Paradoxes on Chinese Dice and Magic Square

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Abstract

Green (1981) has described a set of dice with *non-transitive* paradox where they are referred to as Chinese Dice. Other interested cases are the *pairwise-worst-best* paradox (Kolz and Stroup, 1983) and *pairwise-best-worst* paradox. In this paper, we use the equal-sum property of the columns, rows and diagonals on *magic squares* to map them to sets of equal-expectation dice, and discuss the paradoxes on these dice. Java programs for pairwise comparison and simultaneous comparison of dice are available on http://letitbe.math.ncue.edu.tw/ibl/paradox/

1. Introduction

A magic square is defined as an square matrix within which : (i) all the elements are distinct, and (ii) the sums of each row, column, and diagonal (up-left-to-down-right and up-right-to-down-left) are all equal. Most of the magic squares are made of consecutive integers starting from 1, see the figure 1 and 2 for example. There are many fancinating properties of magic squares. In this paper, we focus on some probability paradoxes while the numbers in a square is treated as sets of number of dots on dice.

4	9	2
3	5	7
8	1	6

Figure 1. A 3 by 3 magic square

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 2. A 4 by 4 magic square

Giving a magic square, the numbers in each row, column or diagonal are assigned as the number of dots on faces on a dice. For a 3 by 3 magic square (Figure 3), an easy way is to as assign each of the three numbers to two faces of a regular dice. Figure 4 illustrates an example of assigning the first column to a dice.

	4	9	2	R1
	3	5	7	R2
	8	1	6	R3
D2	C1	C2	C3	D1

Figure 3. A 3 by 3 magic square assigned to 8 dice.

The two diagonals (up-left-to-down-right and up-right-to down-left) are D1 = (4,5,6), D2 = (2,5,8) respectively.

	4	
3	4	3
	8	
	8	

Figure 4. Assigning C1 in Fig. 3 to a 6-face-dice

For a 4 by 4 magic square, each number can be assigned to a face of a tetrahedron dice. For a 5 by 5 magic square each number can be assigned to a rectangular face of a pentagonal prism dice (Figure 5.). In this case we "rolling" a dice instead of "tossing" a dice. A 6 by 6 magic square can be mapped to cubical dice, and 8 by 8 magic square can be mapped to octahedron dice, etc..

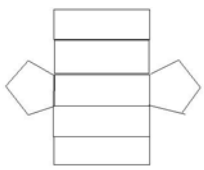

Figure 5. A pentagonal prism dice.

2. Paradoxes

For any two random variables A an B, we define that "A is greater than B in probability" if the probability that A outscores B is greater than 0.5, and "A, B are equal in probability" if A, B have equal chance to outscores each other. The following definitions are for probabilistic inequality and equality.

Definition 1: $A \overset{p}{>} B$ if $\Pr(A > B) > 0.5$

Definition 2: $A \overset{p}{=} B$ if Pr(A > B) = Pr(A < B)

When there are more than two variables in competition, simultaneous comparison is necessary. Two types of the game are considered here: (i) exclusion game: remove the worst from the contest list (and may continue iteratively), and (ii) selection game: select the best only. In an exclusion game, the one with the highest chance being the worst is more likely to be excluded out, and the one with the lowest chance being the worst is more likely to be advanced. On the other hand, in a selection game, the one with the highest chance of being the best is more likely to win, and the one with the lowest chance of being the best is more likely to loss. To simplify the notation, we give the following definitions.

Definition 3: A has the minimal probability to be the unique best, denoted by

$$A \overset{\min b}{<} (B,C),$$

if Pr(A>B and A>C) < *min*(Pr(B>A and B>C), Pr(C>A and C>B)). Here *max(.)* and *min*(.) represent the maximum and minimal of a set of values.

Definition 4: A has the minimal chance to be the unique worst, denoted by

$$A \overset{\min w}{>} (B,C),$$

if Pr(A<B and A<C) < *min*(Pr(C<A and C<B), Pr(B<A and B<C)).

Definition 5: A has the maximal chance to be the unique best, denoted by

$$A \overset{\max b}{>} (B,C),$$

if Pr(A>B and A>C) > *max*(Pr(B>A and B>C), Pr(C>A and C>B)).

Definition 6: A has the maximal chance to be the unique worst, denoted by

$$A \overset{\max w}{<} (B,C),$$

if Pr(A<B and A<C) > *max*(Pr(B<A and B<C), Pr(C<A and C<B)).

Analogously, we can generalize this three-dice comparison to four or more dice comparison, for example,

$$A \overset{\max w}{<} (B,C,D),$$

if Pr(A<min(B,C,D)) > *max*(Pr(B<min(A,C,D)), Pr(C<min(A,B,D)), Pr(D<min(A,B,C))).

Paradox 1. Non-transitivity

Inequality transitivity is an axiom for real numbers, i.e., for any three real number a,b and c, if a> b, b>c, then a>c. However in probabilistic world this may not be true. Non-transitivity paradox for A, B and C can be described by:

$$A \overset{p}{>} B,\ B \overset{p}{>} C, \text{ but } C \overset{p}{>} A.$$

In this case (A, B, C) form a cycle. Examples of this paradox is not rare, especially in ball games: team A has better chance in beating team B, team B has better chance in beating C; however, C has better chance in beating A.

When there are simultaneous comparison, the following paradoxes may occur.

Paradox 2. Pairwise-worst-best-of-all

Assume the following scenario: C is the worst in pairwise comparisons; however, in a simultaneous comparison, C has the highest chance to be the best. I.e.,

$$A \overset{p}{>} C \text{ and } B \overset{p}{>} C, \text{ but } C \overset{\max b}{>} (A,B).$$

Therefore in a selection game, C has the highest chance to win. This scenario is likely to happen in beauty contest.

Paradox 3. Pairwise-worst-least-worst paradox

Variable C is pairwisely the worst; however, in a simultaneous comparison, C has the least chance to be the worst. I.e.,

$$A \overset{p}{>} C \text{ and } B \overset{p}{>} C, \text{ but } C \overset{\min w}{>} (A,B).$$

In an exclusion game, C has the least chance to be the excluded out in the first round, or in other word, has the highest chance to advance.

Paradox 4. Pairwise-best-worst-of-all

Variable A is pairwisely the best; but paradoxically, A also has the highest chance to be the worst in a simultaneous comparison. I.e.,

$$A \overset{p}{>} B \text{ and } A \overset{p}{>} C, \text{ but } A \overset{\max w}{<} (B,C).$$

So in an exclusion game, A has the lowest chance to advance.

Paradox 5. Pairwise-best-least-best

Variable A is pairwisely the best; but paradoxically, A also has the least chance to be selected as the best in a simultaneous comparison. I.e.,

$$A \overset{p}{>} B \text{ and } A \overset{p}{>} C, \text{ but } A \overset{\min b}{<} (B,C).$$

In a selection game, A has the highest chance to loss.

3. Paradox on magic squares

Consider three dice α, β, and γ and let A, B and C be their respective outcome random variables in a toss. For a pairwise comparison we toss two dice and compare the number of dots, and for a simultaneous comparison we toss three or more dice at the same time. Here we set the rule that the one with the most dots will win, and assume there is no tie in either exclusion or selection games. If the tie of the best or the worst does happen, then just roll again until the unique best or worst comes out. Furthermore, we assume the dice are fair (equal probability for each face), then the expectation for each dice are equal.

3 by 3 magic square

In Fig 3., $(C1 \overset{p}{<} C2 \overset{p}{<} C3 \overset{p}{<} C1)$, $(R1 \overset{p}{<} R2 \overset{p}{<} R3 \overset{p}{<} R1)$ are both non-transitive cycle. $D1 \overset{p}{=} C2 \overset{p}{=} R2$, and if we replace C2 or R2 by D1 in the above cycle, the non-transitivity still hold. $D2 \overset{p}{=} Ci$ and $D2 \overset{p}{=} Ri$. for all i=1~3.

4 by 4 magic square

For n by n squares with n >3, there are usually more than one version of magic squares. We found that all the 4 by 4 magic squares have the following property: for columns C1 to C4, and rows R1 to R4, we have $Ci \overset{p}{=} Cj$, $Ri \overset{p}{=} Rj$, i=1~4, j=1~4, therefore no interesting paradox here.

6 by 6 magic square

Fig. 6 shows an example of 6 by 6 magic square, and some example of paradoxes is listed in (1) to (12).

	31	9	2	22	27	20	R1
	3	32	7	21	23	25	R2
	35	1	6	26	19	24	R3
	4	36	29	13	18	11	R4
	30	5	34	12	14	16	R5
	8	28	33	17	10	15	R6
D2	C1	C2	C3	C4	C5	C6	D1

Figure 6 . A 6 by 6 magic square assigned to 14 dice.

Non-transitivity:

$$C1 \overset{p}{<} C2 \overset{p}{<} C3 \overset{p}{<} C1\ldots, \qquad (1)$$

$$C4 \overset{p}{<} C5 \overset{p}{<} C6 \overset{p}{<} C4\ldots, \qquad (2)$$

$$D1 \overset{p}{<} R3 \overset{p}{<} R1 \overset{p}{<} D1\ldots, \qquad (3)$$

$$D1 \overset{p}{<} R6 \overset{p}{<} R4 \overset{p}{<} D1\ldots. \qquad (4)$$

Pairwise-best-least-best:

$$D1 \overset{p}{>} C2 \overset{p}{>} C3, \text{ but } D1 \overset{\min b}{<} (C2, C3), \qquad (5)$$

with Pr(D1 the best)=0.29, Pr(C2 the best)=0.34, and Pr(D2 the best)=0.37, and variance of D1, C2, C3 are 111.5, 221.3 and 224.5 respectively.

Pairwise-worst-best-of-all:

$$D1 \overset{p}{<} C4 \overset{p}{<} C5, \text{ but } D1 \overset{\max b}{>} (C4, C5), \qquad (6)$$

dice	D1	<	C4	<	C5
Pr(best)	*0.37*		0.29		0.34
Pr(worst)	0.43		0.30		0.27
variance	111.5		29.9		37.1

Figure 7. 3-dice comparison (D1, C4, C5).

$$D1 \overset{p}{<} C5 \overset{p}{<} C6, \text{ but } D1 \overset{\max b}{>} (C5, C6), \qquad (7)$$

dice	D1	<	C5	<	C6
Pr(best)	*0.36*		0.31		0.33
Pr(worst)	0.47		0.29		0.24
variance	111.5		37.1		29.9

Figure 8. 3-dice comparison (D1, C5, C6)

$$D1 \overset{p}{<} C6 \overset{p}{<} C4, \text{ but } \quad D1 \overset{\max b}{>} (C4, C6), \qquad (8)$$

dice	D1	<	C6	<	C4
Pr(best)	*0.36*		0.31		0.33
Pr(worst)	0.44		0.27		0.29
variance	111.5		29.9		29.9

Figure 9. 3-dice comparison (D1, C4, C6).

Note that (C4, C5, C6) form a cycle, and D1 is less than each of them in probability paiwisely, but in the simultaneous comparison of (D1, C4, C5, C6), we have

$$D1 \overset{\min w}{>} (C4, C5, C6), \qquad (9)$$

dice	D1	<	(C4	C5	C6)
Pr(best)	0.15		0.27	0.30	0.28
Pr(worst)	*0.17*		0.28	0.30	0.25
variance	111.5		29.9	37.1	39.9

Figure 10. 4-dice comparison (D1, C4, C5, C6)

Pairwise-best-worst-of-all:

$$R5 \overset{p}{<} D1 \overset{p}{<} R3, \text{ but } \quad R3 \overset{\max w}{<} (R5, D1), \qquad (10)$$

with Pr(R3 the worst)=0.350, Pr(R5 the worst)=0.342, Pr(D1 the best)=0.308.

Pairwise-worst-least-worst:

$$D2 \overset{p}{<} C3 \overset{p}{<} C1, \text{ but } \quad D2 \overset{\min w}{>} (C1, C3). \qquad (11)$$

dice	C1	>	C3	>	D2
Pr(best)	0.41		0.38		0.21
Pr(worst)	0.34		0.36		*0.30*
variance	224.3		224.3		96.3

Figure 11. 3-dice comparison for (D2, C1, C3)

Pairwise-best-least-best and Pairwise-worst-best-of-all:

$$D1 \overset{p}{>} C1 \overset{p}{>} C3, \text{ but } D1 \overset{\min b}{<} (C1, C3) \text{ as well as } C3 \overset{\max b}{>} (C1, D1). \qquad (12)$$

dice	D1	>	C1	>	C3
Pr(best)	*0.31*		0.34		*0.35*
Pr(worst)	0.224		0.356		0.39
variance	*111.5*		224.3		224.3

Figure 12. 3-dice comparison for (D1, C1, C3)

4. Discussion and question:

It is amazing that a set of seemingly fair dice (in the sense of equal expectation) can produce so many paradoxes. Among a set of dice with equal expectation, the one with both the largest and the smallest numbers usually has the largest variance.

From our observation, it seems that a dice has the largest variance and is likely to lose in a pairwise comparison games, then it is more likely to win in simultaneous comparison games. For example, among dice (D1, C4, C5, C6) in Figure 6, D1 has the largest number 32 and the smallest number, 6, therefore a relatively large variance. Frome (6)~(9), we see that D1 is dominated by C4, C5, C6 in pairwise comparison; however, has the best chance to win the 3-dice selection games (Figure 7~Figure 9) or to advance in 4-dice exclusion game (Figure 10).

On the other hand, D1 has the smallest variance among (D1, C2, C3) (see (6)) or (D1, C1, C3) (see (12)). Although D1 dominates C1, C2 and C3 in pairwise comparison, however, is most likely to loss in 3-dice selection games; and oppositely, C3 with lager variance is dominated by other but is most likely to win the selection game (see (12)). The one with the smallest variance is not necessary to be favored in pairwise games, for example, D2 has the smallest variance among (D2, C1, C3), but is less favored than C1 or C3 in pairwise games, however, is more likely advance in an exclusion game (Figure 11).

The consequent questions arose are

(i) From the above cases, we observed that of those pairwisely less favored dice, if they have the highest probability to be the best of three (or more), then they also have the highest probability to be the worst as well. Or on the other hand, if they have the lowest probability to be the worst of all three (or more), then they also have the lowest chance to be the best as well. How to prove the above argument?

(ii) Example (12) shows that both paradox 2 and 5 can exist at the same time. Is that possible for paradoxe 2 and 4 to exist in one set of dice? For example, is there an (A,B,C) such that

$$A \overset{p}{<} B \overset{p}{<} C, \text{ but } A \overset{\max b}{>} (B,C) \text{ and } C \overset{\max w}{<} (B,C)?$$

Reference

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