A Study of Dynamic Mathematical Exploring in GSP

Yi-Long Yen

The Master Program of e-Learning
College of Science
National Chiao-Tung University
Hsinchu 30050, Taiwan
mjchen@mail.nctu.edu.tw

Abstract

An advanced version of dynamic “Proofs without Words (PWWs 圖說證明）” in GSP is proposed in this paper. In addition to dynamic presentations, the students can observe the dynamic proofs under various situations simply by modifying those parameters of the graphs under consideration. The students can choose to show step by step either automatically or manually by themselves; moreover, the mathematical proof statements and related figures can be shown simultaneously on the screen on demanding. Some elegant visual demonstrations are provided for Pythagorean Theorem, *AG* inequality, and Cauchy inequality. To realize the geometric as well as the physical roles played by the parameters of functions. Taking advantage of its ability of drawing and computing, dynamic environments are built in GSP so that the students can observe the interrelationship between the parameters and graphs of functions simply by modifying the parameters, or contrarily by dragging the graphs. Special attentions are paid over lines and slopes, and over conic sections and eccentricities. All dynamic environments in a CD-ROM are also available on request.

Keywords: dynamic proofs without words, parameters of functions, graphs of functions, Pythagorean Theorem, *AG* inequality, Cauchy inequality

1. Introduction

To stimulate and to encourage students’ interests in certain mathematical ideas with the help of their own visual intuition is a crucial task. The column of “Proofs without Words（PWWs 圖說證明）” has served this purpose well over the past thirty years. The idea of dynamic visualizing has been added by Huang *et.al*. over the classical PWWs within the environment of PowerPoint . However, it’s reasonable to question further that whether the same conclusion remains true if the positions of some points are moved, or the lengths of some segments are changed?

An advanced version of dynamic PWWs in GSP is proposed in this paper. In addition to dynamic presentations, the students can observe the dynamic proofs under various situations simply by modifying those parameters of the graphs under consideration. Based on their own demands, the students can choose to show step by step either automatically or manually by themselves. The students can also observe the relationship between the figures and their corresponding data on screen; moreover, the mathematical proof statements and related figures can be shown simultaneously on the screen too on demanding. Some elegant visual demonstrations are provided.

To realize the geometric as well as the physical roles played by the parameters of functions is another crucial task. Taking advantage of its ability of drawing and computing, dynamic environments are built in GSP so that the students can observe the interrelationship between the parameters and graphs of functions simply by modifying the parameters, or contrarily by dragging the graphs. Special attentions are paid over lines and slopes, as well as over conic sections and eccentricities.

All works mentioned above are available on web, and can also be used in classes by the instructors interactively; all dynamic environments in a CD-ROM are also available on request.

2. Dynamic Proof without Words in GSP

It is a crucial task to stimulate and to encourage students' interests in certain mathematical ideas by providing a visible environment, so that the students come up the mathematical ideas with the help of their own visual intuition. The columns of "Proofs without Words (PWWs) " [1, 2] in "Mathematics Magazine" published by MAA has served this purpose well over the past thirty years. Parts of those works have been compiled by R. B. Nelson in [1, 2], moreover, a database [3] of this column is now available on web. PWWs are not really proofs, and they indeed are pictures or diagrams that help the observer see *why* a particular statement may be true, and also to see *how* one might begin to go about proving it true. In some an equation or two may appear in order to guide the observer in this process. The emphasis is clearly on providing visual clues to the observers to stimulate mathematical thought.

The idea of dynamic visualizing has been added to PWWs within PowerPoint by Huang et.al. [3]. Each figure in either paper based or Power Point based PWWs usually provides much information appropriately. However, it's reasonable to question further that whether the same conclusion remains true if the positions of some points are moved, or the lengths of some segments are changed? For example, is the Pythagorean Theorem still true if the sides of a right triangle are changed? Toward this goal, an advanced version of dynamic PWWs in GSP is proposed in this paper. In addition to the dynamic presentation, the students can observe the dynamic proofs under various situations simply by modifying those parameters (e.g., the lengths or the angels) of the graphs under consideration. Based on their own demands, the students can choose to show step by step either automatically or manually by themselves. All related data can be shown on screen if necessary, so the students can observe the relationship between the figures and their corresponding data; moreover, the mathematical proof statements and related figures can be shown simultaneously on the screen too on demanding. Some elegant visual demonstrations are provided.

2-1 Options for operations

Two types of dynamic presentations step by step either automatically or manually on users' demand are considered in this section. The buttons of *Graph*, *Automatic*, *mStepwise*, *Data*, *Parameter*, *Reset*, *Make equal* are provided, followed by a few dynamic examples of Pythagorean Theorem, sum formulas of trigonometric functions, Law of Cosines, arithmetic and geometric inequality and Cauchy inequality for illustration.

< Graph >: present automatically on turning on, to understand the whole framework; click <graph> for repeating presentation;

< Automatic >: show the presentations step by step automatically; explanations and corresponding data shown together;

< mStepwise >: show the presentations step by step manually; the procedure and the speed of presentation can be modified;

< Data >: to see the graphs and the corresponding data simultaneously by taking advantage of computing capacity of computers;

< Parameter >: to modify those parameters by clicking and dragging <change a, b>, <radius>, <resize>;

< Reset >: to return to its original graph after some modifications of parameters made;

< Make eq*ual*>: to show the situation when the equal sign holds in the inequality.

2-2 The Pythagorean Theorem

There are over 367 different proofs on the Pythagorean Theorem in the literature including the one drawn by Pythagoras himself on the beach around 600 B.C. as shown below, the Pythagorean Theorem $c^2 = a^2 + b^2$ can be told immediately from the diagrams in Figures 1 and 2 respectively.

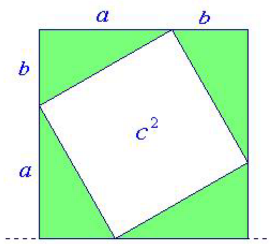

Figure 1: PWW1

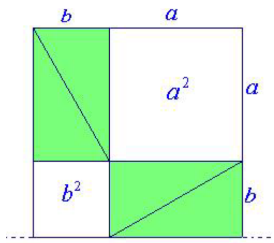

Figure 2: PWW2

In a right triangle with $\angle C = 90^\circ$, three squares are attached outward along the sides of the triangle with areas a^2, b^2, c^2 respectively, see Figure 3, 4. The relationships among a^2, b^2, c^2 will be explored dynamically. Let $\overline{AB} = c$, $\overline{BC} = a$ and $\overline{CA} = b$ in a right triangle, recall that $\angle C = 90^\circ$, if and only if $c^2 = a^2 + b^2$.

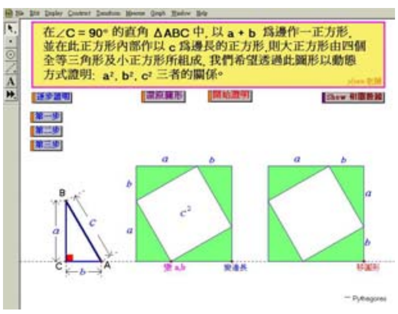

Figure 3: Pythagorean Theorem 1

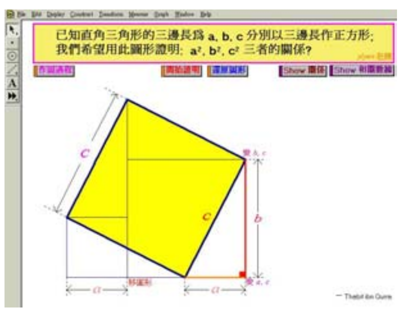

Figure 4: Pythagorean Theorem 2

Instructions:

To modify the values of a, b, c by pressing/ dragging <*change* b, c> or <*change* a, c>; click <*Show*> to see the outcome of this experiment, click <Hide> to erase unnecessary information; and the buttons <*Show*>, <*Hide*> change alternatively; though it is correct, the dynamic proof will not be seen if $a > b$.

Study Guide:

click <*mstepwise*> first and then <*step* 1> to translate a square, click again <*step* 2>; is the line joining the lower right corner of the square after translation and the center of the upper square with length b parallel to a side of the square with length c? Justify your observations.

Recall that the area of a parallelogram is equal to the product of the bottom and the height; the areas of two parallelograms with equal length of bottoms and heights are equal.

Instructions:

to modify the values of *a, b, c* by pressing / dragging <*resize*>; if the length of a square is too short, it is not easy to observe dynamically.

Study Guides:

click <*mstepwise*> and then click < *step* 1>, you will see that the areas of the original square and the parallelogram obtained are identical. Justify your observation; click <*step* 2>, you will see that the areas of the original square and the parallelogram obtained are identical. Justify your observation.

2-3 The Sum Formulas of Sines

There are a few sum formulas involving trigonometric functions, a dynamic presentation for the sum formula of Sines will be included in this section for illustration purpose. Recall that $\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$, and the area of $\Delta ABC = \frac{1}{2}bc\cdot\sin A$;

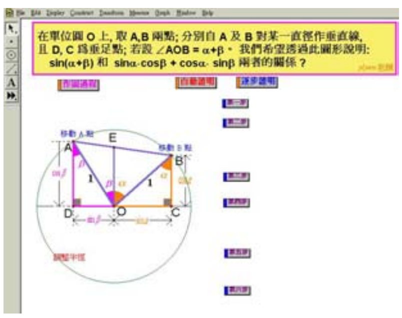

Figure 5: The Sum Formula of Sines

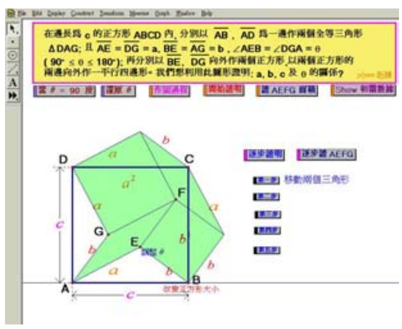

Figure 6: Law of Cosines

Instructions:

to modify α, β by pressing/dragging <*drag A* >, <*drag B* >; to modify the radius by pressing/dragging <*radius* >; this experiment works only if $0 < \alpha,\ \beta < \pi/2$; if click <*drag A*> and drag over $\overline{OH}$, the experiment still works though the graph will not be seen.

2-4 Law of Cosines

A dynamic presentation of Laws of Cosines will be presented in the case of obtuse angels, the conclusion of law of cosines can be observed through simple calculations. If $ABCD$ is a square with length c , $\Delta ABE, \Delta DAG$ are identical triangles, $\overline{AE} = \overline{DG} = a$, $\overline{BE} = \overline{AG} = b$, $\angle AEB = \angle DGA = \theta$ $(0^\circ < \theta < 180^\circ)$; then add two squares outward along $\overline{BE}, \overline{DG}$, followed by a parallelogram with two sides of each of these two squares. The relationships among a, b, c, θ will be explored dynamically as shown in Figure 6.

Instruction:

to modify $\angle C$ by pressing/ dragging <*change* θ>; to see the situation when $\angle C = 90^\circ$ by pressing/ dragging <*make* $\theta = 90^\circ$>; this experiment works dynamically only if ΔABC is an acute triangle.

Recall that if $\angle C = 90^\circ$ in ΔABC , then $\overline{AC} = \overline{AB} \cdot \cos\theta$, and $\overline{BC} = \overline{AB} \cdot \sin\theta$; the area of a parallelogram with two incident edges a, b with angle θ is $a \cdot b \cdot \sin\theta$; $\sin(\theta - 90^\circ) = -\cos\theta$.

Instruction:

to modify the angle $\angle AEB$ by pressing/ dragging <*change* θ>; to see the situation of $\angle AEB = 90^\circ$ by clicking <*make* $\theta = 90^\circ$>; to modify the squares by pressing/ dragging <*resizes square*>; this experiment works only if $\angle AEB \geq 90^\circ$.

2-5 Arithmetic and Geometric Inequality

Let O be a half circle with $\overline{AB}$ as its diameter and P a point on $\overline{AB}$; a line through P and perpendicular to $\overline{AB}$, and meet the half circle at a point M , then $\overline{PM} = \sqrt{ab}$, $\overline{ON} = (a+b)/2$. Note that the corresponding lengths of two similar triangles are proportional each other. A dynamical exploration of comparing these two values is shown in Figure 7. It is interesting to see when the equal sign holds in an equality by clicking <*make equal*>, the figures move gradually till the case that the equal sign holds. Similar situation holds for the Cauchy inequality too as shown in Figure 8.

Instructions:

to modify the radius of the circle by clicking <*radius*>; to modify the magnitudes of a, b by pressing/ dragging <*change* a, b>; because it is not easy to observe $\overline{MP} = \sqrt{ab}$; click <*Show MP*> to see the dynamic proof stepwise either automatically or manually; when a, b are small, the experiment still works though it is not easy to observe dynamically.

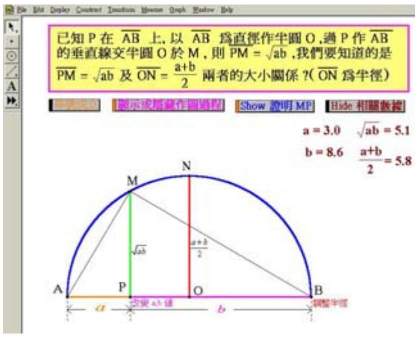

Figure 7: AG inequality

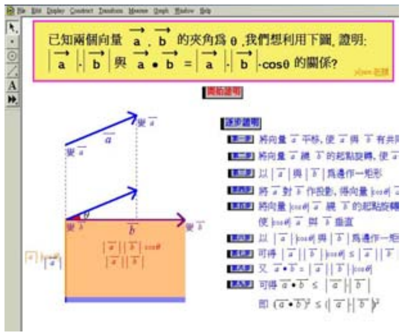

Figure 8: Cauchy inequality

2-6 Cauchy inequality

Let $\vec{a}$, $\vec{b}$ be two vectors with an angle θ, then the projection of $\vec{a}$ in $\vec{b}$ is $(|\vec{a}|\cos\theta)\vec{b}/|\vec{b}|$. The relationship between $|\vec{a}||\vec{b}|$ and $|\vec{a}\cdot\vec{b}|$ will be explored in a dynamical way. The lengths and the angle between these two vectors can be modified on demand, it leads to two rectangles with areas $|\vec{a}||\vec{b}|$, $|\vec{a}\cdot\vec{b}|$ respectively; and then compare their values as shown in Figure 8.

Instructions:

to modify the lengths and the angle among them by dragging <*drag* $\vec{a}$> or <*drag* $\vec{b}$>; click <*Ctrl*>, < B > simultaneously to delete the traces on completion of proof; it is not easy to observe the dynamic proof while the angel between them are small, though the experiment still works.

Study Guides:

Is the relation still true if the angle between them is acute by clicking <*drag* $\vec{a}$>, <*drag* $\vec{b}$>?

3. Interaction between parameters and graphs of functions

Let L be a line and P a point outside of it, the traces of the set of all points satisfying $\overline{PF} = e\cdot d(P,L)$ are parabola, ellipse or hyperbola depending on the values $e=1$, $0<e<1$ or $e>1$ respectively. The relationship between slopes and lines, and between eccentricities and conic sections will be treated separately for illustration purpose. In particular, we can see the two way interactions either from the parameters to the graphs or from the graphs to the parameters.

3-1 Slopes and Lines

An environment is provided for exploring the variations of the line $y = mx + b$ by modifying the slope *m* and the *y*-intercept *b* respectively as shown in Figure 9. Contrarily, the variations of the parameters *m* and *b* in the linear equation $y = mx + b$ can be observed by dragging the line horizontally, vertically or even by rotating with an angle as shown in Figure 10.

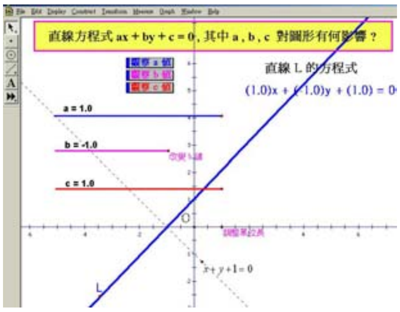

Figure 9: parameter2graph

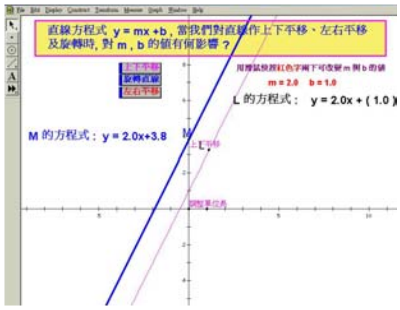

Figure 10: graph2parameter

Instructions:

1. to see the graphs if the parameters are modified:
   to modify the equation $y = mx + b$ by clicking <*observe* m>, <*observe* b>, and pressing/ dragging <*change* m>, <*change* b>, and observe the variations of the graph simultaneously; observe carefully the equation of the line L and the equation $y = x + 1$; to study the graph carefully by pressing/dragging <*change unit*> .
2. to see the parameters if the graphs are dragged:
   to modify the line vertically, horizontally and rotating with an angle by pressing/ dragging <*v-translate*>, <*rotate line*> respectively, both the equation of the line and its graph can be seen simultaneously; to modify the equation of the line L simply click twice red m or b to modify their values. Other data will be modified simultaneously; study the relationship of the equations L and M ; press/ drag <*change unit*> to observe the graphs closely.

All lines in this experiment are expressed in the form of $y = mx + b$, no line perpendicular to x-axis can be seen. If $m = 0$, the line L is parallel to the x-axis, and hence <*h-translate*> cannot be seen.

Study Guides:

1. to see the graphs if the parameters are modified:
   watch the graph carefully by clicking <*observe* m>; similarly for <*observe* b> ; pressing/ dragging <*change* b>, should we increase or decrease the value of m in order to rotate the line L clockwise; click <*change* b>, should we increase or decrease the value of b in order to move the line L upward along the y -axis?
2. to see the parameters if the graphs are dragged:
   modify the parameters m, b in the equation by clicking <*v-translate*> and pressing/ dragging <*v-translate*> ; are there any patterns? similarly for <*rotate line*>; modify the equation of the line L to $y = 2.0x + 1$; by clicking <*v-translate*> , how many units towards up or down should we drag in order to get the equation $y = 2.0x + 3$? modify the equation of the line L to $y = 2.0x + 1$ by clicking <*rotate line*> , how many units towards up or down should we drag in order to get the equation $y = 4.0x + 1$? should we move clockwise or counterclockwise to get the equation $y = 4.0x + 1$?

3-2 Eccentricities and Conic Sections

The variations of the three types of conic sections can be observed by modifying the values of eccentricity as shown in Figure 11. The conjugate point is the intersection of the asymptote with line through the vertex and is perpendicular to the transverse axis. The observation of the equations of hyperbola while dragging their vertices, center and conjugate points are shown in Figure 12. The equation of a hyperbola is $(y-k)^2/a^2-(x-h)^2/b^2=1$ when its transverse axis is parallel to the y-axis, where $2a$ is the distance between two vertices along its transverse axis; $2b$ is the distance between two vertices along the conjugate axis; and $2c$ is the distance between the two foci; (h,k) is the coordinate of vertex; moreover $b^2=c^2-a^2$

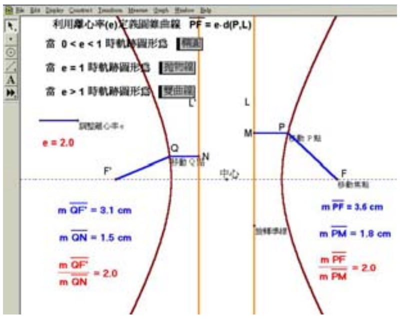

Figure 11: eccentricity

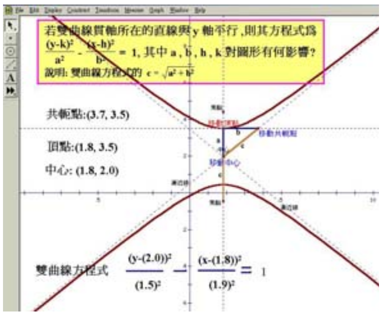

Figure 12: hyperbola

Instructions:

1. to see the graphs if the parameters are modified
   click on the button <*parabola*>, <*ellipse*>, <*hyperbola*>, we can find the graphs of parabola, ellipse, hyperbola respectively, the eccentricity *e* will be shown too; to seethe change of the eccentricity as well as its graphs by clicking and dragging the button <*change e*>; observe the variations of $\overline{PF}$, $\overline{PM}$, $\overline{QF'}$, $\overline{QN}$, $\overline{PF}/\overline{PM}$ and $\overline{QF'}/\overline{QN}$ respectively carefully by pressing/ dragging <*drag* P> or <*drag* Q>.

2. to see the parameters if the graphs are dragged:
   to modify the positions of center, vertex, congugate by pressing/ dragging <drag center>, <drag vertex>, <drag congugate> respectively; the equations and graphs of the hyperbola can be seen simultaneously; click <drag Center> and move gradually vertically or horizontally to study their effects over the parameters; to study carefully the relationship between the coordinate and its equation; to observe the figure in detail by clicking <*drag unit*>.

Study Guides

1. to see the graphs while modifying the parameters:
   study the value $\overline{PF}/\overline{PM}$ by clicking <*parabola*> and pressing/ dragging <*drag P*>; study the value $\overline{PF}/\overline{PM}$ by clicking <ellipse> and then pressing/ dragging <*drag Q*>; study the values $\overline{PF}/\overline{PM}$ and $\overline{QF'}/\overline{QN}$ respectively by clicking <hyperbola>, and then pressing/ dragging <*drag P*> or <*drag Q*> respectively.

2. to see the parameters while dragging the graphs:
   for the hyperbola $(y-k)^2/a^2-(x-h)^2/b^2=1$, study the parameters a,b,h,k carefully by pressing/ dragging <*drag vertex*>; are there any patterns? study the parameters a,b,h,k carefully by pressing/ dragging <*drag center*>; are there any patterns? where are the center, the vertex, the conjugate point in order to get an equation $(y+2)^2/3^2-(x-1)^2/4^2=1$ for a hyperbola? Find the coordinate of another vertex and the foci?

If the eccentricity e approaches to 0, the experiment works though no data will be shown; On the other hand, if e is large enough, the experiment still works though it is not easy to adjust. The operation <*drag vertex*> and <*drag conjugate*> works only for vertical and horizontal movements.

Acknowledgements: The author would like to thank Professor Ming-Jang Chen for many constructive discussions during the preparations of this work.

References

[1] Roger B. Nelsen, *Proofs without Words- Exercises in Visual Thinking*, Classroom Resource Materials Number 1, The Mathematical Association of America (MAA) 1993.
[2] Roger B. Nelsen, *Proofs without Words II- More Exercises in Visual Thinking*, Classroom Resource Materials, The Mathematical Association of America (MAA) 2000.
[3] Kuo-Chung Huang, Yuan-Feng Peng and Wen-Lung Hsu, Some examples of dynamic proof without words in PowerPoint, ACTM 2003 Hsinchu Taiwan.
[4] The Home of Dr. Yen for High School Mathematics (in Chinese) http://www.nehs.hc.edu.tw/~ylyen/gsp4ile/gsp4ile.htm