. Calculator graph for $\mathbf{r}(t) = 10\sin t\,\mathbf{i} + 5\sin 2t\,\mathbf{j}$.

Each of the activities involved observation of point-by-point plotting and the calculation of co-ordinates of points on the $\mathbf{r}(t)$ trajectories, as in the first lesson, and operational understanding of position vectors was salient. Other operational (procedural) views that were discussed were that components can be evaluated separately (e.g., the $\mathbf{j}$ component was used for obtaining maximum height in activity 3) and can be obtained using the trace facility on the calculator graph.

The prediction in activity 2) involved evaluating $\mathbf{r}(t)$ for different values of t and observing the change in position. Thus, procedures that were relevant to calculating velocity were touched upon, so, operational understanding of velocity was promoted. The motion was described as anticlockwise/clockwise, a structural conception of (angular) velocity. Activities 3) and 4) also resulted in students differentiating the **j** components of $\mathbf{r}(t) = 2t\mathbf{i} + (10 + 12t - 4.9t^2)\mathbf{j}$ and $\mathbf{r}(t) = 10\sin t\,\mathbf{i} + 5\sin 2t\,\mathbf{j}$ in order to obtain time for the turning points on the trajectories. Differentiating the **j** components to locate particular positions laid the foundation for differentiating $\mathbf{r}(t)$ expressions to obtain velocity vectors, two lessons later. Homework was to copy the bowtie graph (Figure 3) onto graph paper, mark the positions for $t = 0$, $\pi/8$, $\pi/4$ . . . 2π, and draw vector arrows from the origin to the points. Thus, procedures (and therefore operational understanding) pertaining to the geometric (arrow representation) of $\mathbf{r}(t)$ were addressed.

Dynamic graphing was also used the third lesson. The teacher projected the graph of $\mathbf{r}(t) = 10\sin t\,\mathbf{i} + 5\sin 2t\,\mathbf{j}$ (Figure 3) from his calculator onto the whiteboard. He drew arrows for the position vectors $\mathbf{r}(\pi/4)$ and $\mathbf{r}(\pi/2)$ on the static graph, using co-ordinates that students provided from their homework. So, he demonstrated and re-articulated the process for constructing the arrow representation of position. Later, he used the dynamic graph of $\mathbf{r}(t) = 10\sin t\,\mathbf{i} + 5\sin 2t\,\mathbf{j}$ to explain how the periods of the components (2π for the **i** component and π for the **j** component) determined the shape of the trajectory (thus promoting structural understanding of $\mathbf{r}(t)$).

Investigation of velocity and acceleration on the bow-tie graph (Figure 3) occupied a major part of the next four lessons. In brief, students differentiated position vectors $\mathbf{r}(t)$ to obtain velocity vectors $\mathbf{v}(t)$, and predicted what velocity vectors would look like on the trajectory. They checked their predictions by calculating $\mathbf{v}(t)$ at $t = 0$, $\pi/8$, $\pi/4$ . . . , and drew the vectors on their graph paper graphs to start at points on the trajectory, according to the teacher's instruction. Some calculated the components of velocity by-hand. Others entered the $\mathbf{v}(t)$ component expressions into the parametric aplet of their calculators and obtained the numeric components from the table of values.

The teacher constructed the vectors on a bowtie graph drawn on a grid on an overhead transparency, using students' component values. The class discussed magnitude and direction of velocity on the circuit, as evidenced by the length and direction of the vectors, and located approximate positions for maximum speed. They spoke of velocity and speed as though they were phenomena and not just mathematical objects. The teacher asked students to formulate a general equation for speed and to use it to calculate speed for the velocity vectors that they had drawn. Led by the teacher, the class entered speed as a function of time into their calculators, and identified the points on the bow-tie graph that corresponded to the relative minima and maxima points on the speed-time graph. They followed a similar sequence for direction as a function of time. Thus, rich, precise, mathematical descriptions of speed, direction, and velocity were articulated.

Another activity was predicting what acceleration vectors $\mathbf{a}(t)$ would look like for the bow-tie trajectory (see Forster & Taylor, 2003 for a full account). Students referred to their graphs on graph paper that showed the velocity vectors, and the teacher projected the graph he had drawn on the transparency onto the whiteboard. Discussion centred on changes in speed, which could be deduced from the lengths of the vectors. Students individually checked the predictions by calculating the acceleration vectors and drawing them on their graphs. The teacher drew the vectors on the transparency using students' values. The class explained changes in speed and direction in terms of the angle between the acceleration and velocity vectors. The teacher demonstrated the variation in the angle using a computer program. He entered the trajectory in algebraic form. Segments for velocity and acceleration and the angle between them were displayed dynamically on the trajectory. More activities on angle properties followed.

In summary, dynamic graphing in the second and third lessons afforded students opportunities to broaden their operational understanding of position vectors. Motion as evidenced on the dynamic graphs remained relatively undefined but procedures that were relevant to obtaining velocity vectors were introduced, as was the property that (angular) velocity can be clockwise or anticlockwise.

The hand-drawn bow-tie graph, with hand-drawn vector arrows for $\mathbf{r}(t)$, $\mathbf{v}(t)$ and $\mathbf{a}(t)$ at selected times, was the main resource for advancing the mathematics of velocity and acceleration. Tables of values on the calculator and static speed-time and direction-time graphs were also used.

Conclusion

Dynamic point-by-point graphing allowed students to view position on a trajectory, and view position in relation to Cartesian axes. In light of the findings of others (e.g., Lapp & Cyrus, 2000; Roth et al., 1996), experiencing phenomena and simultaneously viewing mathematical representations of the phenomena assists conceptualisation. Thus, it is likely that the dynamic graphing helped students conceptualise position.

Mathematical understanding of position was also promoted through calculating components as part of choosing scales on the graphs and through the retrieval of co-ordinates from the graph and table of values. Anomalous graphs (the jagged parabolas and the circles that did not coincide) provoked discussion, and addressing the anomalies resulted in further definition of procedures and properties to do with position vectors. As Artigue (2002) has noted, the pursuit of appropriate use of technology and addressing limitations of the outputs on technology can foster mathematical understanding. It was also the case that a truncated table of $\mathbf{r}(t)$ components on the calculator resulted in an incorrect generalisation—the table was misleading because few components pairs were showing, and this problem is not easily addressed.

The graphing also allowed students to experience velocity and acceleration phenomena, but there was no lasting mathematical representation of velocity and acceleration on the calculator screen. Hence, mathematical relationships pertaining to velocity and acceleration had to be inferred from the trajectories as they were plotted, and inference was imprecise. A program that draws arrows (or segments) to represent velocity and/or acceleration on the graph, with appropriate lengths and direction as per the students' hand-drawn graphs and the Roth et al. (1996) software, would address the shortcoming of the calculator display.

Another aspect of dynamic graphing in the Year 12 class that warrants recognition is that graphing $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ on non-symmetrical axes produced an ellipse, which was not suitable for judging velocity or acceleration: symmetrical axes are required so that spatial and rate of change relationships are replicated. One other outcome was the graphs were used to verify change in position calculations. Systematic investigation of change in position over time, using the trace facility to obtain time and co-ordinate values which are available on the screen after plotting is complete, could be used to introduce velocity formally and this would complement the introduction in the Year 12 class which involved differentiating position vectors.

In conclusion, the analysis in this paper was driven by the theoretical views that: dynamic graphs allow 'co-ordination of the phenomenal and conceptual' (Roth et al., 1996), and that conceptualisation of mathematics constructs involves understanding the procedures of their production and their properties (for operational and structural understanding). Keeping the theory in mind helped me to identify the mathematics learning that was possible through the dynamic graphing activities in the Year 12 class. In addition, although mentioned only briefly in this paper, use of the view screen with students' calculators attached, discussion between students, and the teacher's questions in class discussion, were important means through which mathematics procedures and properties were defined and understandings were shared (see Davis & Forster, 2003; Forster (in press); Forster & Taylor, 2003). Therefore, in summary, implications of my inquiry for teaching practice are that having students generate dynamic graphs on their calculators can be valuable for developing understanding of position in two-dimensional space, the scope for defining motion with the graphs is limited, addressing the limitations of the calculator outputs can assist mathematical understanding, and any benefit for learning of dynamic graphing hinges on effective communication in the class.

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