Using Computer Algebra Systems in Secondary School Mathematics: Issues of Curriculum, Assessment and Teaching.

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Abstract

In 2001 and 2002, a group of students in Victoria, Australia, studied a final year secondary mathematics course for university entrance that permitted use of a computer algebra system (CAS) for the first time in our region. This paper summarizes the changes to curriculum, assessment and teaching that occurred and discusses some major issues emerging. Teachers found they could give students a better overview of topics and saw benefit in some students moving more quickly through lengthy calculations that would otherwise frustrate them. The four teachers adopted different teaching styles with CAS, but they all stressed by-hand procedures as the basis of understanding. Teachers spent time developing students' appreciation of the need to use CAS in a discerning way and the algebraic insight that is needed to deal with the sometimes surprising answers provided by the machine. Some new topics could be added to the curriculum, but greater explicitness about the value of learning skills and concepts is required. Assessment with CAS raises many issues for question design and marking schemes. The final examination needed careful design, especially so that users of different CAS were treated equitably. Students demonstrated good achievement on all aspects of the course. In summary, the trial was a success and is now expanding to more schools.

Introduction

In 2001 and 2002, a group of students in Victoria, Australia, studied a secondary mathematics subject in the final two years of school that permitted use of a computer algebra system (CAS) for the first time. This paper will summarize the changes to curriculum, assessment and teaching that occurred and discuss the findings and some of the major issues that needed to be addressed.

The students were part of the "CAS-CAT" project, a trial conducted by the Victorian Curriculum and Assessment Authority (VCAA), the state government agency responsible for senior secondary school programs and for the end-of-school assessment, in conjunction with a research team from the University of Melbourne. A committee of experts established by the VCAA designed the curriculum and assessment for a new mathematics subject for years 11 and 12 which was undertaken in

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three schools, using Casio Algebra FX 2.0, Hewlett Packard HP 40 G and Texas Instruments TI-89 calculators respectively. These "CAS calculators" have all the features of a standard graphics calculator and in addition have powerful symbolic manipulation facilities for algebra and calculus. Some of these facilities are illustrated in Figure 1. With some differences which will be summarized below, the content and assessment of the new subject *Mathematical Methods (CAS*) paralleled the existing subject *Mathematical Methods*. This is the main mathematics subject preparing students for the mathematical requirements of a wide variety of university and technical qualifications in science, engineering, commerce and many other courses. About one third of the total cohort of students across the state enrolls in this subject in Year 12. It is basically a functions and calculus course, with a relatively low degree of rigor (e.g. no formal proofs are required). A more advanced subject and a less advanced subject are also available, alone or in combination with *Mathematical Methods* or *Mathematical Methods (CAS).*

In summary, this trial of CAS use took place within a subject designed to cater for a broad cohort of students with a wide range of mathematical needs and abilities and within a regime of very high-stakes assessment, where small differences in final total score can have a major impact on students' university entrance and other vocational options. As will be described below, the trial was a success and is now expanding to more schools. Further details, including papers, discussion documents and all the material prepared for schools, are available from the project website (HREF1) and from the VCAA (HREF2), which gives regulations, sample examination questions, details of the extended pilot study etc. This paper will report and reflect on changes that occurred to the teaching, the curriculum and the assessment, and draw some lessons for the future.

Sample item	Graphics Calculator	CAS Calculator
$3e^{0.5t} = 4$	$t = 1.96$	$t = 1.96$ or $t = 2.\ln(4/3)$
$3e^{at} = b$	no answer	$t = \ln(b/3)/a$ or $t = [\ln(b)\text{-}\ln(3)]/a$
$x^2 – 6x + 7 = 0$	5.828 or –0.1716	$x= 3 + 2\sqrt{2}$ and $x= 3 – 2\sqrt{2}$ or numeric
$x^3 = 16$	2.5198	2^(4/3) and two complex roots
$\sin 3x = 0$	Numerical values	$x = 2k\pi/3$ or $(2k\pi–\pi)/3$

Figure 1: Comparison of capability of graphics and CAS calculators in equations.

Teaching

Observations by the teachers

The project worked closely with the teachers in both 2001 and 2002, interviewing them on several occasions, visiting their classrooms and holding regular meetings. The data reported here is from the four teachers (Ken, Lucy, Meg and Neil) who taught the Year 12 students in 2002, all of whom had also taught Year 11 students in the project in the previous year. Three of them had taught

Year 12 *MM* many times. Participation in the project was a major professional undertaking for the teachers and their schools, as they needed to work with new technology and new curriculum, submit their students to new assessment procedures and provide research data. As have been noted often before (see for example Guin and Trouche, 1999), CAS calculators are complex machines, and learning to use them skillfully is a major undertaking for teachers as well as for students.

All the teachers were committed to teaching mathematics with understanding and to avoid any tendency for students to feel that they could simply press buttons to do maths. They all paid attention to developing effective use of CAS (Pierce and Stacey, in press), as they were concerned to ensure that students became discerning users who did not rely too much on the technology. From the beginning, it was clear that students needed a strong understanding of algebra to successfully use the CAS beyond the most routine tasks and so teachers aimed to develop the algebraic insight (Ball, Stacey and Pierce, 2003) that is needed to deal with the sometimes surprising answers provided by the machine. Dealing with unexpected algebraic forms of answers supplied by the CAS was a source of frustration for many students and required classroom discussion at every stage of the trial, although especially at the beginning of Year 11, when initial testing demonstrated that many of the students had only rudimentary familiarity with algebraic equivalence. For example only 54% of students were able to quickly recognize the equivalence of $(b+a)^2$ and a^2+b^2+2ab. This frustration diminished over the two year course as students' algebraic insight improved (Ball et al, 2003) but it always remained an issue for the weaker students. One teacher, Lucy developed this situation into a sort of classroom game: *"[Often] I'll write up three or four answers and ask the kids which one did the calculator give. … I try to guess which one [..] based on my understanding of the conventions but I am not always right"*. In summary, the capacity of all three brands of calculators to provide unexpected answers continued to alternatively exasperate and delight the whole team throughout the project.

New opportunities for teaching

The teachers were generally pleased with the new opportunities which CAS provided for their teaching. They and their students used it functionally (to get answers) and pedagogically (to support exploration of ideas). The experienced teachers reported moving more quickly through the curriculum than they had expected. All of them noted time saved when they were able to move through some lengthy calculations quickly. Finding the area between the x-axis and a curve which cuts it in several places is a typical instance where CAS use saved time. Students could automatically find roots of the function and integrate with these as endpoints with only a few CAS commands. Lucy, for example, recalled an occasion when she labored for minutes to find a minor error in her by-hand calculations of such an area on the board, whilst her students quickly completed this and other exercises using the technology. She felt that using CAS for long calculations reduced the level of frustration that some students feel with mathematics. There was some evidence that students were more aware of the overall plan of their solutions and did not get as often lost in the detail of calculation as students doing the algebraic manipulations by hand.

The teachers did reduce the amount of by-hand practice that they expected students to do. A typical compromise was to try half of the routine textbook questions by hand and half using CAS for *new* skills. When *previously acquired* skills were involved (e.g. using integration in continuous probability questions), teachers were generally happy for students to choose the quickest method, which meant some students used CAS on all but the easiest questions. All of the teachers were concerned that students would become dependent on CAS, and were concerned to encourage them to think ahead, selecting CAS only when it was likely to be the fastest method, an orientation in accordance with the need for speed on the final examination.

All of the teachers noted that using CAS enabled them to begin a topic with "the big picture" and two teachers, Neil and Meg, observed that this saved considerable time in teaching concepts. Meg noted that she covered almost all of the conceptual work involved in exponential and logarithmic functions in just two lessons, by demonstrating key ideas with CAS graphical and symbolic features (e.g. linking key features of graphs of simple and compound functions such as $y = \exp(kx)$ and $y = \ln(2x^2 + 3x - 1)$ to their derivatives found using CAS and by-hand). Meg also described her teaching of applications of differentiation, where she observed that without the cognitive distraction of dealing with technical details, students could focus more clearly on why the derivative is used and the overall process. For example, she felt that students were more aware that for finding a maximum value there are three macro steps: obtaining a derivative, finding when it is equal to zero and testing the nature of the identified points. In summary, she commented that "*I've probably gone a lot further than what I have done with any [Mathematical Methods] teaching this year, with the help of the calculator*".

Neil reported that he developed a new "holistic" style for introducing topics, using graphical and symbolic features of CAS to provide an overview that emphasized the purpose of the new material, demonstrating ideas numerically, graphically and symbolically and observing rules and generalizations from the information arising. For example, Neil's first lesson on calculus discussed instantaneous rates of change, demonstrated graphically with tangents and with tables of values of gradients, and linked to the results of symbolic differentiation. Patterns observed in the symbolic answers led students to induce rules such as that the derivative of x^n is nx^{n-1}. Previously, this would have been taught as a separate rule, as a generalization of the derivative of x^2, without proof. Neil reported that this new style was more time efficient and gave improved understanding.

In contrast, Lucy and Ken generally began from by-hand work. Lucy, for example, said: "*My personal preference is that for every new procedure they see it first by hand, they learn to do it by-hand for simple examples and then progressively as the manipulations get more difficult or as the manipulation is not the core part of the problem, you ask the calculator to do it. …. I don't think that you can start effectively by just saying that this is what the CAS does and [later] let's go back to how you do it by-hand.*" Lucy also stressed, however, that in the examination, students should choose the quickest method. On a few occasions, Lucy used CAS to advantage in a black box way. For example, she demonstrated the power of integration to find areas by using CAS to calculate the area of a circle from $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$, an integral well beyond the course.

In terms of the black box/white box distinctions, Neil (and to a lesser extent Meg) adopted a black box/white box approach for teaching new material using CAS as a generator of examples and an explorative tool, whereas Lucy and Ken adopted the white box/black box approach close to that advocated by Buchberger (1990). All of these teachers were generally successful with their own students: there is no data in the project to say that one approach was more successful than the other. The four teachers adopted quite different teaching styles with CAS, although they all stressed by-hand procedures as the basis of understanding.

The curriculum

The need for change

The curriculum was designed to run parallel to the existing subject *Mathematical Methods*, which allows use of graphics calculators without symbolic manipulation features, but permits user-stored programs. The increasing sophistication of the user-stored programs on graphics calculators provides one reason why a project investigating the use of computer algebra systems in schools and examinations was timely. Programs with increasingly sophisticated symbolic manipulation are available which can solve certain classes of equations algebraically (e.g. linear, quadratic), differentiate and integrate standard functions etc. One example is the downloadable program "Symbolic" for the TI 83 graphics calculator (HREF3). Students with more advanced graphics calculators or whose teachers are more knowledgeable about resources for calculators are therefore likely to gain an advantage over others. The current rules permitting graphics calculators (those without built-in symbolic manipulation – i.e. not what we call "CAS calculators") are therefore likely to become increasingly inequitable.

In designing the new curriculum, a number of fundamental questions arose. The first decision was that the new subject should have a great deal of content in common with the existing subject. This allowed students to move between *MM* and *MM(CAS)* from Year 11 to Year12 (as they would need to do if they shifted from the experimental school, for example), enabled reasonable use to be made of the existing textbooks for *MM*, gave teachers familiar material to teach whilst other things changed etc.

How, though, and why should the existing content be modified? To create a sensible curriculum and to take advantage of new opportunities, there were several reasons for change:

- Some topics becoming more accessible when CAS is available;
- Including more topics to use curriculum freed by using CAS;
- Changes in the pragmatic value of topics when CAS is available;
- Changes in the epistemic value of topics when CAS is available.

CAS makes some topics more accessible than before, principally by eliminating certain technical obstacles. (Many authors, for example Drijvers (2003), note that CAS by no means eliminates all technical obstacles). The new content introduced into *MM(CAS)* generally fitted into this category – for example, the probability content was extended to include work on transition matrices since technology can readily calculate high powers of transition matrices in exact or numerical form. (Unless long runs, and hence high powers are involved, there is no benefit in the transition matrices over tree diagrams). Similarly, in the new curriculum the range of functions which could be used in many settings (e.g. graphing, differentiating) was extended, relaxing previously tight constraints that limited technical difficulty of examination questions.

It is reasonable to expect that if students can use automated procedures to carry out certain routine calculations, then some curriculum time which would otherwise be spent on by-hand practice may be freed for other purposes: possibly to include more topics, to study topics in greater depth or to spend more time on applications of what has been learned. Since time needs to be allocated to learning how to use the machine and some by-hand work is essential for developing a strong understanding of each topic, the amount of curriculum time freed is limited, as has been observed by many authors (e.g. Guin and Trouche, 1999). However, some moderate increase in the content of the curriculum can be seen to have been justified because, as noted above, the teachers in the trial generally reported that they were able to cover topics more quickly. A particularly successful addition was to include continuous probability distributions, whereas previously only finite probability distributions were covered. This topic was successful because it revisited the ideas of finite probability distributions in a new context, it provided an area of application for integration (a previously studied topic) and students dealt confidently with the technical features of integration either by-hand or by-CAS if they preferred.

Pragmatic, epistemic and pedagogical values

Topics earn a place in a mathematics curriculum because they are useful (have *pragmatic* value) or because they hold an important place in the structure and development of mathematical content (have *epistemic* value) or because studying them serves a purpose not related to the content itself, such as providing an opportunity to practice skills (i.e. *pedagogical* value). Artigue (2002) discusses the first of these two categories and we are adding the third. The framework is illustrated in Figure 2.

Changes in technology can change both pragmatic and epistemic value. For a few topics, having CAS increases the pragmatic value. For example, since CAS is able to work with exact values, these became more useful in the *MM(CAS)* course. Solving problems with exact values has epistemic value, but in a graphics calculator environment with its orientation to the numerical, the pragmatic value is very low. CAS, on the other hand, is able to calculate with exact values and this seemed to increase their value to students, who were able to use them as intermediate as well as final results.

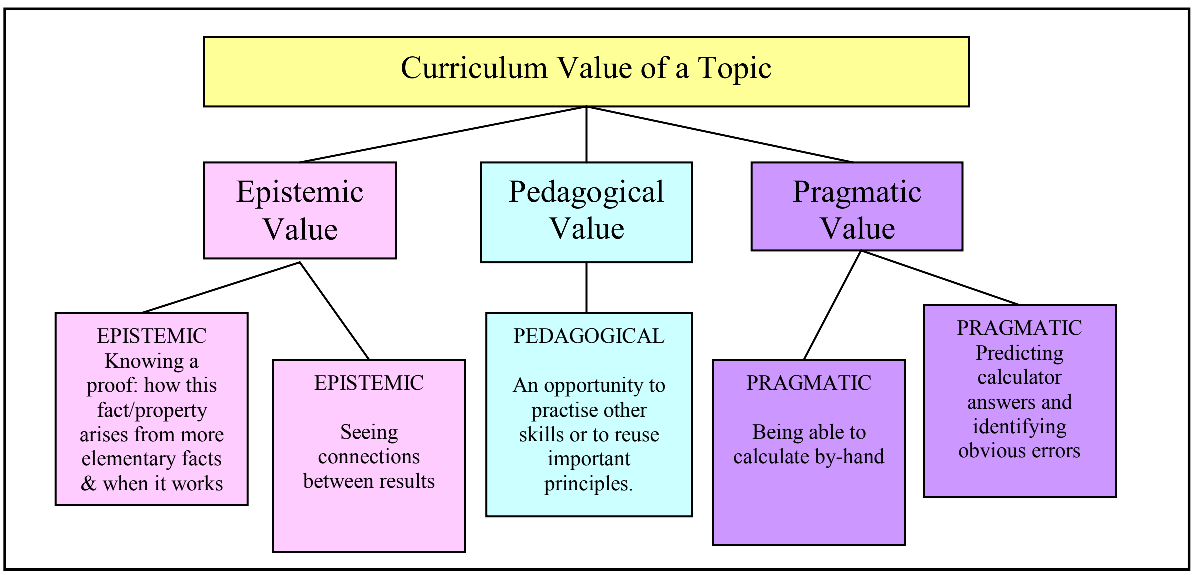

Figure 2: *Topics can have epistemic, pragmatic and pedagogical value.*

More often, though, technology reduces pragmatic value. For example, using calculus to make a linear approximation to a function with the formula $f(x+h) \simeq f(x) + h.f'(x)$ once had a pragmatic value in permitting ready approximation to the values of complicated functions. One could, for example, readily approximate the square root of 26.1 as 5.11 (with correct value 5.1088 to four decimal places) based on $x = 25$, whereas a by-hand calculation is very long. Questions of this nature have been asked in our school examinations until recently. When students have immediate access to a scientific calculator, this approximation formula is no longer of pragmatic value. However, this formula still has immense epistemic value, being central to the principles underlying calculus. More subtly, it also retains a pragmatic value as the basis of many algorithms used within the calculator, but this is not of pragmatic value *to the student*. In a course of generally low rigor such as *MM* (and hence *MM(CAS)*), it is difficult to retain an emphasis on key ideas such as this, when there are no accessible problems to solve using them.[2]

The rules for differentiation of complex functions also lose their pragmatic value when they are automated by readily available technology. If they are only taught as rules without reasons (as they tend to be in our low-rigor courses), they also have little epistemic value and seem good candidates for elimination from the curriculum. To retain a place, these rules need to be used to enhance the understanding of the structure of calculus and connections between topics. For example, the product rule now has little pragmatic value. CAS calculators are highly reliable in differentiating both simple and complex functions (in contrast, for example to their lower reliability in equation solving, where minor variations in algebraic form can make a major impact on whether a CAS can solve a set of equations). There is some pragmatic value in knowing the product rule in order to monitor results of CAS calculations and identify errors of entry or syntax and there may be some pragmatic value in knowing that the derivative of a product is not the product of the derivatives, hence avoiding a temptation to differentiate by hand erroneously. There may also be some pragmatic/pedagogical value in teaching the product rule in case students require it for other study where CAS is not available. Altogether though, the case for inclusion of the product rule on pragmatic grounds is not strong.

On epistemic grounds, there is a strong case for the product rule related to demonstrating the connections in the conceptual web that is calculus. The sense of understanding mathematics arises in large measure from seeing it as a related whole, rather than as a series of disconnected facts. As is shown in Figure 3, the derivative of a product can be found using the linear approximation of the function obtained from the derivatives. In this way, the two terms of the product rule are exposed as the "cross-product" terms of the expansion of the large brackets on the first line. From this informal proof, I feel that I understand why the product rule holds. Without a proof, there is still some lesser epistemic value in being able to connect results together – for example, to link the derivative of x^3 to the derivatives of x and x^2:

$$\frac{d(x^3)}{dx} = \frac{d(x.x^2)}{dx} = x.(2x) + 1.x^2 = 3x^2$$

This epistemic value will, however, only be realized if the links and reasons are explicitly made, which requires a different orientation to using the product rule only in sets of differentiation exercises.

Our experience with the trial program has underlined two facts. Firstly, the curriculum value of topics is markedly changed by the introduction of CAS. Old justifications for teaching topics, especially pragmatic justifications, will not necessarily still apply. Secondly, the educational community now needs to build up sophisticated rationales for curriculum decisions related to areas that were not debated in the past. Justifications may be on pragmatic, epistemic or pedagogical grounds. Meg commented that she often asked herself "*What am I doing this for, when the kids can just put it into their calculators and do it?"*. Nowadays, a curriculum with or without CAS needs a strong rationale to help teachers understand the value of the topics included, so that they can be treated in a way which enhances their curriculum purposes.

[2] One possibility is to regain some of the pragmatic value by introducing basic calculator algorithms as a topic of study.

$$
\begin{aligned}
&(f(x+h).g(x+h) \simeq (f(x)+hf'(x)).(g(x)+h.g'(x)) \\
&\simeq f(x).g(x)+f(x).h.g'(x)+hf'(x).g(x)+h^2 f'(x).g'(x) \\
&\lim_{h\to 0}\frac{(f(x+h).g(x+h)-f(x).g(x))}{h} \\
&=\lim_{h\to 0}\frac{f(x).g(x)+f(x).h.g'(x)+hf'(x).g(x)+h^2 f'(x).g'(x)-f(x).g(x)}{h} \\
&= f(x).g'(x)+f'(x).g(x)
\end{aligned}
$$

Figure 3. *The product rule arises from the use of the derivative to approximate functions.*

Students' achievement

This section presents some observations on the achievement of the *MM(CAS)* trial students. It is not possible to use the examination data to make an overall judgment about whether learning mathematics with CAS promotes higher achievement than learning mathematics with just a graphics calculator or whether the students in *MM(CAS)* did better than the students in *MM*. This is because the trial could not be set up to allow valid conclusions of this nature to be made. Firstly, all three schools had teachers of above average competence and commitment, which was essential to work in the difficult experimental situation (e.g. without an appropriate textbook and with complex new technology to master). For the same reason, the schools involved are not representative. Secondly, for ethical reasons the students were not randomly allocated to *MM* or *MM(CAS)* – the decision was either made by the school when it decided to be involved in the trial or by the students and their families. Thirdly, the learning objectives of the two subjects are different (the content and methods differ) and whilst there are common questions on the exams, the assessment is not the same. Despite the inability to compare achievement, it was essential to monitor the achievement in the CAS course was satisfactory and this was done as outlined below.

The greater general ability of the *MM(CAS)* pilot group was confirmed by the fact that the *MM(CAS)* "scaled score" which is used for university entrance was higher than the *MM* scaled score. The scaled score for each subject is calculated from the results of the student population[3] in all the *other* examinations that these students sat (HREF4). For the 2002 examinations, the scaled mean scores were 38.8 for *MM(CAS)* and 36.6 for *MM*. For comparison, the subject English had a scaled mean score of 28.5. The hardest mathematics subject, Specialist Mathematics, had a scaled mean score of 41.2.

Analysis of the responses to questions asked on both the *MM* and *MM(CAS)* examinations confirmed that the *MM(CAS)* cohort performed better than the *MM* cohort, as expected. *MM(CAS)* students outperformed *MM* students on almost all of the questions which were asked on both papers

[3] For technical reasons, the calculations of the scaled score involved only 70 of the 78 *MM(CAS)* students.

(66% of the total marks) (Leigh-Lancaster, 2003). Since CAS was generally not useful in these common questions, the higher performance of the *MM(CAS)* group is unlikely to be due to the different technology permitted in the examination. VCAA testing of important algebra and calculus skills without technology, which was carried out shortly before the examinations, also showed that students had similar by-hand skills to students learning without CAS, reflecting the attention that the pilot group teachers had given to by-hand skills for developing understanding.

The official assessment reports noted that students generally used CAS well. The VCAA-appointed Chief Examiners for *MM(CAS)* were both highly experienced in writing and marking external mathematics examinations, including *MM*. As part of the formal process of the examination, they commented (positively and negatively) on achievement in each question, noting that students overall found the examination papers accessible, that students generally used the symbolic facility of the calculators well, generally wrote answers in standard mathematical notation, and that many students did very well. Analysis by the research team of the students' scripts showed that they omitted fewer questions than is normally the case. They used CAS very widely in the examinations, probably more than their teachers expected and for purposes which had not been expected.

Further confidence that the students in *MM(CAS)* learned well comes from the observations of three of the four teachers who were highly experienced in teaching *MM*. Asked 4 months before the final examination to summarize how the students were progressing, the teachers were all positive, although all reported some degree of difficulty in teaching the new subject and using the CAS calculators for the first time. Meg contrasted the students' enjoyment with her own apprehension and commented: "*I think I'm doing a lot more conceptual and high-powered things than I would normally [do] in the Mathematical Methods course. I think I've gone further with these kids than what I have in the straight Mathematical Methods course*". Lucy commented that: "*relative to a normal MM course, the kids seem to continually surprise me with a depth of understanding that I wouldn't have expected in the traditional course*." Neil, whilst noting a Hawthorne effect for students who feel special because they are involved in the experimental course, commented that students have a "*better holistic view of the topic. ... Rather than just overtaking their mathematics, it [CAS] has just given then another way of looking at it, a very powerful way of looking at every piece of maths that they would want*."

Assessment

Assessment of mathematics has an important part to play in the design of an excellent system of mathematics education for any country. It reflects the curriculum and also plays an important role in showing what is valued. It defines in detail what is regarded as acceptable and what methods for solving problems are preferred. The final *MM(CAS)* examinations needed to be very carefully designed, especially to ensure that users of different CAS were treated equitably. Amongst the key challenges of CAS to assessment are:

- That CAS "gobbles up" the solving phase, so that intermediate steps are not available for inspection by the examiner or examinee;
- That there is an explosion of possible methods that students can use, including making some conceptually simple numerical and graphical methods feasible, where previously only algebraic methods were practical.

- That the lower intrinsic value of manipulative procedures demands a shift away from questions that test routine procedures towards questions that test formulating mathematical models and interpreting what answers mean in context;
- That many of the devices used to avoid questions which have been "trivialized" by CAS (e.g. the introduction of parameters) can in fact make questions very much more difficult conceptually;
- CAS often causes shifts in what mathematical knowledge and abilities are assessed so that careful consideration must be given as to what is assessed in an examination question;
- That different models and brands of CAS have distinctly different capabilities, raising heightened issues of equity.

All of these phenomena have consequences for both examination question design and for marking schemes. It is expected that examination practices will take some time to evolve and to develop powerful techniques for examining important mathematics in this new environment. Our creativity is challenged to design questions that will show how deeply held mathematical values can be adequately tested, in a way which encourages preparation through good learning and teaching.

The consequences for examinations are outlined below. However, it is important to note that extending high-stakes assessment beyond the traditional examination format will also be required, in order to gain any benefit from the way in which CAS can amplify what students can do.

Our recommendations for examinations are that:

- Examinations should include both questions where CAS is necessary for nearly all students, where it may be chosen by some students and where it is not useful (i.e. the examination as a whole should be "CAS-active", although not all individual questions will be);
- The overall level of difficulty of CAS-permitted examinations must be monitored closely to prevent them becoming too difficult, and this will require inclusion of some routine questions that an expert regards as "trivialized" by CAS;
- Examinations need to be carefully monitored to check that they are "brand-neutral" overall (i.e. users of no permitted CAS models and brands have an unfair advantage over others), but it is not feasible to set an examination where each of the questions are individually "brand-neutral";
- Opportunities exist to set newer style questions that test valued mathematical knowledge and focus on using CAS to apply algebra and calculus-based techniques.
- Examination questions require explicit indications to students of which intermediate answers should be shown;
- Students need assistance to write adequate records of solutions assisted by CAS (Ball and Stacey's (2003) RIPA scheme is one approach);
- Examiners need to be aware of emerging capabilities of the technology that students use and adjust questions accordingly;
- Marking schemes will need to reflect new solution approaches offered by students and will need to accommodate the explosion of methods that is a consequence of technology availability;
- Examiners can expect solutions containing non-standard mathematical syntax;
- The complexity of modes and other means of personalizing a CAS means that students should not have to clear memory or reset machines to default settings before an examination;
- The opportunities to assess a wider range of mathematical knowledge and abilities beyond routine calculation should be taken carefully but deliberately.

The next sections provide examples of the points above, and further details are given in Stacey (2002).

Example 1. Finding the maximum value of $f(t) = \exp(-kt).\, t^n$

This question illustrates how CAS gobbles up intermediate steps, how technology can change what a question assesses (in this case shifting the demand from differentiation to finding equivalent algebraic forms), the need for good understanding of algebra (especially algebraic insight) and how a careful watch needs to be carried out to minimize inequities for users of different models and brands of CAS.

By hand, finding the maximum value of the function $f(t) = \exp(-kt).\, t^n$ is a substantial task for our students. They have to differentiate using the product rule and then simplify the expression $f'(t) = -k.\exp((-kt).t^n + n\exp(-kt).t^{n-1} = (n-kt).\exp(-kt).\, t^{n-1}$ before solving $f'(t) = 0$ using the null factor law in an unfamiliar situation. In contrast a CAS user can define the function $f(t)$ and then give a single command in appropriate syntax such as

solve (d/dt (f(t)) = 0, t)

to receive the solution $t = n/k$. CAS has "gobbled up" the technical work, leaving only the overall plan of the solution to be inputted.

This question also illustrates the point: how using new technology can change what a question assesses. If the student works with the CAS step by step with a standard CAS (the Hewlett Packard HP40G), instead of using the compound command, they find that

$$f'(t) = \exp(n.\ln(t)) * (n-kt)/t.\exp(kt)$$

and it is very difficult to use the calculator to change the expression *exp (n.ln(t))* into the simpler form t^n. A user of another model or brand of CAS may get the form involving t^n immediately. Surprises like this demonstrate that using computer algebra requires good algebraic insight, but that the demand of questions can be significantly altered.

Example 2: The problem of assessing connections with CAS solution

When using CAS for assessment, calculation can no longer be used to assess understanding and knowledge of patterns. Instead, we need to create new strategies for highlighting reasoning, explanations and connections in mathematical work. Currently, except in the highest level courses, our Year 12 students mostly display their mathematical reasoning and demonstrate their understanding of the connections between mathematical facts when they carry out algebraic manipulations in the context of trigonometry, calculus etc. For example, the argument shown in Figure 4 links two different types of

By-hand solution	By-CAS Solution
$(\sin x + \cos x)^2$	Enter: $(\sin x + \cos x)^2$
$= \sin^2 x + 2\sin x \cos x + \cos^2 x$	Output: $1 + 2\sin x \cos x$
$= \sin^2 x + \cos^2 x + 2\sin x \cos x$	Highlight second term and select
$= 1 + 2\sin x \cos x$	TLIN or TCOLLECT
$= 1 + \sin 2x$	Output: $1 + \sin 2x$

Figure 4: *Connections evident in a by-hand solution are missing in a by-CAS solution.*

trigonometric expressions, by applying algebraic expansion, Pythagoras's theorem and a double angle formula. It is an ordinary result of mathematics, but quite beautiful and obtaining it by carrying out the steps one by one by-hand is immensely satisfying. In contrast, using CAS for the simplification (see Figure 4 for the HP40G steps) takes away the charm and satisfaction. Much of what many people like about school mathematics is contained in small instances like these. In Gila Hanna's terms (Hanna & Jahnke, 1996), the proofs that explain, rather than simply prove, are the ones which we do not want to lose. My hope is that we can find creative ways of retaining this even when we adopt the technology.

Example 3: How algebraic form opens and closes gateways to solution paths

The question in Figure 5, which is written for a non-CAS environment, has been analyzed by Flynn and Asp (2002). This question demonstrates how working with a CAS might open some gates in a solution path and close others, in a way that is different to by-hand solutions. The answers in parts (a) and (b) can be obtained in one step with some CAS and in two steps with others (first differentiate, then factor to get in the required form). For part (c), however, there are considerable inequities which would arise for students assisted by different CAS. All CAS have strengths and weaknesses in this regard: with some questions one brand is better, with others a different brand has an advantage.

With one CAS (the TI89) solving the equation $e^{2a}(3\cos a - 4\sin a) = 0$ for a gives

$$a = n\pi + \tan^{-1}(3/4) \text{ where } n \in J .$$

and applying the function "tan" to both sides of the equation gives the output $\tan a = 3/4$. Substituting into the derivative gives $e^{2\tan^{-1}(3/4)}$ which is e^{2a} (with some care about the $n\pi$ term).

{question introduction omitted} . . . [Let] $y = e^{2x}\cos x$

(a) Show that $\dfrac{dy}{dx} = e^{2x}(2\cos x - \sin x)$

(b) Find $\dfrac{d^2y}{dx^2}$

(c) There is an inflexion point at P(a,b). Use the results from (a) and (b) to prove that

(i) $\tan a = 3/4$ and (ii) the gradient of the curve at P is e^{2a} .

Figure 5. *International Baccalaureate Mathematical Methods Standard Level Paper 2, Question 7, 2000*

With another CAS (in this case the Casio FX2.0), the problem is considerably harder and the CAS user needs considerable flexibility moving between by-hand and CAS techniques as the unexpected CAS output interrupts the flow of the question as planned by the examiners. With this CAS solving the equation $e^{2a}(3\cos a - 4\sin a) = 0$ for a gives $a = 2k\pi + 2\tan^{-1}(1/3)$ where $k \in J$ and it is very difficult to get to $\tan a = 3/4$ in the exact mode, although the approximate mode readily evaluates tan *a* as 0.75. Substituting $\tan^{-1}(3/4)$ into the expression for the derivative gives $e^{2\tan^{-1}(3/4)}(2\cos(\tan^{-1}(3/4)-\sin(\tan^{-1}(3/4)))$ and the second bracket can be evaluated to 1 in the CAS, giving the required answer. This example shows how calculator outputs are dependent on internal simplification algorithms. Different outputs lead solutions most easily along different paths. Intermediate results are like gateways that are provided in the question to enable passage from one area of a question to the next. In this question, we see how a gateway meant to easily lead the solver from one part of the solution to the next, may in fact be very badly placed for some CAS users and thus can serve to make the question very much harder.

Example 4: Assessing patterns and connections

One example of how we may be able to adapt assessment to emphasize the connections and patterns in school mathematics is shown in Figure 6, which was created by Peter Flynn of the University of Melbourne. With a CAS, differentiating $e^{2x}\cos x$ is a trivial task, although as we note above getting the answer into a requested algebraic form may be more complex. Whereas solving this question without CAS demonstrates knowledge of the interaction between multiplication and differentiation (the "product rule"), with CAS this knowledge is not needed. The question in Figure 6 attempts to assess the appreciation of the pattern of the product rule. Since this can no longer be directly assessed by computation, an indirect approach is needed to assess understanding of patterns such as the product rule.

(a) Find the derivative of $e^{2x}\cos x$.
(b) The product rule states that $(uv)' = uv' + vu'$ Given that $u' = -\sin x$, show how your answer to (a) fits the product rule by correctly identifying *u*, *v* and *v* '

Figure 6: *A possible new approach to assessing understanding of the product rule.*

Future directions

This paper has outlined some of the major findings of the CAS-CAT project. Overall the trial was a success, due in great measure to the dedication of the participating teachers who had to learn how to teach with a complex and demanding technology. In this process, there were frustrations, but there were also opportunities for enhanced learning. The presence of CAS offered many challenges for the teachers, in terms of both content and pedagogy. The symbolic capability of CAS provided opportunities to treat topics in a more algebraic way, although this was not always an easy thing to do. In concert with the graphical capability, the CAS offers a powerful tool to enhance school mathematics. Students required good understanding of algebra to work with CAS, and this understanding was able to grow in its presence. Overall the results of the students indicated that they had learned well and that further trials can confidently proceed.

CAS has implications for curriculum and for assessment, whether one is considering a school system about to adopt new technologies for learning and assessing mathematics, or a more conservative school system which nevertheless has an eye to the future. An exclusive emphasis on routine procedures is not defensible: instead the curriculum and the assessment of the future must give greater weight to formulating real world problems in mathematical terms and interpreting mathematical solutions in real problem contexts. This has been a goal of mathematics curriculum reform for many years but more needs to be done and the rationale is becoming stronger and stronger as technology takes over mathematical procedures from the simplest shopping applications to advanced algebra and calculus. This affects questions in a traditional examination but it also requires that other modes of assessment should be used.

Effective assessment with CAS is a challenging task and it will take some years and considerable creativity for new practices to be established. A series of "hot-spots" have been illustrated in this paper, along with some guidelines for responding to them. They demonstrate that doing mathematics with a technology tool is not the same as doing it by-hand. Throughout the CAS-CAT project, we have had many occasions when the answers thrown up by the CAS have caused us to look at mathematics in new ways. In this way the project has been challenging and rewarding mathematically and pedagogically and has provided strong guidelines for policy changes.

References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentalisation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*, 245-274.

Ball, L., & Stacey, K. (2003). What should students record when solving problems with CAS? Reasons, information, the plan and some answers. In J. Fey (Ed.), *Computer Algebra Systems in Secondary School Mathematics Education, (*pp. 289-303). Reston, Va: NCTM

Ball, L., Stacey, K., & Pierce, R. (2003). Recognising equivalent algebraic expressions: An important component of algebraic expectation for working with CAS. In N.A. Pateman, B.J. Dougherty and J.T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol 4, 15 - 22). University of Hawai'i: PME.

Buchberger, B. (1990) Should students learn integration rules? *Sigsam Bulletin*, 24(1), 10 – 17.

Drijvers, P. (2003) *Learning algebra in a computer algebra environment. Design research on the understanding of the concept of parameter*. Published dissertation. Center for Science and Mathematics Education. University of Utrecht, Netherlands.

Flynn, P. & Asp, G. (2002). Assessing the potential suitability of "show that" questions in CAS-permitted examinations. In B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.), *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematical Education Research Group of Australasia, pp. 252-259). Sydney: MERGA.

Guin, D. & Trouche, L (1999) The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning*. 3, 195-227.

Hanna, G. & Jahnke, N. (1996). Proof and Proving. In. A.J. Bishop et al (Eds) *International Handbook of Mathematics Education.* (pp. 877–908).Kluwer, Dordrecht.

HREF1 http://www.edfac.unimelb.edu.au/DSME/CAS-CAT *CAS-CAT project*, Department of Science and Mathematics Education, The University of Melbourne, Australia. (Accessed August 27, 2003)

HREF2 Victorian Curriculum and Assessment Authority Accessed 3Sep 2003 http://www.vcaa.vic.edu.au/vce/studies/MATHS/

HREF3 Texas Instruments http://www.ticalc.org/pub/83plus/flash/ Accessed 1 Sep 2003.

HREF4 Victorian Tertiary Admissions Commission Accessed 3 Sep2003 http://www.vtac.edu.au/library/stats/2002-2003/section-H.pdf

Leigh-Lancaster, D. (2003). The Victorian Curriculum and Assessment Authority Mathematical Methods Computer Algebra pilot study and examinations. Paper presented at 2003 symposium of *Computer Algebra in Mathematics Education*. Available http://ltsn.mathstore.ac.uk/came/events/reims/ Accessed 8 Sept 2003.

Pierce, R. & Stacey, K (in press) A Framework for Monitoring Progress and Planning Teaching towards Effective Use of Computer Algebra Systems. *International Journal for Computers and Mathematics Learning.* (in press)

Pierce, R. U., & Stacey, K. C. (2002). Algebraic insight: The algebra needed to use Computer Algebra Systems. *Mathematics Teacher, 95*(8), 622-627.

Stacey, K. (2002). Challenges to mathematics assessment from new mathematical tools. In D. Edge & B. H. Yeap (Eds.), *Mathematics Education for a Knowledge-Based Era* (Keynote address during the proceedings of the second East Asia Regional Conference on Mathematics Education and the ninth Southeast Asian Conference on Mathematics Education, *1*, 11-16). Singapore: Association of Mathematics Educators.