Curvature for everyone

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Abstract. The first derivative is about approximation by a linear object. Curvature is a measure of the distance of a smooth nonlinear object from being linear, straight, flat, ... Curvature has been studied at least as long ago as the ancient Greeks, and it remains a very tumultuous research area today under the names of differential geometry and its applications. While having been a core subject of study in most university mathematics curricula for many decades, its overly precise language and humongous formulas have always been a major obstacle to making curvature and differential geometry widely accessible.

The advent of computer algebra systems (CAS) is completely changing the playing field: The fundamental concepts and insights related to curvature are becoming accessible to a much larger part of the populace. At the same time, CAS allow one to shift from mostly very abstract theories with extremely few computable, nontrivial examples to lots of interactive experimentation largely based on computer visualization, but also on computer algebra. The opportunities for undergraduate research projects appear endless at this time!

This article is to present some background material surveying selected basic notions and also objectives. It accompanies a live presentation whose heart are interactive implementations into the CAS MAPLE of some of the most intriguing topics of elementary differential geometry. The main objective is to demonstrate how much technology has changed the access to fundamental notions of curvature and closely related topics.

1 Introduction

First derivatives are about approximation by linear objects. Curvature measures the *distance* from being linear. Formal studies of curvature go back at least as far as the 3rd century BC in ancient Greece when Apollonius of Perga studied normals, centers of curvature, and the evolutes of elementary curves [12]. The last three centuries have seen differential geometry bloom and reach ever higher – all an outgrowth of studying curvature. This expansion continues until today – e.g., the last few years have seen ever deeper probing into whether the universe is *flat* on a large scale or whether it will eventually reverse its expansion. Only this year we are seeing that one of the most famous open problems in mathematics, the Poincare conjecture [3, 8] may have been proved using very sophisticated methods that arise form curvature.

The language of geometry has developed into one of the most precise ones in all branches of mathematics – which at the same time makes it ever more daunting for the non-full-time

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geometer to access the topic. The language of geometry has also become inextricably intertwined with algebra, but unfortunately many of the resulting formulas are sheer monstrosities that again make access difficult.

Now add computer algebra systems (CAS) which are destined to take care of all bookkeeping of repeated differentiations and no matter how messy algebra. Together with powerful numerical and graphical capabilities they provide new access for everyone to curvature, and differential geometry. We understand that this subject is inherently beautiful, a core part of classical and modern mathematics, and replete with applications from the mundane everyday environment (e.g. turning radius of a car) to intriguing modern discoveries such as gravitational lensing of star light that we read about in the newspapers. It is a major challenge to think about a curved universe that is curved without having some "surrounds that it is curved in". In a completely different direction we read about superstructures of the DNA which locally looks like a helix but which is coiled up in intricate ways that may well be of major biological importance. Curvature and differential geometry are seen everywhere by the one who knows how to recognize them – but in recent decades the role differential geometry has slipped in many undergraduate curricula. There are good reasons for this as new subjects emerge and prove their value. We believe that modern computer technology not only will bring back differential geometry to its formerly much more prominent role inside university curricula, but it will make it accessible to a much larger part of the population than it ever was before.

This article shall provide some theoretical background and outlines of objectives and purpose for a presentation given at the 8$^{\text{th}}$ Asian Technology Conference in Mathematics. The heart of the presentation will be a live demonstration of MAPLE worksheets that illustrate that curvature can indeed become accessible for everyone through modern technology. While we have related JAVA-based projects in progress, this article shall focus on the computer algebra system implementations. The MAPLE worksheets referenced here and used in the presentation will remain available at the author's web-site [6] where they will be maintained and updated to match new releases of the CAS system. This WWW-site shall also be mirrored at the atcm's web-site.

This article and presentation shall survey selected topics related to curvature, starting with curves in the plane, then curves in 3-space, surfaces in 3-space and elementary Riemannian manifolds. By *"selected"* we mean that we will leave out many major subjects related to curvature such as minimal surfaces, compare e.g. [5]. Similarly, we will not go into any depth of the many classic curves that are superbly treated in JAVA in the Famous Curves Applet [11]. We try to give a reasonably clean survey without being overly rigorous – our goal is to be accessible to the mathematician whose closest contact with this area is occasional teaching of multi-variable calculus. In particular, we decided to stay mostly within the realm of curves and surfaces in Euclidean 3-space, taking the *Euclidean connection* for granted, and making only isolated remarks about generalizations to manifolds.

We do not claim to present any new theorems – instead we shall focus on how technology, here CAS, has completely transformed the playing field. In traditional studies the sheer monstrosity of most formulas made it prohibitive to demonstrate more than a few truly elementary examples. This certainly propelled forward the abstract theoretical work by a few, while at the same time it drove away hordes of students who wanted to see more nontrivial examples and applications. This lopsided history is exemplified by the dearth of nontrivial examples in traditional texts – which naturally made choices based on what would be reasonable for paper-and-pencil calculations (which is very rare in differential geometry). Now, with CAS at our finger tips, the universe is open – there are uncounted examples which have never been looked at simply because they are too hard and too messy for manual calculations. Yet many of these "examples" provide for very rich geometry, allowing many an undergraduate student to come up with new conjectures, even new theorems. In our eyes, the advent of CAS means that *playtime* is just beginning in differential geometry – and playing and experimentations are the starting point for all new discoveries.

2 Curves in the plane

Intuitively, curvature measures the rate at which a curve changes direction. More formally, the curvature is the (signed magnitude of) the rate of change of the normalized tangent vectors with respect to distance along the curve. We review some basic terminology and introduce notation that is used in the CAS worksheets.

We consider curves $\gamma\colon I \mapsto \mathbf{R}^2$ defined on an interval $I \subseteq \mathbf{R}$ with nonempty interior whose *speed* is nowhere zero, i.e., $\|\gamma'(t)\| > 0$ for all $t \in I$. Such a curve can always be *reparameterized by arc length*, that is, there exists a smooth bijection $\varphi\colon J \mapsto I$ for some interval $J \subseteq \mathbf{R}$ such that the curve $\sigma = \gamma \circ \varphi\colon J \mapsto \mathbf{R}^2$ has unit speed $\|\sigma'\| \equiv 1$. For theoretical purposes it is convenient, and usual to assume that any curve under consideration is parameterized by arc length. However, for practical calculations it is usually more convenient to work with an arbitrary parameterization as the change of coordinate $s = \varphi(t) = \int_{t_0}^{t} \|\gamma'(\tau)\|\, d\tau$ rarely can be written explicitly in terms of elementary functions. This is a first place for CAS experiments: Explore for how few curves the arc length integral can be expressed in terms of elementary functions: Obviously, straight lines and circles yield simple formulas. The catenary $\gamma\colon t \mapsto (t, \cosh t)$ and the cycloid $\gamma\colon t \mapsto (t + \cos t, \sin t)$ are well-known examples for nice arc-length integrals. The parabola $\gamma\colon t \mapsto (t, t^2)$ is barely doable, ellipses and the graph of the sine lead to Elliptic integrals, and almost every other curve that has not been carefully cooked up by a math-teacher of book-author leads to nonelementary arc length integrals, compare the MAPLE worksheets associated with this article. Fortunately, most analysis of curves never requires an explicit formula for $s = \varphi(t)$, instead one only needs the differential $\frac{ds}{dt} = \varphi'(t) = \|\gamma'(t)\|$ of the change of coordinate.

The normalized velocity $T \circ \varphi = \gamma'/\|\gamma'\|$ is a unit tangent vector field along the curve γ. If the curve is a straight line, then T is constant. In advanced language, one says that T is parallel along itself – referring to the Euclidean connection on $\mathbf{R}^2$. If $T' \neq 0$, write $T' = \kappa\, N$ where $N\colon J \mapsto S^1 \subseteq \mathbf{R}^2$ is a unit vector field along σ and the magnitude $\kappa\colon J \mapsto \mathbf{R}$ is called the *curvature*. In the plane one usually requires that $\{T, N\}$ has positive orientation and, in return, allows κ to take negative values.

Some simple, but fun CAS exercises contrast animations of the geometric vector fields T,

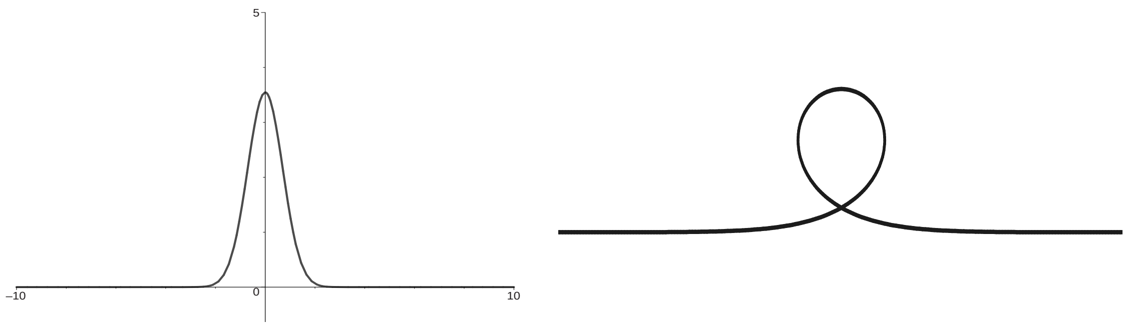

Figure 1: *The curvature determines the curve*

N and/or κN and animations of the physical velocity γ', acceleration γ'' and its parallel and perpendicular components $a_{\|} = \langle \gamma'', T \rangle T$ and $a_{\perp} = \langle \gamma'', N \rangle N$. Our beginning students found it helpful if the animation used color to indicate *accelerating* and *braking*, e.g. using a continuous change from green to red according to the size and sign of $a_{\|}$.

The next simple animations show the *osculating circles* moving along the curve. These are circles which are tangent to the curve and which have the same curvature as the curve at that point. It is a common misconception that these circles should not intersect the curve: Generically the osculating circles cross the curve at the point of contact!

The third CAS exercise is to plot the evolutes of the curve, that is the curve formed by the centers of the osculating circles.

The beauty in all these exercises is that the CAS takes care of manipulating quite messy formulas while the student can concentrate on the structure. The user enters formulas into the CAS worksheet that appear to be purely abstract, theoretical. Yet at the same time the CAS works specific examples – allowing the user to make a nice transition from the concrete example to the abstract analysis.

We conclude this section with some CAS-aided analysis that relies on the important observation: The curve σ is completely determined by its initial values $\sigma(0)$ and $\sigma'(0)$ and the curvature $\kappa\colon J \mapsto \mathbf{R}^+$, in the case that the latter is never zero. In modern language, the curvature forms a complete set of invariants under reparameterizations, rotations and translations. To break the ice, we encourage the student and the reader to sketch the graphs of curvature versus arc length for various familiar curves, not necessarily graphs of functions. These propositions are readily confirmed or rejected by simple plots by the CAS. Again, in the pre-CAS days, few instructors would have dared to plot these graphs for nontrivial curves!

The next step is the inverse question: Given the curvature as a function of the arc length, sketch the corresponding curve. It makes sense to separately consider curves of infinite length, and curves of finite length – especially those that are closed. The latter translates into a simple condition $\int_0^L \exp\left(i \cdot \int_0^u \kappa(s)\, ds\right)\, du = 0$. Finding the curve determined by the curvature takes two successive integrations. In many cases, this will require numerical methods – and as such provides an excellent backdrop for an explicit discussion when to use the symbolic and when to use the numeric capabilities of the CAS.

As a system of differential equations this inverse problem may be written as $\dot{x} = \cos\theta$, $\dot{y} = \sin\theta$, and $\dot{\theta} = u$, where the curvature $u = \kappa$ may be also be understood as an input, or control. A practical application is the *Dubins' car* [4], variations of which remain popular subjects of study in optimal control and robotics. In the most simple case consider a boat on a lake moving at constant speed. The control is the angle of the rudder, which directly translates into the curvature of the path of the boat.

One naturally assumes finite bounds on curvature, corresponding to maximal angles between the rudder and the central axis of the boat. A typical problem asks for the optimal control strategy that steers the boat from any given initial position and orientation to any given final position and orientation. The analysis of optimal paths for this and similar problems routinely requires determining the curve from the curvature, see [13] for some recent related work and many references to recent and current work. See also [2] for the role of curvature in modern mechanics and control.

A natural follow-on experiment is motivated by considering loops made of an elastic material that naturally *wants* to straighten out. The most simple model considers a family of curves $\sigma(s,t)$ parameterized by arc length and *time*, and imposes the diffusion equation $\frac{\partial}{\partial t}\kappa = c\frac{\partial^2}{\partial s^2}\kappa$ (for some diffusion constant $c > 0$). In the associated CAS worksheet we provide implementations of these models, including captivating animations of simple closed curves which tend to *round out* to become circles.

On the side note that this model is the most simple analogue of the Ricci flow that has been employed in the (at this time still: "likely") proof of the Poincare conjecture [8], one of the most famous open problems in mathematics. The Clay Institute [3] included it as one of the seven *Millennium Prize Problems*, each worth one million dollars.

3 Curves in 3-space

For curves $\sigma\colon J \mapsto \mathbf{R}^3$ define the unit tangent vector T and principal normal vector N as before with the additional requirement that $\kappa \geq 0$ (this makes a choice between N and $-N$.) Define the *binormal* $B = T \times N$ (cross-product) to obtain a right-handed orthogonal 3-frame $\{T, N, B\}$ associated to the curve. Again, this is only well defined at points where $\frac{d}{ds}T \neq 0$. From an advanced point of view this associates to any curve σ (with $\|T'\| > 0$) a curve in the Lie group SO(3). Differentiation of this curve yields the Frenet-Serret formulas

$$\frac{d}{ds}\left(\begin{array}{c|c|c} T & N & B \end{array}\right) = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \left(\begin{array}{c|c|c} T & N & B \end{array}\right) \tag{1}$$

which define the *torsion* τ as the second invariant of the original curve. Graphically speaking, the torsion determines the rate at which the *osculating plane* spanned by $\{T, N\}$ *twists* as it moves along the curve.

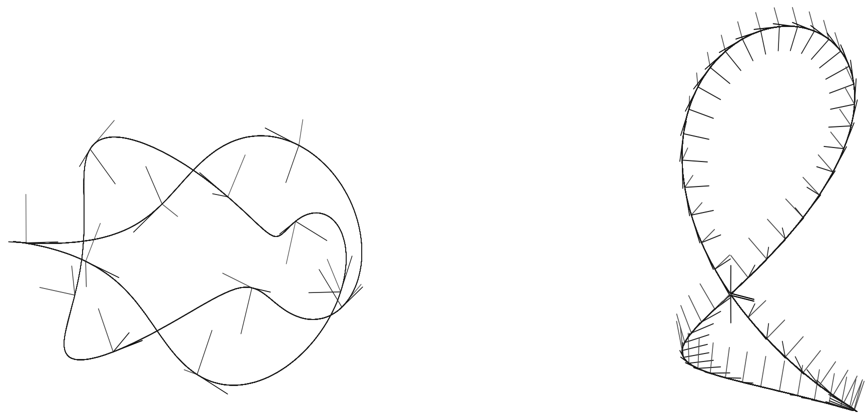

Figure 2: *Frenet frame animations: Torus knot and Viviani's curve*

In the pre-computer days all we had were mental images of the $\{T, N, B\}$ 3-frame moving along the curve – but rarely could we do any calculation that was much more than e.g. a helix $\gamma\colon t \mapsto (\cos t, \sin t, kt)$ With modern CAS we now actually can visually explore this moving frame along any curve – and how many surprises we find! It is one of the most rewarding small programming exercises using CAS. Writing a procedure that generates such an animation of the Frenet frame is a challenging project, but well within reach of most multi-variable calculus level students. It is only such fine points such as automatically choosing suitable lengths of the T, N, B vectors compared to the diameter of the curve which require some advanced programming. In our worksheet we provide both simple student-level syntax and more advanced syntax (suitable for demonstrations).

Having such nice procedures, one needs exciting curves! The traditional textbooks are poor resources for this as they are oriented towards formulas which result in reasonable paper-and-pencil calculations – not in interesting curves. Our worksheets provide a list of more exciting curves: Very simple are analogues of Lissajous figures in 3-space defined by formulas like $\gamma\colon t \mapsto (\sin kt, \sin mt, \sin nt)$ for integers k, m, n, where any one or all of the sines may replaced by cosines or combinations thereof. The next nice set of curves results from torus knots, that is, skew lines with rational slope on the torus, e.g. $\gamma\colon t \mapsto ((R + r\cos mt)\cos nt, (R + r\cos mt)\sin nt, r\sin mt)$ again for integers m, n. The first interesting example is the trefoil knot obtained from $(m, n) = (3, 2)$. Other nice curves are the result of intersections of simple surfaces. A nice example is Viviani's curve which is the intersection of a circular straight cylinder and a sphere of twice the radius whose center lies on (the wall of) the cylinder.

The moving normalized Frenet frame hides some key information. An informative alternative is to animate $\{T, \kappa N, \tau B\}$. Here, appropriate scaling becomes very important as there is no reason why the unit-length of T, the diameter of the curve σ, the curvature κ and the torsion τ should have even comparable magnitudes. Our worksheet illustrate some simple preprocessing that automatically selects reasonable scales. Since the torsion and/or curvature may be close to zero on some intervals, we prefer to still plot thin gray arrows indicating N and B on those intervals where the fat colored arrows κN and/or τB become too short to be seen.

It is very easy to modify this Frenet frame animation to obtain animations of the osculating planes and osculating circles along the curve. These are the plane spanned by T and N, and the circle with center $\sigma + \varrho N$ and whose radius ϱ is the *radius of curvature* $\varrho = \kappa^{-1}$ of σ.

Just like in the planar case, the inverse question makes for very attractive explorations: Given curvature and torsion functions κ, $\tau \colon J \mapsto \mathbf{R}^+$, calculate, and plot, the curves determined by these. In our classes we found that both instructor and students had very poor intuitive understanding what the resulting curves may look like, even for very simple curvature and torsion functions. We encourage the reader to try out, and improve, our sample procedures. Such exploration quickly lead to analytic questions such as under which conditions on κ and τ the resulting curve is closed, when it will be a knot, eventually leading to linking numbers and other advanced subjects.

Finally, we may consider the evolution of curves when their curvature and torsion are controlled by some partial differential equations – in analogy to the case of planar curves. So far we have played mostly with closed loops in space which may flatten and round out to become circles, or which undergo vibrating motions. This is an exciting area for small undergraduate research projects – in addition to the mere beauty, more than one student was intrigued by the, admittedly far-flung, association to string theory of particle physics.

4 Surfaces

There are many exciting aspects of curvature of surfaces in 3-space that deserve to be studied. Due to limited space and time, we here only consider Gaussian curvature and its interplay with geodesics. Already Euler considered notions of curvature for surfaces. But his *sectional* curvatures were basically curvatures of curves that were obtained by as intersections of a normal plane with the curve. A little later Meusnier considered also intersections with planes that were not necessarily normal. We provide some simple animations helping demonstrate these subjects, and suggesting how the sectional curvatures may be easily computed from the *principal curvatures* which correspond to the eigenvectors of some quadratic form.

It was Gauss who developed the first intrinsic notions of curvature for surfaces. A key function is the Gauss map which maps every point on the surface to a the unit normal vector to the surface at that point, i.e. a point on the unit sphere. The curvature at a point is simply the infinitesimal magnification factor of area for this map at that point, usually expressed as the ratio of the second and first fundamental forms. While such theoretical description seems quite simple, the formulas for surfaces described as graphs of function $x = f(x, y)$ or as parameterized surfaces $S \colon (u, v) \mapsto (x(u, v), y(u, v), z(u, v))$ are quite forbidding. In classes without technology, one may compute maybe one or two simple examples – but in general the formulas yield expressions that are not at all enlightening or useful.

Now enter CAS: The formula for Gaussian curvature only involves several first and second order partial derivatives, and quite a bit of algebra – but this is precisely what CAS were

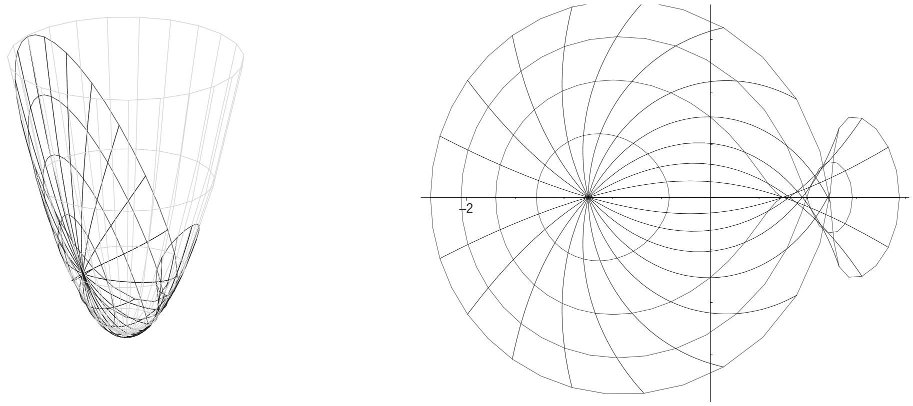

Figure 3: *Geodesic spheres and conjugate points: Surface and top-view*

designed for! The user may concentrate on the structure and geometry and may let the CAS handle the straightforward, but usually voluminous book-keeping! The first item on the agenda is to get a good feeling for the Gaussian curvature. A well-proven standard is to color-code the curvature: Write it as the difference of its positive and negative parts $\kappa = \kappa_+ - \kappa_-$ with $\kappa_+, \kappa_- \geq 0$ and $\kappa_+ \cdot \kappa_- = 0$. Color the background of the surface in deep blue (or yellow) and use κ_+ and κ_- to determine the red and green values (suitably scaled). In MAPLE write `plot3d( ...,color=COLOR(RGB,kappaplus,kappaminus,0.7));`. Having such a nice routine, it is surprisingly hard to try it out on exciting surfaces – old textbooks are a poor source as they focused on examples whose algebra was reasonable by hand, not on examples with intriguing geometry. Our worksheets will provide several intriguing examples, from the features of the monkey-saddle $z = \mathrm{Re}(x + iy)^3$ to the Gaussian curvature of various minimal surfaces.

Arguably one of the most important features of Gaussian curvature is its interplay with geodesics, that is, critical points (curves) of the *length* functional. Calculus of variations provides an analogue of the first derivative test of first year calculus for critical points. In the case of minimizing the length of curves that connect a given pair of points on the surface, this necessary condition takes the form of a system of second order coupled quadratic differential equations. While straightforward, it already takes significant effort to write out these differential equations in detail by hand. Again, generating these equations, as well as (usually numerically) solving them is a simple task for a CAS. We provide CAS syntax that asks the user to enter no more than some formula for surface and a desired starting point, and it will generate animations of the full geodesic spray – that is the collection of geodesics emanating from this starting point in all directions. Except for some trivial surfaces, without modern technology there was no hope of getting ones' hands on these amazing animations. Now they are available to everyone at a fingertip.

One of the first observations is that on some surfaces the geodesics never again cross each other, while on other surfaces they form envelopes and intersect at *conjugate points* (where they generally loose their minimality). Similarly, the geodesic spheres, that is, the loci of points of equal distance from the starting point, sometimes are simple circles, sometimes they are curves with self-intersections. Conjectures are abound in every class that tried this out.

The next step is to overlay the geodesic spray and geodesic spheres with the surface colored by curvature. The fundamental observation is inescapable: Regions of positive curvature *focus* geodesics, while in regions of negative curvature geodesics spread further apart, preserving their minimality.

On the side we note that CAS also provide dramatic animations of the Frenet frame moving along geodesics on surfaces: A curve is a geodesic only if its principal normal vector is at all times perpendicular to the surface. On the screen this property is easier to observe by watching the binormal vector along the curve: If the principal normal is perpendicular to the surface, and the unit tangent to the curve is tangent to the surface, then $B = T \times N$ clearly must again be tangent to the surface which makes it easy to distinguish geodesics from non-geodesics using the Frenet frame animation from the previous section.

5 Riemannian manifolds

The key step from classical differential geometry of curves and surfaces to modern differential geometry is to consider manifolds, that is generalized surfaces, that are not necessarily imbedded into any Euclidean space. The key observation is that one may still characterize the curvature of such objects in an entirely intrinsic way, that is without any reference to being curved in some surrounding space! It is a real challenge for most students and teachers alike to think of our universe as being curved, but curved by itself, without having to be curved in some surrounding space. Yet this thought defined a wonderful target for a long course of study.

Without going far, we provide a few simple examples of how CAS can much help to visualize the curvature on such manifolds. The key is to think of a manifold as locally being homeomorphic to a Euclidean space $\mathbf{R}^n$, for our purposes $\mathbf{R}^2$. In the setting of Riemannian manifolds, the notion of distance is defined by a smoothly varying quadratic form, represented in coordinates by a positive definite matrix G such that the length of any curve $\gamma\colon [y_0, t_1] \mapsto \mathbf{R}^2$ in such a surface patch is given by the $L(\gamma) = \int_{t_0}^{t_1} \langle \gamma'(t), G(\gamma(t))\, \gamma'(t) \rangle \, dt$.

CAS now allow us to visualize such Riemannian metrics directly, our favorite representations of such matrix valued fields being collections of images of squares or of circles under the linear transformation of the (tangent) plane defined by $\sqrt{(G)}$. It is a nice challenge to match such plots of the metrics to images of familiar surfaces such as cones, paraboloids, saddles, cylinders – each of which may be using different sets of common coordinate systems. The next step is to use these plots to understand how a length-minimizing curve will navigate this patch connecting any given p[air of points.

From the metric it is a simple step to a notion of curvature. Again we code the curvature by color – but unlike viewing colored surfaces imbedded into 3-space, we now only view colored patches of the plane. The first step is to again match such curvature plots to familiar surfaces (with usual coordinate systems). Once this is achieved, one may take the full step to modern geometry and forget about imbedded surfaces, and concentrate entirely on the patches in the plane. Our nicest images here are again images of the geodesics and of the geodesic spheres

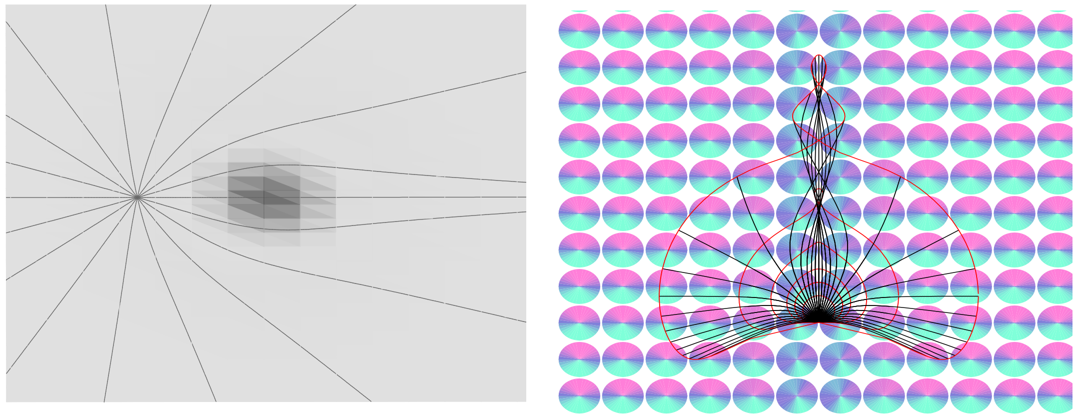

Figure 4: *Positive curvature focuses geodesics: Bump in the plane and in optimal control*

overlaid over color-coded plots of the curvature. Ideally we want to be able to drag the initial point of the geodesic flows using the mouse, and instantaneously see the corresponding change of the geodesic spheres and conjugate points – this is a typical task for JAVA programs. We are working on free-standing such programs. Another hope is that the new MAPLE release 9 with its new JAVA interface will allow us to directly implement this new kind of interactivity directly into the CAS.

What matters most in these visualization efforts of curvature without any ambient space is again the key insight that positive curvature focuses geodesics – exactly like the gravitational lensing of light rays by super-heavy stellar objects observed by astronomers. Conversely, negative curvature spreads out all geodesics.

As the final icing on the cake, our worksheets also contain a brief visual introduction to the curvature of optimal control [1]. The sample problem is similar to the previously discussed boat on a lake, trying to steer from one point to another point. However, the more exciting is when there is a steady nonzero wind blowing that may vary in direction and magnitude from point to point. The analysis of this simple question requires modern tools from optimal control theory that have been developed only in the last decade. One particularly beautiful notion is that of a curvature for optimal control. The key difference is that now the analogue of Gaussian curvature depends on both the point and the direction in which the point is traversed. Technically speaking it is a scalar function on the circle sub-bundle of the cotangent bundle. Yet with CAS one does not need to first understand such fancy language, or even master the humongous formulas that are even larger than those for the classical Gaussian curvature. Now one can explore first – and there is still room for many discoveries. The technical language comes later – after one's interest has been piqued, and one wants to take the next step – proving the conjecture to make it into a theorem. Our worksheets demonstrate a few simple examples that illustrate that in this setting the old theorem still holds true: If the curvature along a geodesic is always negative, then there will be no conjugate points and the geodesic is globally length minimizing.

References

[1] A. Agrachev and Y. Sachkov, *Lectures on Geometric Control Theory* Preprint SISSA 38/2001/M (2001).

[2] A. Bloch, J, Ballieul, P. Crouch, and J. Marsden,, *Nonholonomic Mechanics and Control*, Appl. Math. Sciences Series, (2003), Springer.

[3] Clay Institute, *Millennium Prize Problems* http://www.claymath.org/Millennium_Prize_Problems/.

[4] L. Dubins, *curves wof minimal lengths with a constraint on average curvature and with prescribed initial and terminal conditions and tangents*, Amer. J. Mathematics **79** (1957) pp.497–516.

[5] A. Gray *Modern Differential Geometry of Curves and Surfaces with Mathematica*, (1997) CRC press.

[6] M. Kawski, http://math.asu.edu/~kawski/diffgeom/actm03/

[7] J. Lodder *Curvature in the Calculus Curriculum*, American Math. Monthly (2003)

[8] J. Milnor, AMS Notices **50** (2003) pp. 1226–1233.

[9] L. Noakes *Duality and Riemannian cubics*, (preprint).

[10] Mathworld at Wolfram Research, http://mathworld.wolfram.com/Evolute.html

[11] B. Soares and J. O'Connor, St. Andrews, Famous Curves JAVA applet, http://www-gap.dcs.st-and.ac.uk/~history/Java/.

[12] St. Andrews, History pages, http://geometryalgorithms.com/history.htm.

[13] H. Sussmann, *The Markov-Dubins problem with angular acceleration control*, Proc. 36th IEEE Conf. Decision Control, (1997) 2639–2643.