Approaches to the Derivative and Integral with Technology and Their Advantages/Disadvantages

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Abstract

The introduction of CAS and graphics calculators into mathematics teaching has had a strong impact on the way in which derivative and integral are introduced and developed. This paper provides a review of pen-and-paper and technology approaches identified in the research literature. They range from object-based to local, and all incorporate student investigation. We discuss their advantages and disadvantages with reference to research findings.

Introduction

This paper gives an overview of the approaches in which the derivative is introduced and looks at the analogous approaches for the integral. We distinguish local and global, object-based (functional) approaches with technology for both derivative and integral. The local approaches are to regard the derivative as a *gradient or rate of change at a point* and the integral as an *area over a restricted domain*, while an object-based approach would regard that differentiation and integration yield *derivative and integral functions*. There are a variety of ways in which these points of view can be implemented. We discuss these below in relation to $f(t) = \sin t$, quoting the literature, which is more extensive for differentiation than for integration.

Local approaches

Local approaches to the derivative include using graphs to make sense of the limiting process that defines the gradient of a function at a point, or they involve calculating average rates of change and inferring instantaneous rates of change. In the recent literature the use of graphical and numerical features on a graphics calculator or a CAS are discussed for the illustration of the concept "derivative at a point". We will consider four approaches. The first exploits zooming and involves inferring that a function can be approximated at any point by a straight line whose slope is the derivative at that point. The second and third involve the use of secant lines to determine the tangent line at the point of interest and hence the gradient at the point. The fourth is a rate of change approach. We finish this section of the paper with analogous approaches to integration.

Local linearity

Kawski (1997) recommends introducing the derivative at a point through zooming in on the graph of a function. Students choose a function for the class to work with, points are assigned to students, and they zoom in repeatedly, using the same magnification factor in each direction. Comparison of the resulting screen displays leads to the realisation that everyone has ended up with the display of a straight line, which can be characterised by its gradient (see Figure 1). Kawski regards the emphasis on local linearity as the key advantage of the zooming method. However, the fact that the gradient is generally point dependent, so that the gradient at nearby points could be quite different, is not evident from the graphical display. This problem can be partially addressed by having students estimate the gradient for their point or access the gradient on the technology they are using.

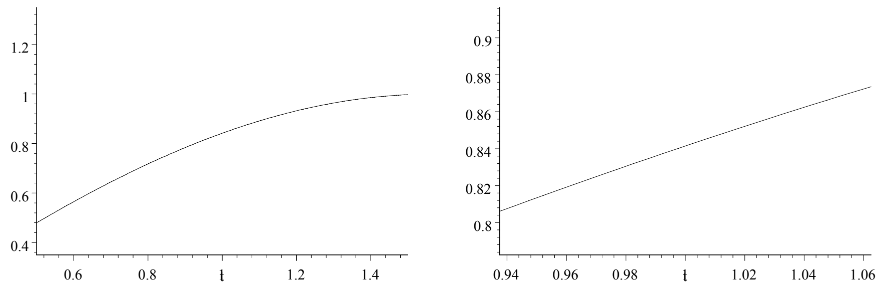

Figure 1. Repeated zooming in, exemplified for $f(t) = \sin t$ near t=1, magnification 8×8

Secant lines

Another approach to the derivative is to infer the tangent line to the graph at a point on the basis of inspection of secant lines constructed through the point, all with different gradients (Lagrange, 1999). We consider obtaining the tangent line to the graph of $f(t) = \sin t$ at (1, sin 1) by graphing the equation $y = m(t-1) + \sin 1$ with different values of m. The method is illustrated in Figure 2.

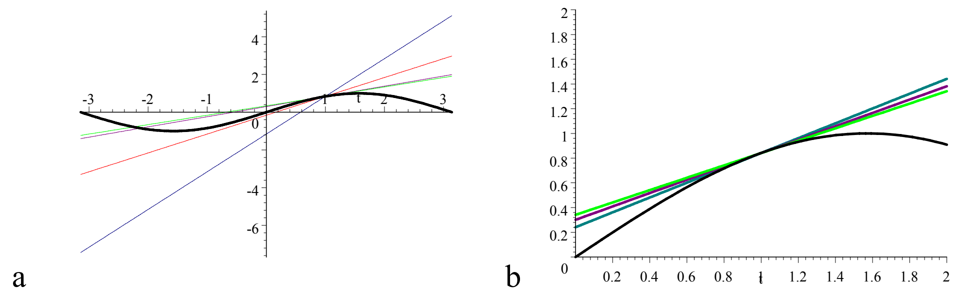

Figure 2. Graphs of $f(t) = \sin t$ and $y = m(t-1) + \sin 1$ for (a) m = 0.5, 1, 2 (b) 0.5, cos(1), 0.6

Lagrange recommends the tangent approach because students generally have an appreciation of tangential behaviour. However, usually no line appears tangential partly because of the screen resolution of calculator and computer technologies, and visual determination of the appropriate line becomes difficult (see Figure 2(b)). To overcome the problems associated with a purely visual determination, Lagrange describes how the estimates for m can be checked on a CAS by computing the differences between the y value for each line and the value of the given function, at a point near the chosen tangency point. To us, a disadvantage is the mathematics of the check is complex.

A second secant line approach is to define the derivative at $(x, f(x))$ as the limiting value as $h \to 0$ of the gradient through the points $(x, f(x))$ and $(x+h, f(x+h))$ yielding the definition: $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$. Figures 3(a) and 3(b) show diagrams for negative and positive h respectively. However, in Kawski's (1997) view, the secant line can be unconvincing as an approximation for the local linearisation of the curve and the dynamic process of taking a limit is not captured well by the static secant images. Other problems are that often line segments, not secant lines are used in illustrations of the approach, and the segments disappear in the limit (ibid).

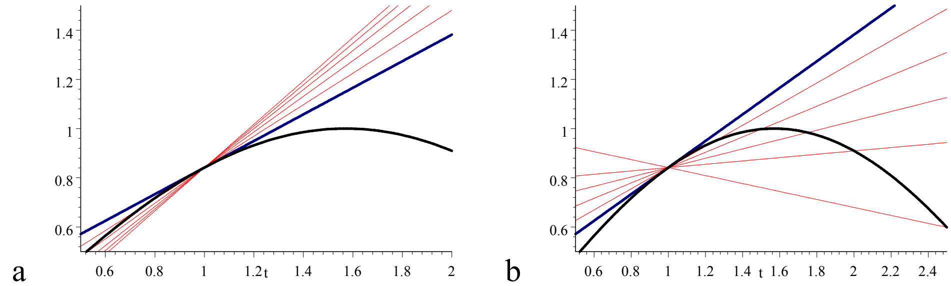

Figure 3. The function $f(t) = \sin t$, secant lines through $(1, \sin 1)$ and $(1+h, \sin(1+h))$, and the tangent line shown in bold.

Rate of change

Speiser and Walter (1994) describe a rate of change approach where students generated (*time*, *distance*) data from a sequence of photo frames for a cat running in a straight line, then calculated average speed values and deduced instantaneous speed for some of the frames. Hence, the derivative was introduced as an instantaneous rate of change. A graphical analysis of the data and interpretation of the results in relation to the context highlighted that instantaneous rates cannot always be reliably predicted from discrete values and this may be beneficial to understanding. Others (e.g., Lapp & Cyrus, 2000) describe similar activities where students generated data with CBL (computer based lab) devices. Benefit seemed to lie in the simultaneous collection and plotting of the data, so that students could see the effect on the slope of the graph at the same time as they experienced changes in the speed of the object/person whose motion was being logged.

Integration

The secant approach to the derivative, where the gradient through $(x, f(x))$ and $(x+h, f(x+h))$ is considered, has a direct counterpart in integration, in that limiting values are considered. The method is to treat the definite integral as the area of the region enclosed by the graph of a function and the x axis, and to construct left, then right rectangles over a given interval and sum their areas. See Figure 4 for the function $f(t) = \sin t$ on $[0, \pi/2]$.

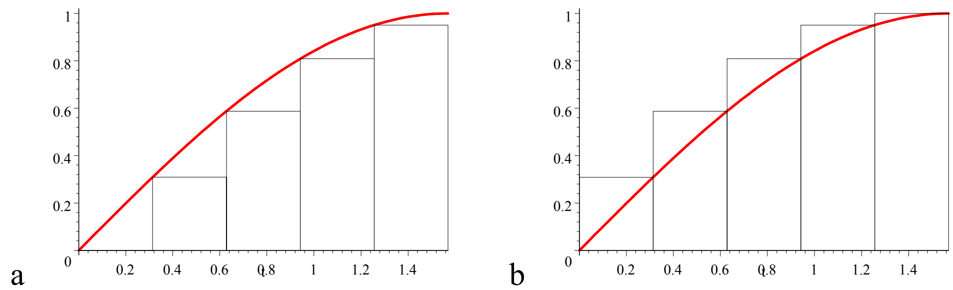

Figure 4. (a) Left and (b) right rectangles for estimating $\int_0^{\pi/2} \sin t \, dt$

Then, rectangles with heights given by the function values at arbitrary points in the subintervals (e.g., midpoints) of the subintervals are constructed and the areas are summed to show visually and numerically that the result is a better approximation for the area under the curve (Figure 5a). Next, the method is extended to show that an increase in the number of subintervals improves the area estimate. This can be done using circumscribed and inscribed rectangles, a method which captures the essence of convergence to a limiting value from above and below, or the midpoint rule can be applied (Figures 5b). In the case when only graphics calculators are available and the rectangles cannot be shown, an explanation of the method together with a program to calculate the sum of the rectangular areas for different numbers of subdivisions can be used (Stick, 1997).

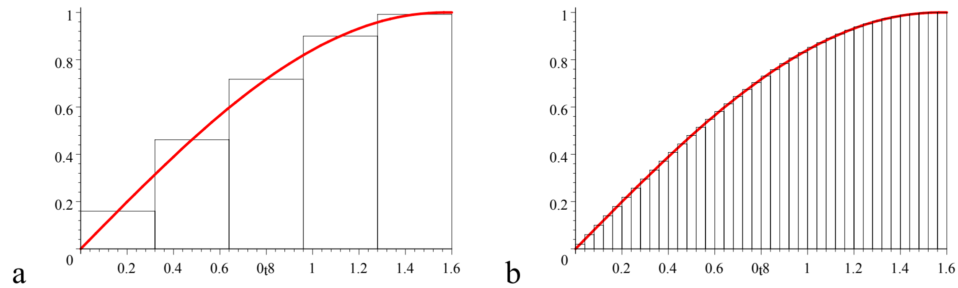

Figure 5. Approximating $\int_0^{\pi/2} \sin t \, dt$ using the midpoint rule with (a) 5 and (b) 40 subintervals.

Another approach is to introduce the integral and derivative together. Schnepp and Nemirovsky (2001) describe opening activities for introductory calculus which involve considering numerical data or a symbolic function rule for the rate at which some quantity changes. The task is to determine how much of a changing quantity has accumulated over various time intervals. One possible implementation is a computer package that allows students to draw the rate graphs and the computer communicates with a mechanical device that moves according to the graph. Students are led to considering accumulation in terms of the area of rectangles drawn under the rate graphs, as in Figure 4a, and to sketching the accumulation at each point in an interval, for instance the accumulation at $t = 0, \pi/10, \pi/5, 3\pi/10....$ in Figure 4a, which would yield rough approximations for $\int_0^0 \sin t \, dt$, $\int_0^{\pi/10} \sin t \, dt$, $\int_0^{\pi/5} \sin t \, dt$, $\int_0^{3\pi/10} \sin t \, dt$ and so on. The purpose is to enable students to recognize the link between integration (accumulation) and differentiation; and, more specifically, that they develop numerical methods for approximating integrals, relate these to area under a graph, and begin to discuss instantaneous rate.

Object-based approaches

Yerushalmy and Schwartz (1999) suggest an object-based approach to the derivative where students plot the difference quotient $y = (f(x+h) - f(x))/h$ for decreasing values of *h*, to obtain a global view of estimates to the derivative function (see Figure 6a). The structure of the graph can be described in terms of the gradients of the secants to the graph of *f*.

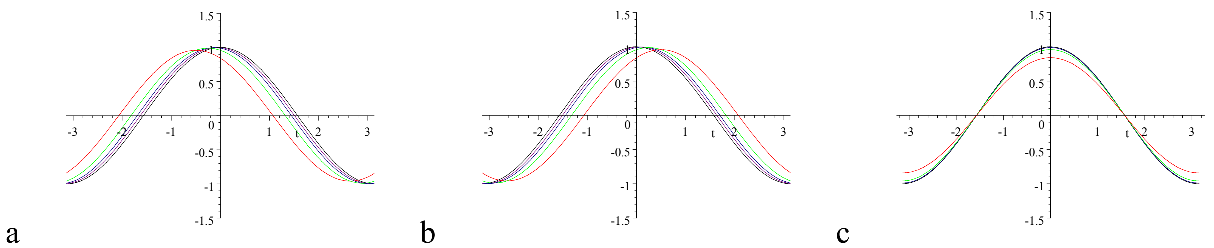

Figure 6. The functions defined by the difference quotients (a) $(\sin(t+h) - \sin t)/h$, (b) $(\sin t - \sin(t-h))/h$ and (c) $(\sin(t+h/2) - \sin(t-h/2))/h$ for $h = 1, 0.5, 0.25, 0.125, 0.0001$.

They suggest also to use $y=(f(x)-f(x-h))/h$, and $y=(f(x+h/2)-f(x-h/2))/h$ (Figure 6 (b) and (c)). In their view, the three approaches may allow a deeper understanding of convergence. Tables of values for all three functions for small increments *h* would also illustrate the convergence and would most likely enhance understanding if they were used in conjunction with the graphs.

Speiser and Walter (1994) suggest plotting each component of the difference quotient in succession, to highlight the effect of the numerator and denominator on the structure of the difference quotient curve. They describe the activity in relation to their rate of change approach (see above). For $f(t)=\sin t$, the quotient $(\sin(t+h)-\sin(t))/h$ is obtained, and then each component is graphed for a small increment *h*. So, if $h=0.5$, then $y=\sin(t+0.5)$, $y=\sin t$, $y=\sin(t+0.5)-\sin t$, and $y=(\sin(t+h)-\sin(t))/h$ are graphed in succession (see Figure 7). The graph of $y=\sin(t+0.5)$ is the graph of $y=\sin t$ shifted to the left, and the difference in height for any given *t* is the numerator $\sin(t+0.5)-\sin t$ (see Figure 7(a)). The structure of $y=\sin(t+0.5)-\sin t$ reflects the structure of the gap between $y=\sin(t+0.5)$ and $y=\sin t$ (Figure 7(b)). The graph of $y=(\sin(t+0.5)-\sin(t))/0.5$ is the graph of $y=\sin(t+0.5)-\sin t$ rescaled by a factor of 1/0.5 (i.e., $1/h$) (see Figure 7(c)). Thus, the method stresses the role of the functional relationship $f(t)=\sin t$ in the computation of the derivative, and the rescaling effect of *h* (the time increment).

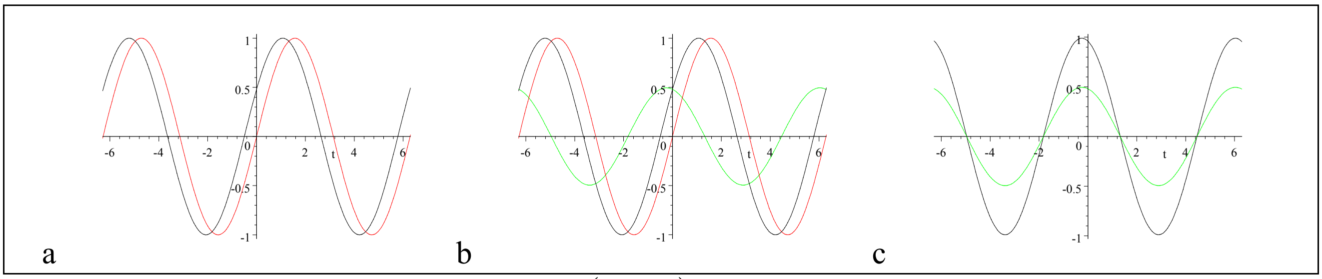

Figure 7. (a) $y_1(t)=\sin t$ and $y_2(t)=\sin(t+0.5)$, (b) y_1 and y_2 with the difference function y_2-y_1, (c) the difference function together with the difference quotient.

The method is repeated for different increments, for example *h* =1, 0.25, 0.125 and 0.0001 and the function in the limit, that is, the derivative function $(\cos t)$ may be deduced. Figure 8(a) shows the result over $t\in[-\pi,\pi]$ and Figure 8 (b) shows the screen after zooming out.

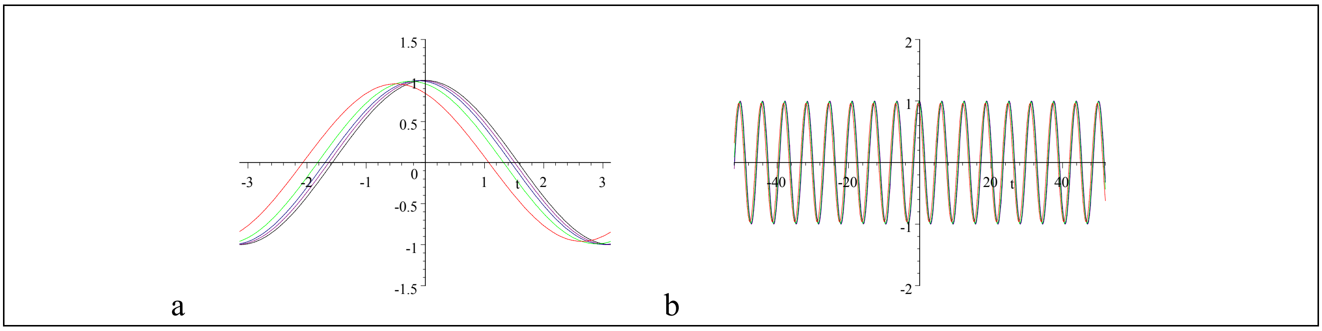

Figure 8. The function defined by the difference quotient $y=(\sin(t+h)-\sin t)/h$ for $h=1$ (leftmost graph), 0.5, 0.25, 0.125, 0.0001 over (a) $t\in[-\pi,\pi]$ and (b) $t\in[-16\pi,16\pi]$

The analogous approach for integration takes as its starting point the approximation of the signed area of the region bounded by the graph of the function, the x-axis, the line $x = 0$ and the line $x = t$. The midpoint rule will be used for the approximation and a fixed number n of subintervals. In the case of $f(t) = \sin t$, the definite integral $\int_0^t \sin x \, dx$ is approximated by $g_n(t) = \frac{t}{n}\sum_{i=0}^{n-1} \sin \frac{(2i+1)t}{2n}$. The approach is illustrated for $f(t) = \sin t$ with $n = 5$ and for different values of t in Figure 9.

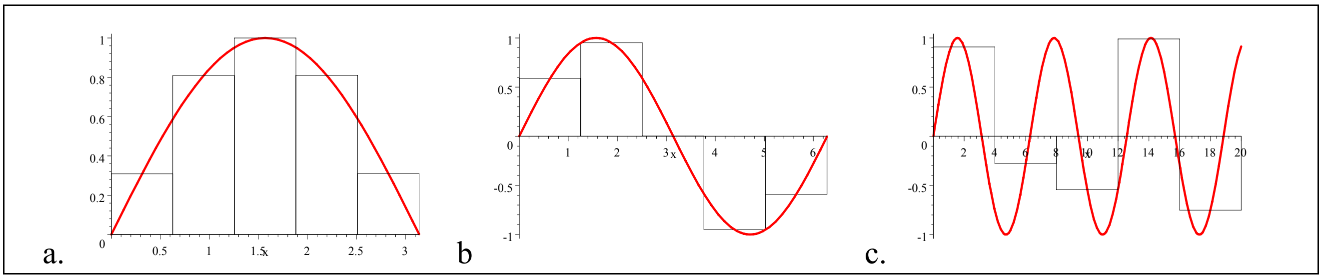

Figure 9. $g_5(t)$ for (a) $t = \pi$, (b) $t = 2\pi$ and (c) $t = 6\pi$

The graphs for the resulting values of $g_n(t)$ for $n = 5, 10, 20$ and 40 are shown in Figure 10, for the interval $t \in [0, 3\pi]$ (Figure 10(a)) and the interval $t \in [-16\pi, 16\pi]$ (Figure 10(b)). For small values of t there is little difference between the graphs, however, as the distance from the lower limit of integration increases, the functions g_n based on small numbers of subintervals exhibit large oscillations. These diminish in magnitude as the number of subintervals increases. In the limit, as $n \to \infty$, the graph of $\int_0^t \sin x \, dx = -\cos t + 1$ is obtained.

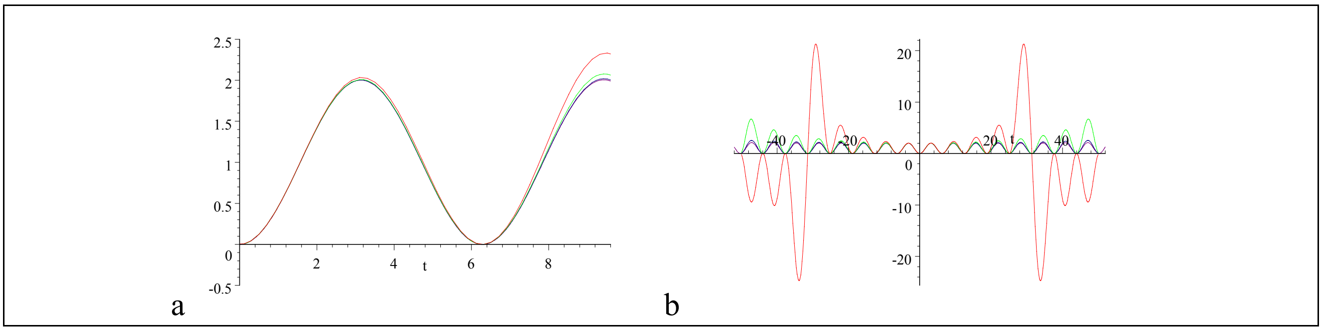

Figure 10. The functions g_5, g_{10}, g_{20} and g_{40} for (a) $t \in [0, 3\pi]$ and (b) $t \in [-16\pi, 16\pi]$.

In order to account for the fact that the integral of a function is only determined up to a constant an investigation of the function $\int_a^t f(x) \, dx$ and its approximations for different initial points a could be carried out (see Figure 11).

The use of different lower limits of integration can lead to the recognition that there is a family of functions associated with each integrand. However, the use of the definite integral to exhibit this feature will not lead to the full family of functions.Another limitation of the method is that as the width of the interval of integration increases, a fixed number of intervals leads to wider and wider subintervals which introduces convergence problems (see Figure 10(b)). This could be remedied by using a fixed mesh-size instead, however, it then becomes impossible to evaluate the function $\int_a^t f(x)\,dx$ for arbitrary values of t

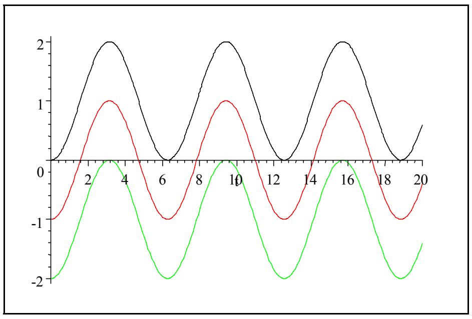

Figure 11. Approximations for $\int_a^t \sin x\,dx$ using the midpoint rule with 100 subintervals and $a = 0,\ \pi/2$ and π (top to bottom graph)

Another object-based approach is to treat integration as the reverse operation to differentiation. In other words, treat the function whose integral is to be found as a derivative. So, for the integral $\int f(x)\,dx = F(x)$ it is recognised that $f(x) = F'(x)$. A graphical approach uses slope fields, borrowed from differential equations. The slopes $f(x)$ along the integral function $F(x)$ are indicated by slopes of line segments. Once an initial value is chosen, the graph of the integral is obtained by tracing the function using the trend of the segments. We illustrate this approach for $f(x) = \sin x$ with initial conditions given by $F(0)=1$, and $F(0)=-3$ (see Figure 12). A strength of this approach is to emphasize the fact that for any given integrable function there is an entire family of antiderivatives, all having the same shape, but parallel-translated along the y-axis. In other words, the calculation of $\int f(x)\,dx$ requires the inclusion of a constant of integration. A problem with the visual approach is the line segments are usually placed so their centres are spaced evenly on points on a grid, which could be confusing for it is incorrect to draw a curve by joining the segments.

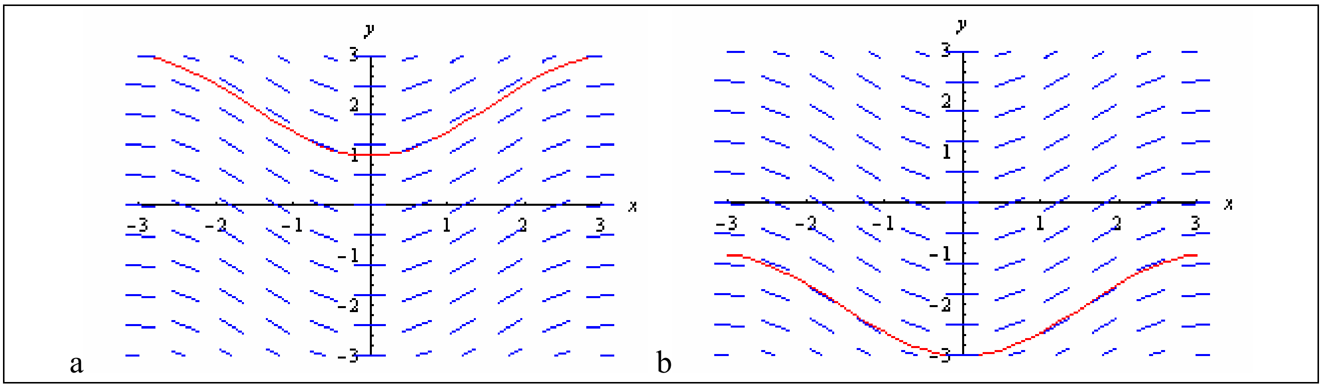

Figure 12. Slopefield for $f(x) = \sin x$ together with antiderivative for (a) $F(0)=1$, and (b) $F(0)=-3$

Schnepp and Nemirovsky's (2001) accumulation approach to integration, given functions or data for rate (see above), can also be used for exploring indefinite integrals. They describe how students realized that, while a given rate function allows the calculation of accumulation in a given time interval (that is calculation of definite integrals), a constant needs to be added for total accumulation at an arbitary time *t*.

Advantages, Shortcomings and Extensions

The main advantages that technology offers are computational power and consequent quick access to graphs, and hence the display of the integration and differentiation processes. Problems and properties associated with each of the approaches have been described above and would need to be taken into account in teaching. In brief, these are poor resolution in the graphical display, which may be confusing in relation to the derivative at a point, and computation difficulties with object-based approaches to integration. The problems with object-based integration perhaps explain why the approaches are not referred to much in the literature on teaching and learning. Another facet of many of the approaches is the absence of real-life contexts, although some methods are embedded in contexts. It is relevant that learning can be more meaningful if contextualised.

The different approaches variously lend themselves to the exploration of differentiation and integration for particular functions. For instance:

- the difference quotient $y = (f(x+h) - f(x))/h$ yields an easily recognizable function if $f(x)$ is a sine or cosine function but for higher order polynomial functions the resulting derivative function is not so easily identifiable.
- provided the position of a discontinuity is known, Kawski's (1997) zooming approach on a CAS could highlight that the derivative at a point does not exist at a point discontinuity--the graph no longer appears linear near the discontinuity. See Figure 13(a) for the function $f(x) = \sin x/(x-\pi)$ at $x = \pi$. However, the behaviour may not be apparent on a graphics calculator because of the limited resolution of the calculator screen. Another problem is the graph may appear the same on a CAS package once the function has been redefined to remove the discontinuity--in that case, local linearity should be seen but is not.
- the tangent approach advocated by Lagrange (1999) would fail to highlight the non-existence of the derivative of $f(x) = \sin x/(x-\pi)$ at $x = \pi$ unless students realized it was inappropriate to draw lines through $(\pi, -1)$. However, if the function $f(x) = \sin x/(x-\pi)$ is redefined to have a value of -1 at $x = \pi$, Lagrange's approach is viable (see Figure 13(b)).

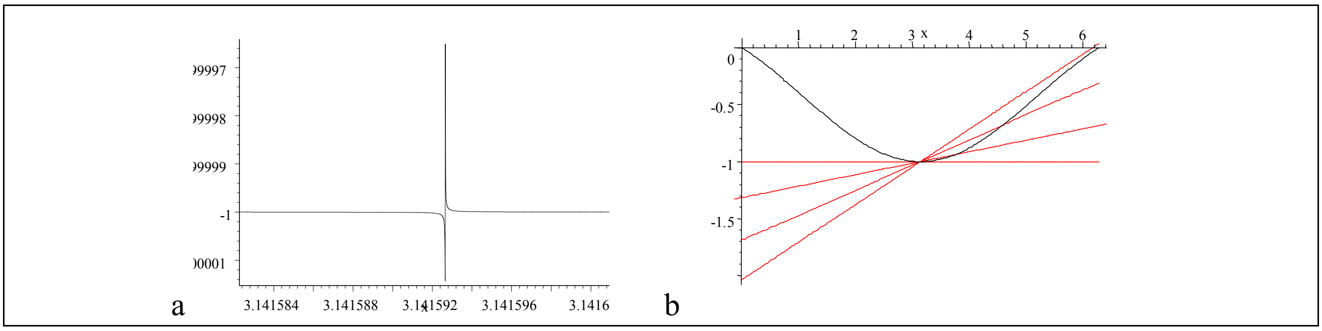

Figure 13: (a) Kawski's approach for $f(x) = \sin x/(x-\pi)$ and (b) Lagrange's approach for $f(x) = \sin x/(x-\pi), x \neq \pi$ and $f(\pi) = -1$

- the non-existence of an integral, such as $\int_0^1 (1/x)\,dx$, can be illustrated graphically by considering upper and lower sums for a sequence of integrals $\int_{10^{-k}}^{1} (1/x)\,dx$, $k > 0$, and using different numbers of subintervals. The sum can be explored visually via the rectangles and numerically using tables. As k increases, the upper sum grows without bound, what-ever number of subdivisions is used.

The approaches described above can also be used for deducing rules for differentiation and integration for different classes of functions and properties that link functions, derivatives and integrals. If the purpose is to deduce differentiation rules and the graph of the difference quotient is used, the value for h would be kept constant and small (e.g., $h = 0.001$), while for integration rules, if a limiting sum of areas of rectangles is used, the value of n would typically be kept large and constant. Some other approaches to inference are:

(a) Derivative function and integral function *values* can be produced on calculators using commands for the derivative (or slope) and integral respectively, and the algebraic expressions inferred, then, differentiation and integration rules for classes of functions deduced. Otherwise, the algebraic expressions for derivatives and integrals can be evaluated on CAS calculators and the rules deduced. Lagrange (1999) observes that students are likely to be more reflective than in cases when the teacher gives the rules. Lumb, Monaghan and Mulligan (2000) report some students lost interest with such approaches, while they stimulated the interest of others.

(b) Graphs of functions and derivative functions, or functions and integral functions can be plotted on the same screen and the properties that link them inferred, from the graph or associated tables of values.

Stroup's (2002) findings in relation to a qualitative approach to calculus concepts are relevant to the discussion. He notes that traditionally analysis starts with linear functions, for which rate of change (i.e, the derivative) is constant. He observes that students have difficulty with the properties associated with the linear functions, and have greater success in understanding the fundamental properties of derivatives and integrals when starting with situations where the rate varies.

We also note that building in requirements for students to make links between representations in investigative/instructional activities can result in students being able to coordinate the use of different representations effectively in subsequent problem solving (Porzio, 1999). There is scope to prescribe that connections be made between algebraic expressions, table of values and graphs in many of the above introductions to the derivative. Otherwise students may access the different representations spontaneously, which Pierce and Stacey (2001) observed in a CAS environment, when students were predicting rules from algebraic expressions.

Concluding discussion

Lagrange's approach to the introduction of the derivative, and the secant line approach are adaptations of traditional approaches that originally arose in a paper and pencil setting. Both easily translate into a technology setting and could be enhanced if the diagrams are programmed dynamically. In our view, the approach using the secant lines is more powerful, as it better encapsulates the key components of the definition of the derivative.

Methods such as zooming and treating the different quotient as a function in its own right are approaches that cannot be implemented without the use of technology. With zooming we note that, while it emphasizes local linearity, the nonconstancy of the derivative is lost even if different points are used in the exploration of the concept. The difference quotient function allows the move from a local point of view to a global one, namely looking at the function as a whole, and so might allow a move from a purely procedural point of view to deeper, conceptual understanding. Approaches that use CAS functions for the prediction of integration and differentiation rules do not address fundamental aspects of the concepts so it would be best to implement them after the methods identified above. The availability of CBL devices and specialised computer applications, such as the one used by Nemirovsky and Schnepp (2001) allow new approaches to the derivative and integral concepts, where an advantage is inference is grounded in students' real-life experience.

References

Kawski, M. (1997). Computer visualisation and vector calculus. In Yang, W-C and Abu-Hassan, Y. (eds.), *Proceedings of the Second Asian Technology Conference in Mathematics (ATCM) 1997* ACTM Inc., USA, 100-114.

Lagrange, J-B. (1999). Complex calculators in the classroom: Theoretical and practical reflections on teaching precalculus. *International Journal of Computers for Mathematical Learning, 4*(1), 51-81.

Lapp, D. A., & Cyrus, V. F. (2000). Using data collection devices to enhance students' understanding. *Mathematics Teacher, 93*(6), 504-510.

Lumb, S., Monaghan, J., & Mulligan, S. (2000). Issues arising when teachers make extensive use of computer algebra. *International Journal of Computer Algebra in Mathematic Education, 7*(4), 223-240.

Pierce, R., & Stacey, K. (2001). Observations on students' responses to learning in a CAS environment. *Mathematics Education Research Journal, 13*(1), 28-46.

Porzio, D. (1999). Effects of differing emphases in the use of multiple representations and technology on students' understanding of calculus concepts. *Focus on Learning Problems in Mathematics, 21*(3), 1-29.

Schnepp, M. J. & Nemirovsky, R. (2001). Constructing a foundation for the fundamental theorem of calculus. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics, 2001 yearbook*. Reston, VA: National Council of teachers.

Speiser, B. & Walter, C. (1994). Catwalk: First semester calculus. *Journal of Mathematical Behavior, 13*, 135-152.

Stick, M. E. (1997). Calculus reform and graphing calculators: A university view. *Mathematics Teacher*, 90(5), 356-361.

Stroup, W. M. (2002). Understanding qualitative calculus: A structural synthesis of learning research. *International Journal of Computers for Mathematical Learning*, *7*, 167-215.

Yerushalmy, M., & Schwartz, J. J. (1999). A procedural approach to explorations in calculus. *International Journal of Mathematical Education in Science and Technology, 30*(6), 903-914.