

The Center of Gravity of Polygonal Regions and Conic Sections

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Abstract: In this paper, we will consider the center of gravity, or the center of mass, of several types of polygonal regions in the XY -plane. In order to calculate the center of gravity of such regions, instead of techniques from calculus, one can conveniently use geometric ideas. Our regions are of variable nature, so one can consider the locus of their center of gravity in the XY -plane. The locus of the center of gravity of a certain region that we will consider turns out to be a hyperbola with axes parallel to the coordinate axes. This observation can be reversed to come up with a new definition for a hyperbola, without involving eccentricity. We will use this new definition along with the dynamic geometry software *Geometer's Sketchpad* to present a novel construction of a hyperbola. The locus of the center of gravity of some other polygonal regions yield other types of interesting curves as well. In addition to *Geometer's Sketchpad*, we have also used the computer algebra system *Mathematica* to facilitate the center of gravity calculations. The paper also shows the importance of using different types of software hand-in-hand to experiment with mathematical problems.

1. Introduction

This paper is a continuation of our studies of center of gravity of plane regions. Previously, in [1] and [2], we considered the center of gravity of a variety of fixed and variable regions. Some of those regions included parabolic, exponential, and logarithmic regions. In order to calculate the center of gravity of these types of regions, one had to use methods of calculus (see [4]). In [1] and [2], we showed how this was done, with the aid of the computer algebra system (CAS) *Mathematica*. In addition to the regions described above, we also considered one example of a polygonal region, a trapezoid in [1] and [2]. One does not have to use methods of calculus to calculate the center of gravity of such regions: Any polygonal region can be divided into triangles, and hence geometric methods can be used to find their center of gravity. Because of the geometric nature of the situation, the dynamic geometry software, *Geometer's Sketchpad* can be used to carry out our investigations.

In [2], we discovered that the locus of the center of gravity of a certain variable trapezoidal region was a hyperbola, the axes of the curve being oblique. In this paper, we will discover a simpler variable triangular region whose locus of the center of gravity is a hyperbola, with axes parallel to the coordinate axes. This discovery can be used to formulate a new definition of a hyperbola without involving the concept of eccentricity.

We will also consider the loci of other types of polygonal regions with variable boundaries. It is of interest to note that ellipses as well as other interesting types of closed curves can be obtained as the locus of the center of gravity of variable polygonal regions.

To facilitate our discoveries and calculations, we have used the CAS *Mathematica* as well as the dynamic geometry software *Geometer's Sketchpad* (see [3] and [6]).

2. The Locus of the Center of Gravity of a Variable Triangular Region

Consider the two straight lines passing through the origin, given by the following equations:

$$y = ax \quad (2.1)$$

$$y = bx \quad (2.2)$$

In the above, a and b are fixed real constants with $a \neq b$. Let $P(0, c)$ be a point on the Y -axis where $c > 0$ is a fixed constant. Consider the variable straight line l with slope t , passing through the point P , intersecting the straight lines $y = ax$ and $y = bx$ at A and B respectively. We will assume that the real parameter t varies such that the triangle OAB is well-defined. As the parameter t changes, the center of gravity G of the triangular region OAB changes. We are interested in finding the locus of G in the XY -plane. See the following figure:

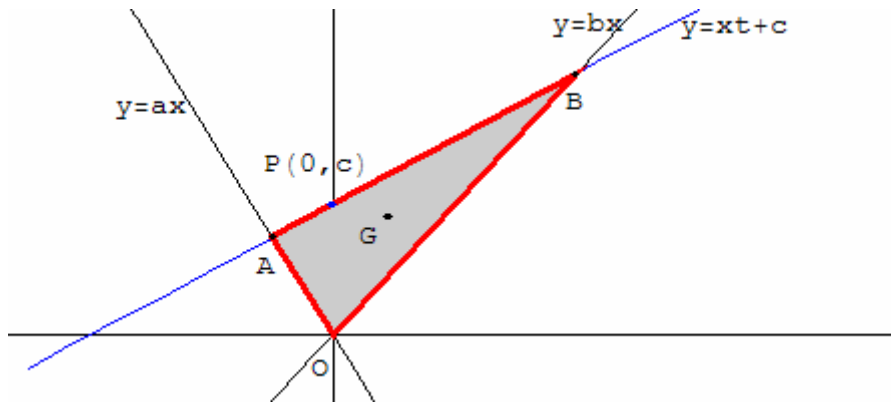


Figure 2.1 The Center of Gravity of a Variable Triangular Region

One can write the equation of the line l as

$$y = xt + c \quad (2.3)$$

One can solve the equations (2.1) and (2.3) to find the coordinates of the point A to be $(c/(a-t), ac/(a-t))$. Similarly, by solving the equations (2.2) and (2.3), the coordinates of the point B are given by $(c/(b-t), bc/(b-t))$. We now know the coordinates of the three vertices of the triangle OAB , where $O(0,0)$ is the origin. The center of gravity $G(\bar{x}, \bar{y})$ of the triangular region OAB can be obtained by averaging the x -coordinates of the vertices, and averaging the y -coordinates of the vertices, separately (see [5]). Thus, the following equations provide the coordinates of the center of gravity G of the region OAB :

$$\bar{x} = \frac{c(a+b-2t)}{3(a-t)(b-t)} \quad (2.4)$$

$$\bar{y} = \frac{c(2ab-at-bt)}{3(a-t)(b-t)} \quad (2.5)$$

To find the locus of G , one can eliminate the parameter t between the equations (2.4) and (2.5). This can be either done by hand, or by using the following *Mathematica* command (see [6]):

Eliminate{x, y} == {c(a + b - 2t)/(3(a - t)(b - t)), c(2a*b - a*t - b*t)/(3(a - t)(b - t))}, t]

Press “**Shift-Enter**” to execute the command. According to the output, the locus of G is given by the following equation:

$$3abx^2 + x(ac + bc - 3ay - 3by) = (2c - 3y)y \quad (2.6)$$

It is not hard to analyze the equation (2.6) using the theory of general equation of second degree in two variables (see [5]). However, we will confine ourselves to the special case where $a + b = 0$. The advantage of this special case is that when $a + b = 0$, the xy -term of the equation (2.6) vanishes. With this condition, the equation (2.6) now reads

$$3a^2x^2 - 3y^2 + 2cy = 0 \quad (2.7)$$

One can complete the square in equation (2.7) to obtain the locus of G as given by the following equation:

$$\frac{(y - c/3)^2}{(c/3)^2} - \frac{x^2}{(c/(3a))^2} = 1 \quad (2.8)$$

The above equation (2.8) represents a hyperbola centered at $(0, c/3)$ with axes parallel to the coordinate axes. Thus, we can now state the following theorem.

Theorem 2.1 Let $c > 0$ be a fixed real constant, and let l be the line through the point $P(0, c)$ with slope t , where t is a real parameter. Consider the triangular region OAB bounded by the line l and the lines given by the equation $y = \pm ax$ where $a \neq 0$ is a fixed real constant. We will assume that the parameter t changes so that the region OAB is always well-defined. Then, for changing t -values, the locus of the center of gravity G of the region OAB is a hyperbola given by the equation (2.8). This hyperbola has center at $C(0, c/3)$, vertices at $V_1(0, 0)$ and $V_2(0, 2c/3)$, axes parallel to the coordinate axes, and eccentricity $\sqrt{1 + a^2} / a$. Moreover, the asymptotes of the hyperbola are parallel to the lines given by $y = \pm ax$ (see the following figure).

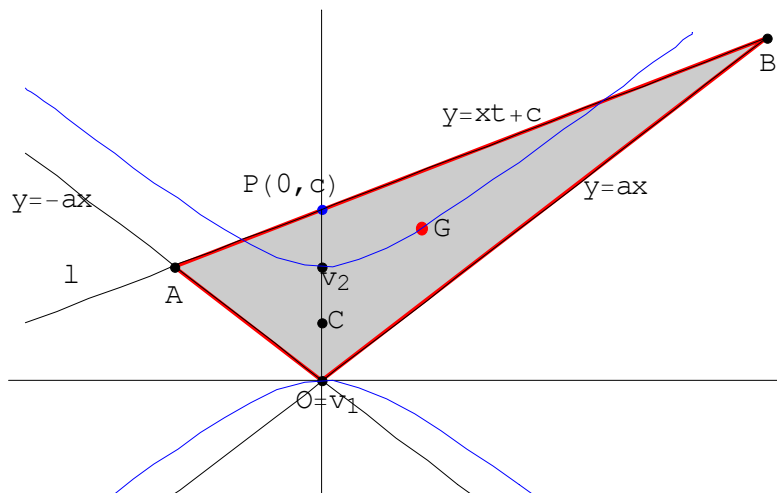


Figure 2.2 A Hyperbola as the Locus of the Center of Gravity of a Triangular Region

Proof. Most of the assertions follow from the discussion preceding the statement of the theorem. Recall that for any hyperbola of the type $(y - k)^2 / a^2 - (x - h)^2 / b^2 = 1$, the vertices are given by $(h, k \pm a)$, the eccentricity e is given by the equation $b^2 = a^2(e^2 - 1)$, and slopes of the asymptotes are given by $\pm a/b$. Using these facts, the rest of the theorem can be verified as well. ■

Mathematica can be used to create a dynamic visualization to illustrate our findings. The following *Mathematica* program creates an animation of the center of gravity G of the triangular region, as the upper boundary l of the region changes.

Program 2.1

```
<< Graphics`ImplicitPlot`
Clear[a, b, c, t]
a = 2; c = 3; b = -a;
{x1, y1} = {c/(a - t), a*c/(a - t)};
{x2, y2} = {c/(b - t), b*c/(b - t)};
{xbar, ybar} = {c(a + b - 2t)/(3(a - t)(b - t)), c(2a*b)/(3(a - t)(b - t))};
p1 = ImplicitPlot[(y - c/3)^2/(c/3)^2 - x^2/(c/(3a))^2 == 1, {x, -5, 5}, {y, -6, 6},
  PlotStyle -> {Thickness[1/200], RGBColor[0.4, 0.7, 0.99]}, DisplayFunction -> Identity]
p2 := Plot[{a*x, b*x, x*t + c}, {x, -5, 5}, PlotStyle -> RGBColor[1, 0, 0],
  DisplayFunction -> Identity, Prolog -> {{Thickness[1/80], RGBColor[1, 0, 0],
  Line[{{0, 0}, {x1, y1}, {x2, y2}, {0, 0}}]},
  {RGBColor[1, 1, 0], Polygon[{{0, 0}, {x1, y1}, {x2, y2}]}},
  {PointSize[1/50], Point[{xbar, ybar}]},
  {PointSize[1/60], RGBColor[0, 0, 1], Point[{x1, y1}]},
  {PointSize[1/60], RGBColor[0, 0, 1], Point[{x2, y2}]} } ]
Do[Show[{p2, p1}, PlotRange -> {-6, 6}, DisplayFunction -> $DisplayFunction],
  {t, -0.8, 0.8, 0.1}]
```

Press “**Shift-Enter**” to execute the program. A few frames of the animation are given below.

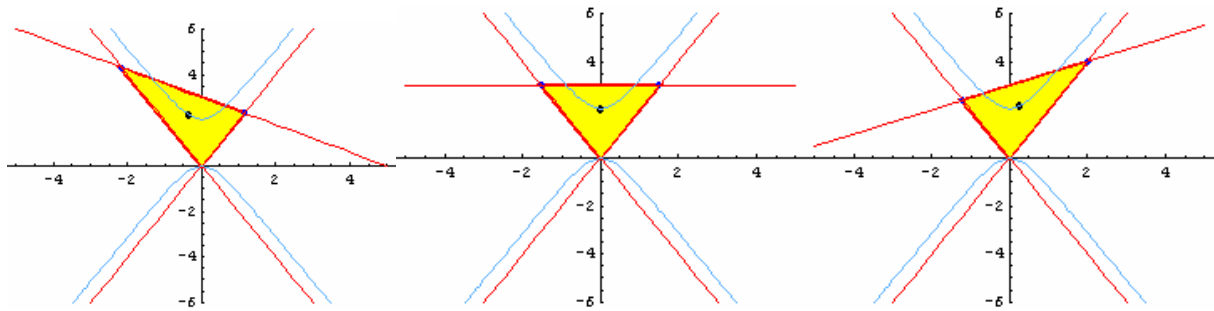


Figure 2.3 An Animation of the Center of Gravity of the Triangular Region OAB

Our theorem says a whole lot more indirectly! In fact Theorem 2.1 implies an important property of a hyperbola. Refer to Figure 2.2. Recall that the center of the hyperbola is given by $C(0, c/3)$ and the vertices are given by $V_1(0,0)$ and $V_2(0, 2c/3)$. The crucial observation is to note that how the position of point C relates to the positions of P and V_1 . The relationship is indeed $CP = 2CV_1$. Also observe that how the sides of the triangle OAB are related to the hyperbola. One side is the variable line with slope t passing through the point $P(0, c)$ on the x -axis. The other two sides of the triangle pass through the vertex V_1 of the hyperbola, and are *parallel to the two asymptotes*. The observation is that the center of gravity of a triangle formed this way, again belongs to the same hyperbola. The following theorem records this observation in precise terms:

Theorem 2.2 Consider any hyperbola with center U and vertices V_1 and V_2 . Let P be the point on the extended transverse axis V_1V_2 so that $UP = 2UV_1$ and V_2 is the vertex closest to the point P . Consider the triangle formed by a variable straight line through the point P , and two straight lines through vertex V_1 parallel to the two asymptotes. Then for changing positions of l , the locus of the center of gravity of this triangle belongs to the same hyperbola (see the following figure).

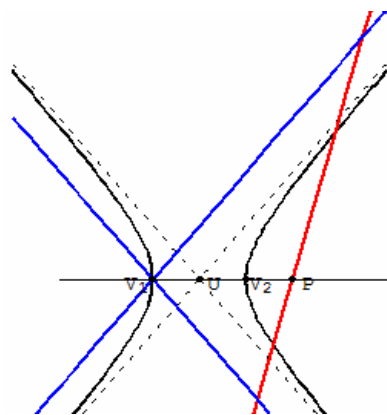


Figure 2.4 An Important Property of a Hyperbola

Proof. Without loss of generality, one can write the equation of the hyperbola as

$$x^2 / a^2 - y^2 / b^2 = 1 \quad (2.9)$$

Then we have the coordinates $U(0,0)$, $V_1(-a,0)$, $V_2(a,0)$ and $P(2a,0)$. The equation of the variable line through point P can be written as

$$y = xt + 2a \quad (2.10)$$

where t is a variable parameter. Since the slopes of the asymptotes of the hyperbola are equal to $\pm b/a$, we can write the equations of the other two sides of the triangle in the theorem as follows:

$$y = -b(x + a)/a \quad (2.11)$$

$$y = b(x + a)/a \quad (2.12)$$

One vertex of the triangle under consideration is clearly $V_1(-a,0)$. By solving the equations (2.10) and (2.11), the second vertex of the triangle can be obtained as the point $((2a^2t - b)/(at + b), -3abt/(at + b))$. Similarly, by solving the equations (2.10) and (2.12), the third vertex of the triangle can be found as the point $((2a^2t + ab)/(at - b), 3abt/(at - b))$. Then one can average the three x -coordinates, and the three y -coordinates, separately, to obtain the following coordinates of the center of gravity (\bar{x}, \bar{y}) of the triangle:

$$\bar{x} = a(a^2t^2 + b^2)/(a^2t^2 - b^2) \quad (2.13)$$

$$\bar{y} = 2ab^2t/(a^2t^2 - b^2) \quad (2.14)$$

Then it is easy to verify that the point (\bar{x}, \bar{y}) satisfies the equation (2.9). This means that the center of gravity of the triangle described in the theorem lies on the given hyperbola. This proves the theorem. ■

One can use the above Theorem 2.2 to formulate a new definition for a hyperbola without involving the concept of eccentricity. Suppose that you are given two intersecting lines l_1 and l_2 (as asymptotes), and a point V_1 not on these two lines (as one of the vertices). In accordance with the Figure 2.4, let U be the point of intersection of these two lines. Locate point P as described in Theorem 2.2, and consider variable lines l through P . Then, for changing positions of P , the center of gravity of the triangle defined in Theorem 2.2 traces a hyperbola. In other words, this provides a convenient way of constructing a hyperbola if the two asymptotes and one of the vertices are given. The following construction utilizes this idea to construct a hyperbola, using *Geometer's Sketchpad* (see [3]).

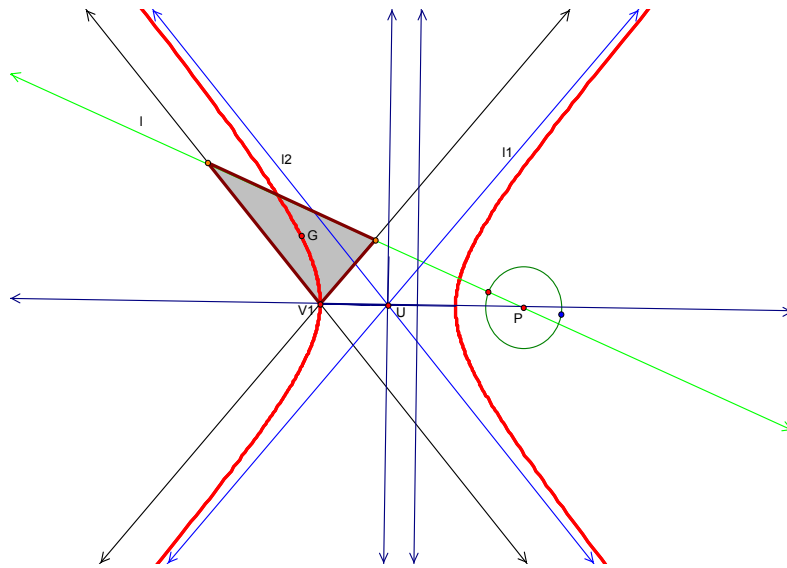


Figure 2.5 Constructing a Hyperbola if the Two Asymptotes and One Vertex is Given

3. The Ellipse as the Locus of the Center of Gravity of a Triangular Region

In the previous section, we obtained the conic section hyperbola as the locus of the center of gravity of a triangular region. Likewise, one can also obtain the ellipse as the locus of the center of gravity of a suitable region. There are several ways to do this, some obvious, but some not so. We will first state a theorem:

Theorem 3.1 Consider the two concentric circles C_1 and C_2 given by the equations $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ respectively, with $0 < b < a$. Let P be an arbitrary point on the larger circle C_1 , and let Q be the mirror image of P on the x -axis. Suppose that the line segment OQ meets the smaller circle at the point R between O and Q , where O is the origin. Then for changing points P , the locus of the center of gravity of the triangular region OPR is given by the ellipse $9x^2/(a+b)^2 + 9y^2/(a-b)^2 = 1$ (see Figure 3.1).

Proof. One can write the point P as $P(a\cos\theta, a\sin\theta)$ for some $0^\circ \leq \theta < 360^\circ$. Then we can write the coordinates of the points Q and R as $Q(a\cos\theta, -a\sin\theta)$ and $R(b\cos\theta, -b\sin\theta)$. We now have the coordinates of the three vertices of the triangle OQR . By averaging the coordinates, one can find that the center of gravity $G(\bar{x}, \bar{y})$ of the triangular region OQR is given by the equations $\bar{x} = (a+b)\cos\theta/3$ and $\bar{y} = (a-b)\sin\theta/3$. By eliminating the variable θ between these last two equations, one can obtain the locus of G as the ellipse $9x^2/(a+b)^2 + 9y^2/(a-b)^2 = 1$. Hence the theorem. ■

The following construction using *Geometer's Sketchpad* illustrates the above theorem:

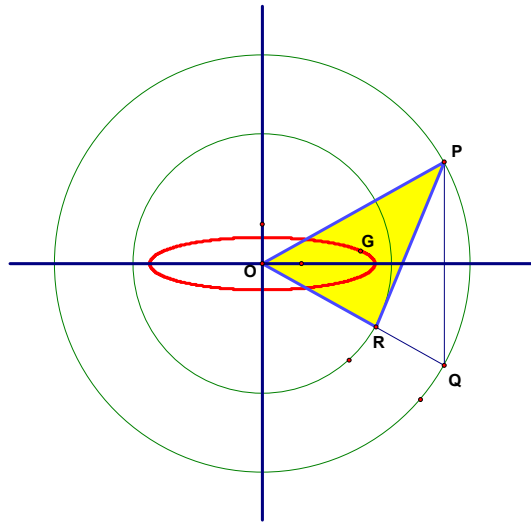


Figure 3.1 Construction of an Ellipse as the Locus of the Center of Gravity of a Triangular Region

One can use the idea learned from the above theorem to construct an ellipse, if the lengths of the major and minor axis are given. Here are the details of the construction:

Construction 3.1 Suppose that the semi-lengths p and q of the major and minor axes of the ellipse are given, along with their direction (position), respectively. Construct the line segments of lengths $a = 3(p + q)/2$ and $b = 3(p - q)/2$. Draw two concentric circles with radii a and b , with center as the center O of the ellipse (see Figure 3.1). Pick an arbitrary point P on the larger circle, and follow the steps described in Theorem 3.1 to obtain the triangle OPR . Find the centroid (center of gravity) G of this triangle in the usual manner, by constructing the medians. Then for changing positions of P , the point G will lie on an ellipse with the semi-lengths of the major axis and minor axis equal to p and q respectively. This is so, because of Theorem 3.1, the semi-length of the major axis is equal to $(a + b)/3 = (p + q)/2 + (p - q)/2 = p$, and the semi-length of the minor axis is equal to $(a - b)/3 = (p + q)/2 - (p - q)/2 = q$.

4. The Locus of the Center of Gravity of a Variable Quadrilateral

We can also consider the locus of the center of gravity of a variable quadrilateral using similar geometric methods. For example, consider the quadrilateral $OAPB$ with four vertices at $O(0,0)$, $A(p,q)$, $B(0,b)$ and P , where P is a variable point on a circle with center at $(x_0, 0)$ and radius r . We will assume that p, q, b, x_0 , and r are fixed quantities, and that the four vertices will define a quadrilateral $OAPB$. See the following figure:

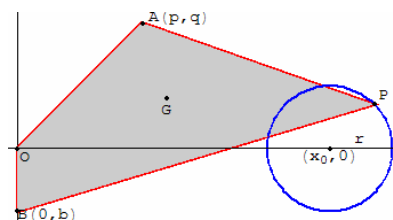


Figure 4.1 The Center of Gravity of a Variable Quadrilateral

We can find the center of gravity of the region $OAPB$ using geometric methods: Write the point P as $P(x_0 + r \cos \theta, r \sin \theta)$ for some $0^\circ \leq \theta < 360^\circ$. First, divide the region using the line segment AB . Using the averaging process, the center of gravity of the triangles OAB and ABP are given by $G_1(p/3, (b+q)/3)$, and $G_2((p+x_0+r\cos\theta)/3, (q+b+r\sin\theta)/3)$ respectively. It then follows that the equation of the line G_1G_2 is given by

$$3xr \sin \theta - 3y(x_0 + r \cos \theta) + (b+q)(x_0 + r \cos \theta) - pr \sin \theta = 0 \quad (4.1)$$

Secondly, divide the region using the line segment OP . It is easy to verify that the center of gravity of the triangles OAP and OBP are given by $G_3((p+x_0+r\cos\theta)/3, (q+r\sin\theta)/3)$ and $G_4((x_0+r\cos\theta)/3, (b+r\sin\theta)/3)$ respectively. Therefore, the equation of the line G_3G_4 is given by

$$3x(q-b) - 3yp + p(b+r\sin\theta) + (b-q)(x_0+r\cos\theta) = 0 \quad (4.2)$$

Now, the center of gravity G of the region $OAPB$ must lie on both lines G_1G_2 and G_3G_4 , so the coordinates of $G(\bar{x}, \bar{y})$ can be obtained by solving the equations (4.1) and (4.2). Either by hand calculations, or by using the *Mathematica* “**Solve**” command, one can obtain the following (see [6]):

$$\bar{x} = [-x_0(pq - bx_0 + qx_0) + (b-q)r^2 \cos^2 \theta + pr(p+x_0) \sin \theta + r \cos \theta (-pq + 2bx_0 - 2qx_0 + pr \sin \theta)] / D \quad (4.3)$$

$$\bar{y} = [(b^2 - q^2)x_0 + r(pq + bx_0 - qx_0) \sin \theta + pr^2 \sin^2 \theta + (b-q)r \cos \theta (b+q+r \sin \theta)] / D \quad (4.4)$$

where

$$D = 3[(b-q)x_0 + (b-q)r \cos \theta + pr \sin \theta] \quad (4.5)$$

In order to find the locus of G , one has to eliminate the variable θ between the equations (4.3) and (4.4). This is a non-trivial problem even with the help of a CAS such as *Mathematica*! However, an indirect approach can be used. One can make the substitutions $\sin \theta = 2t/(1+t^2)$ and $\cos \theta = (1-t^2)/(1+t^2)$ for some real t . With these substitutions, the equations (4.3) and (4.4) become rational functions of t . Then, the “**Eliminate**” command of *Mathematica* can be used to eliminate t to obtain the locus of G (see [6]). Though not included here because of the space limitations, the result, in general, is a fourth-degree polynomial equation in x and y .

As a specific example, for $p = 1/2$, $q = 1$, $b = -1$, $x_0 = 5$, and $r = 1$, the locus of G is given by the following:

$$5184x^4 - 2592x^3y + 5508x^2y^2 - 2592xy^3 + 324y^4 - 19872x^3 + 9720x^2y - 2052xy^2 + 216y^3 + 21492x^2 - 9576xy + 1089y^2 - 5604x + 1338y + 425 = 0 \quad (4.6)$$

The “**ImplicitPlot**” command of *Mathematica* can be used to graph the above equation (4.6):

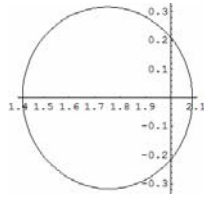


Figure 4.2 The Locus of G for $p = 1/2$, $q = 1$, $b = -1$, $x_0 = 5$, and $r = 1$

According to the above diagram, the locus of G is *similar* in shape to an ellipse or a circle, at least for the values $p = 1/2$, $q = 1$, $b = -1$, $x_0 = 5$, and $r = 1$. However, we do not yet have a proof regarding the exact shape of the above graph. We have also used *Geometer’s Sketchpad* to construct the locus of the center of gravity of the region $OAPB$. Unlike *Mathematica*, *Geometer’s Sketchpad* constructions are very interactive in the sense that one can instantly see the updated locus as values for p , q , b , x_0 , and r are changed by dragging the point A , B , etc. (see [3]). Depending on these values, the locus of G can take a number of interesting shapes, but because of the space limitations, we are unable to include further details. We conclude by saying that when considering geometric problems such as described in the paper, it is important to use more than one type of software to get different insights.

References

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