Graphs of Threshold Functions of up to Five Variables

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Abstract

Threshold functions are Boolean functions that model neurons, processing units of an artificial neural network. These functions can be represented geometrically as a unit hypercube in which each vertex is labeled either by 1 or 0. These representations can be simplified into graphs without losing the relationship between minterms and vertices. A graph labeling can be used to store threshold functions as well as to keep the relation between minterms and vertices. There are 119 different graphs that represent all of the 94,572 threshold functions of five variables. This paper presents graphs of threshold functions and some patterns followed by these graphs for small number of minterms and variables and some open problems.

1. Introduction

Threshold functions are restricted class of Boolean functions that model neurons. Each neuron can be regarded as a processing unit of an artificial neural network. A formal neuron consists of n input lines, each of them having a synaptic weight, a threshold, and a single output line. The output lines of neurons can be connected to some other neurons to assemble a neural network. At random time, each input line may or may not carry signal. Signal will be present at the output line if and only if the weighted sum of the input lines is equal to or greater than the threshold.

Definition:

A Boolean function f of n variables $X = (x_1, x_2, \ldots, x_n)$ is a threshold function if and only if there exists a set of real numbers $w_1, w_2, ..., w_n$, called (*input*) *weights*, and a real number q, called *the threshold*, such that the following conditions are satisfied:

$$f(X) = 1 \text{ if } \sum_{i=1}^{n} x_i w_i \geq q \qquad (1)$$

$$f(X) = 0 \text{ otherwise.}$$

In this case, $[w_1, w_2, ..., w_n; q]$ is a structure of f.

For example, $f = x_1x_2$ ($x_1 AND x_2$) and the (logical) summation $g = x_1 + x_2$ ($x_1 OR x_2$) are threshold functions, but $h = x_1\overline{x}_2 + \overline{x}_1x_2$ ($x_1 XOR x_2$) is not.

Each Boolean function *f* can be represented uniquely in the minterm expansion form [9]. They can also be represented geometrically. Even though the geometrical representation is a convenient way to represent threshold functions, it is difficult to illustrate when the number of variables is more than three. These representations can be simplified into graphs, the projection of the geometrical representation into 2-dimensional space, except for $f = 0$. Every threshold function can be represented uniquely by a graph. However, a graph can represent several different threshold functions. As a result, if we are given a graph, we may not be able to decide which function it represents. In order to maintain the relationship between minterms and vertices, as well as the relationship between functions and their graphs, a certain graph labeling is defined on the vertices of the graphs.

Graphs of threshold functions of up to five variables follow certain patterns. There is a possibility that these patterns can be generalized into theorems. This paper presents those patterns, conjectures, and facts known for small graphs of threshold functions. We are also interested in exploring the consequences, in terms of graphs, derived from the preserving and closure operations on threshold functions, operations performed on threshold functions that produce other threshold functions. Open problems for future research topics are given in the last part.

1. Threshold Function Representations

2.1. Geometrical Interpretation of Threshold Functions

The representation of Boolean function chosen in this paper is *the minterm expansion form*, abbreviated *mef*, because it is strongly related to the geometrical representations of functions. The minterm expansion form is a disjunction of different minterms. A *minterm* is a conjunction of different *literals*, variables or their complements, in which each variable is involved exactly once For instance, the *mef* of $f = x_1 + \overline{x}_2x_3$ is $f = x_1x_2x_3 + x_1x_2\overline{x}_3 + x_1\overline{x}_2x_3 + x_1\overline{x}_2\overline{x}_3 + x_1\overline{x}_2x_3 + \overline{x}_1\overline{x}_2x_3$. Each Boolean function can be expressed uniquely in the *mef*.

If the number of variables is *n* then there are 2^n minterms. Minterms appearing in the *mef* of *f* are called *true* minterms, otherwise are called *false* minterms. A Boolean function of *n* variables having *m* minterms is called an (*n*, *m*) Boolean-function.

Let $\mathbf{m} = x_1^*x_2^*...x_n^*$ be a minterm, $v = (a_1^*, a_2^*,..., a_n^*)$ be a vector or vertex, $b = a_1^*a_2^*...a_n^*$ be a binary number, and $A = \{x_1 = a_1^*, x_2 = a_2^*,.., x_n = a_n^*\}$ be an assignment with the following relation:

$$a_i^* = 1 \text{ if } x_i^* = x_i,$$
$$a_i^* = 0 \text{ if } x_i^* = \overline{x}_i \quad (2)$$

where $1 \le i \le n$. Under such a relation, the minterms (or their corresponding vectors) are in one-to-one correspondence to the assignments. If **m** is a *true* minterm of *f* then *v* is called a *true* vector or vertex. Otherwise, it is called a *false* vector or vertex of *f*.

Let **m** be a minterm of an (*n*, *k*) Boolean-function *f*. The value of **m** under the corresponding assignment *A* is 1; hence, the value of *f* is also 1. Conversely, if *f* has a value 1 under an assignment *A*, one of its minterms has the value 1 under the same assignment. In other words, the *true* minterms are in one-to-one correspondence to the rows of the truth table for *f* in which *f* = 1. Furthermore, the *false* minterms are in one-to-one relationship to such rows in which *f* = 0. Vector *v*, that relates **m** to *A*, is a vertex of a unit *n*-cube. Therefore, *f* can be explained in terms of geometry as a unit *n*-cube where we label the *true* vertices by 1 and the *false* vertices by 0. Henceforth, such an *n*-cube is called the *geometrical representation of f*. If *f* is given in the *mef*, its geometrical representation can be obtained in a straightforward manner and conversely.

2.2. Graphs of Threshold Functions

The geometrical representations are natural and convenient way to represent Boolean functions. However, if $n \ge 4$ it is difficult to illustrate. An alternative way of representing threshold functions is by using graphs [10]. These graphs are constructed by projecting the *true* vertices of the geometrical representation of *f* together with their adjacency edges into 2-dimensional space.

Given an (*n*, *k*) Boolean-function *f* and its geometrical representation. The *graph representation* G_f of *f* is defined as follows: $V(G_f) = \{v_1, v_2, \ldots, v_k\}$ denotes the set of G_f 's vertices, $E(G_f)$ denotes the set of G_f 's edges, and v_i is adjacent to v_j in G_f, written as $(v_i, v_j) \in E(G_f)$, if and only if v_i and v_j are connected by an edge of the unit *n*-cube, for $1 \le i, j \le k$. It means that b_i and b_j differ in exactly one digit; $\mathbf{m}_i$ and $\mathbf{m}_j$ differ in exactly one literal. For example, the geometrical and graph representations of $f = x_1x_2x_3 + x_1x_2\overline{x}_3 + x_1\overline{x}_2x_3 + \overline{x}_1\overline{x}_2\overline{x}_3$ are the following.

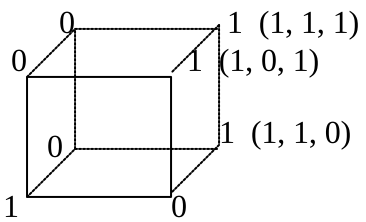


Figure 1: The Geometrical Representation of *f*

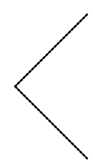


Figure 2: The Graph Representation of *f*

A graph can represent more than one Boolean functions; a graph may not represent any Boolean function at all. Graphs mentioned in this paper are graphs of Boolean functions. If a Boolean function is a threshold function, then the graph representing it is called the graph of threshold function. Graphs of threshold functions are restrictive graphs of Boolean functions.

Recall the definition of threshold function and the inequality (1). The following equation

$$\sum_{i=1}^{n} x_i w_i = q \quad . \tag{3}$$

defines an n-hyperplane. Thus, a Boolean function is a threshold function if and only if there exists an n-hyperplane that separates the *true* vertices from the *false* vertices such that the *true* vertices lie in one side of the n-hyperplane and the *false* vertices are on the other side. It is clear that a graph consisting of two or more components is not a graph of threshold function because we need two or more n-hyperplanes to separate the *true* from the *false* vertices of the geometrical representation of *f*. Graphs representing threshold functions are connected graphs. If a graph of threshold function *f* contains an isolated vertex, it is the only vertex of the graph; *f* contains one minterm.

2. Some Properties of Threshold Functions and Their Graphs

3.1. Preserving Operations and Closure Transformations

If *f* is a threshold function of n variables $x_1, x_2, ..., x_n$, then the (logical) addition and multiplication between *f* and a variable are also threshold functions. In other words,

(1) $f + x_p$,

(2) fx_p

are threshold functions, where $1 \le p \le n+1$ [8]. If $p = n+1$, the operations performed are between *f* and a new variable. Otherwise, the variables involved are those of *f*.

If *f* is an (n, k) threshold-function, then $f + x_{n+1}$ is an (n +1, $2k + 2^n$) threshold-function. The *mef* of $f + x_{n+1} = f(x_{n+1} + \overline{x}_{n+1}) + dx_{n+1}$, where d is an (n-1)-variable threshold function having all possible minterms, represented as an (n-1)-cube.

For $p = n+1$, condition (2) means that an (n, k) threshold function *f* can be regarded as an (n +1, k) threshold-function *g*. Considered geometrically, *f* is viewed in (n + 1)-dimensional space. Note that G_f and G_g are isomorphic.

Conversely, if all minterms of an n-variable threshold function g are identical in the i-th literal, then g can be reduced into an (n-1)-variable threshold function *f* such that $g = fx_i$ (if variable x_i appears un-complemented) or $g = f\overline{x}_i$ (if x_i appears complemented).

Besides the closure transformations of threshold functions there are three preserving operations as follows. Given a threshold function *f*, the Boolean function *f'* that can be obtained from *f* by one or more combinations of the following operations is a threshold function [8]:

(1) Negation of one or more variables;
(2) Permutation of two or more variables; and
(3) Negation of the output function, $\overline{f}$.

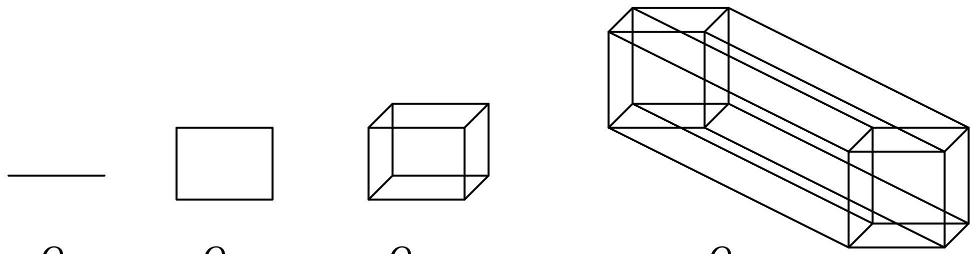
The *NP* equivalent class of *f*, called *NP*-class of *f*, consists of functions that equivalent to *f* under operations (1) and (2). The graph representations of Boolean functions *f* and *g* are isomorphic if and only if they belong to the same *NP*-class. Furthermore, if *f* is a threshold function then so is *g*. This principle can be used to characterize threshold functions. We need a list of graph representations of threshold functions as a table look up.

Let f be an n-variable Boolean function, for $n \le 5$. If the graph of f is isomorphic to a graph of threshold function, then f is threshold function. Otherwise, f is not a threshold function. Some graphs of threshold functions of five variables are given in the next section. They can be used to determine whether or not a given Boolean function is a threshold function. If $n > 5$ and f is isomorphic to one of them, then f is a threshold function since we can reduce it into a five-variable threshold function by preserving operation (2). If $n > 5$ and f is not isomorphic to them, then we cannot decide whether or not it is a threshold function. It is a threshold function if it is isomorphic to graph representations of threshold functions of six or more variables that cannot be reduced to five-variables threshold functions.

As given in the appendix of [10], there are 94,572 threshold functions of five variables grouped into 119 *NP*-classes. Functions of the same *NP*-class are represented by the same graph. Thus, there are 119 different graphs of threshold functions of five variables. Some of them are given in the next section.

3.2. Some Graphs of Threshold Functions

The graphs of threshold functions of up to five-variables follow certain patterns. Every n-variable Boolean function having 2^n minterms, associated with $f = 1$, is a threshold function for $n \ge 1$. Their geometrical representations form n-cubes, called cube graphs. The following are cube graphs for $n = 1, 2, 3, 4$.

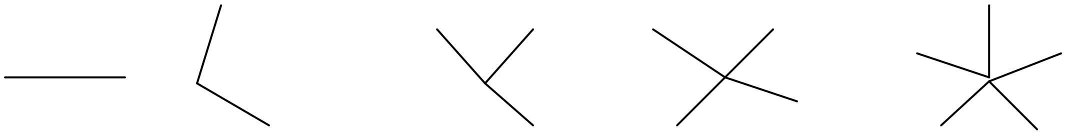


Q_1 Q_2 Q_3 Q_4

Figure 3: Graphs of $(n, 2^n)$ Threshold-Functions, for $n = 1, 2, 3, 4$

Every $(n, 1)$ Boolean-function f is a threshold function. One of its structures is $[a_1^*, a_2^*, ..., a_n^*; n]$, where $a_i^* = 1$ if $x_i^* = x_i$ $a_i^* = 0$ if $x_i^* = \overline{x}_i$. Hence, every $(n, 2^n - 1)$ Boolean-function f is a threshold function. It is the complement of $(n, 1)$ threshold-function. It's graph is obtained by removing one vertex, together with all edges incident to it, from the n-cube graph. For $n = 2$, the graph is given in Figure 2. In fact, it is the only graph of an (2, 3) threshold function; it is called 2-star.

An s-star is a graph of $(s + 1)$ vertices and s edges in which one vertex is adjacent to the rest. Every s-star represents a threshold function having $(s + 1)$ minterms of at least s variables. If the number of variables is n, then the number n-stars is 2^n; each of them represent an $(n, n+1)$ threshold-function. The following are n-stars for $n = 1, 2, 3, 4, 5$.



S_1^* S_2^* S_3^* S_4^* S_5^*

Figure 4: *s*-Stars with *s* = 1, 2, 3, 4, 5

Lemma: Every *s*-star, a graph with *s* vertices and (*s* -1) edges in which one vertex is adjacent to the other (*s* - 1) vertices, is a graph of threshold function.

Proof. Consider $f = x_1x_2x_3...x_n + \bar{x}_1x_2x_3,...,x_n + x_1\bar{x}_2x_3...,x_n + x_1x_2\bar{x}_3...x_n + ... + x_1x_2x_3...\bar{x}_n$. It is an (*n*, *n*+1) threshold-function. We will prove that one of its structures is [1, 1, 1, …, 1: *n*-1]. If the value of each input is equal to 1, then the weighted sum of the inputs is *n* which is greater than *n*-1. Therefore, the value of the output *f* is equal to 1. If the value of an input line is equal to 0 and the rest has the value equal to 1, then there is exactly one minterm has the value of 1 and hence *f* has the value of 1 also.

Other *s*-stars can be obtained from *f* by one or more combinations of negation of one or more variables and permutation of two or more variables.

Threshold functions having four minterms can be represented by either S_3^* or Q_2. There are fourteen (3, 4) threshold-functions, eighty-eight (4, 4) threshold-functions, and four hundred (5, 4) threshold-functions.

Threshold functions having five minterms can be represented by either S_4^* or the following graph.

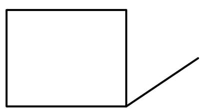


Figure 5: Graph of (*n*, 5) Threshold-Function

There are twenty-four (3, 5) threshold-functions, two hundred and eight (4, 5) threshold-functions, and one thousand one hundred and twenty (5, 5) threshold-functions.

There are three different graphs that represent (5, 6) threshold functions, as shown by the following figure.

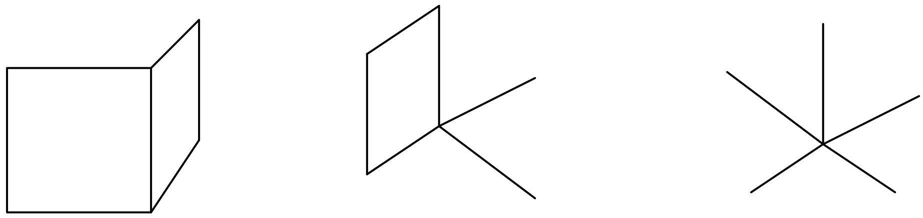


Figure 6: Graphs of (5, 6) Threshold-Functions

They can represent some (n, 6) threshold-functions, for $n \geq 6$; other graphs of threshold functions of six or more variables having seven or more minterms remain an open problem.

3. Graph Labeling to Store Threshold Functions

Consider Figure 2. This graph represents, for examples, the following functions:

$$f_1 = \overline{x}_1 \overline{x}_2 \overline{x}_3 + x_1 \overline{x}_2 \overline{x}_3 + \overline{x}_1 x_2 \overline{x}_3,$$

$$f_2 = \overline{x}_1 \overline{x}_2 \overline{x}_3 + x_1 \overline{x}_2 \overline{x}_3 + \overline{x}_1 \overline{x}_2 x_3,$$

$$f_3 = \overline{x}_1 \overline{x}_2 \overline{x}_3 x_4 + x_1 \overline{x}_2 \overline{x}_3 x_4 + \overline{x}_1 x_2 \overline{x}_3 x_4,$$

and many other functions. When we construct a graph from a geometrical representation of a function, the important relation between minterms and vertices is lost. We can restore the information by using a graph labeling, which is different from the labeling of the geometrical representations of functions.

Let f be a Boolean function and **m** be a true minterm. The label for vertex v_i, associated with **m**, is the decimal conversion of b, see relation (2). Such a labeling on f is an injective function from the set of vertices of f into the set of all possible labels D = {0, 1, 2, 3, …, 2^{n-1}}. For instance, the graph in Fig. 2 has {5, 6, 7} as the set of labels.

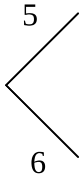


Figure 7: Graph and its Labels

Then, every threshold function can be stored as a set of binary numbers as well as a set of decimal numbers, its labels. For instance, $f = \overline{x}_1 x_2 x_3 + x_1 \overline{x}_2 x_3 + x_1 x_2 x_3$ can be stored as {3, 5, 7}. If the set of f 's labels is given, then the *mef* of f can be obtained and the graph can be reconstructed. First, we convert each label l into its binary representation b, then, find the minterm associated with b using the relation given in (2). The *mef* of f is the conjunction of all minterms of f.

Graphs of (n, 2^n) threshold-functions, cube graphs, for n = 1, 2, 3, 4 (see Figure 3) have the following labels: {0, 1}, {0, 1, 2, 3}, {0, 1, 2, 3, 4, 5, 6, 7}, and {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}.

Stars having up to six vertices (see Figure 4) have the sets of labels respectively {0, 1}, {0, 1, 2}, {0, 1, 2, 4}, {0, 1, 2, 4, 8}, {0, 1, 2, 4, 8, 16}. Therefore, Lemma 1 can be stated as follows.

Corollary: Graphs with the set of labels $\{0, 2^0, 2^1, 2^2, 2^3, 2^4, \ldots, 2^n\}$ are graphs of threshold functions.

Threshold functions of forms

$$\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5,$$

$$\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5 + \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4x_5,$$

$$\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5 + \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4x_5 + \bar{x}_1\bar{x}_2\bar{x}_3x_4\bar{x}_5,$$

$$\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5 + \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4x_5 + \bar{x}_1\bar{x}_2\bar{x}_3x_4\bar{x}_5 + \bar{x}_1\bar{x}_2x_3\bar{x}_4\bar{x}_5, \ldots,$$

$$\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5 + \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4x_5 + \bar{x}_1\bar{x}_2\bar{x}_3x_4\bar{x}_5 + \ldots + x_1x_2x_3x_4x_5$$

have the following sets of labels respectively {0}, {0, 1}, {0, 1, 2}, {0, 1, 2, 3}, …,{0, 1, 2, …, 31}.

Conjecture: Functions given in sets of labels {0}, {0, 1}, {0, 1, 2}, {0, 1, 2, 3}, {1, 2, 3, ..., p} are threshold functions.

Suppose f is an n-variable threshold function and $g = fx_{n+1}$. As mentioned earlier, G_f and G_g are isomorphic. The labels of G_g can be obtained easily provided that the labels of G_f are known. If v_i corresponds to **m** and has label l in G_f, then $2l + 1$ is the label of v_i', associated with $\mathbf{m}x_{n+1}$, in G_g.

If f is an (n, k) threshold-function, negating a variable x_i means changing x_i to $\bar{x}_i$ (or $\bar{x}_i$ to x_i) in every minterms of f. If vertex v, associated with minterm **m**, is changed to v' then the label of v' is the summation of the label of v and -2^{i-1} (or 2^{i-1}) if x_i appeared un-complemented (complemented) in **m**. If f is an (n, k) threshold-function, then $\bar{f}$ is an (n, 2^n-k) threshold-function. The labels of $G_{\bar{f}}$ can be obtained by removing the labels of G_f from D.

5. Future Research Topics

The number of graphs is much less than the number of threshold functions they represent. It is possible to generate n-variables threshold functions by generating their graphs. Based on our investigation, graphs of threshold functions satisfy certain patterns, it is possible that they can be generalized into theorems. The following suggestions for future research topic are related to generating more graphs and finding more properties of threshold functions.

1. Finding a systematic method to generate all graphs of threshold functions having n variables, for $n \geq 6$, by generating their graphs.

2. Finding other patterns of graphs of threshold functions that lead to new theorems.

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