The natural beauty of polynomials.

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Abstract.

The introduction of CAS and symbolic packages in mathematical education makes accessible a much wider domain of mathematical applications. Traditional teaching often limits its scope to linear and quadratic behavior, and a typical course ends at conics. Nonlinear phenomena are often not well understood. Symbolic mathematics packages allow extending the mathematical panorama to problems involving polynomials of any degree, and even arbitrary functions, without additional difficulty. Results that looked strange in the quadratic context become natural when considered in more generality. Polynomial phenomena of surprising beauty are encountered when using graphical capabilities of the CAS.
In the presentation we will give graphical examples of how this occurs at all teaching levels, and discuss what effects introduction of a polynomial viewpoint may have on future curricula.

Introduction.

Things do change over a lifetime. In the early sixties I received education in the old secondary school curriculum that followed Greek mathematics. The Bourbaki mathematics reform was still embryonic and the portable, but heavy, calculators with LED screen had not yet arrived. Curriculum used to cover geometry of lines and conics, algebra of linear and quadratic functions, trigonometry in triangles and circles, and the odd inversion for projective geometry purposes. It was all very stable: roots occurred solitarily or in pairs and most curves were smooth and had curvature in only one direction. Secondary curriculum content made you believe that by the end you knew most of tractable mathematics. Logarithms had to do with numerical tables, and higher degree polynomials and exponentials were all intractable by hand computations.
At university it was a shock to find out that not everything in mathematics showed this symmetry or predictability. I learned about complex roots and cusps where a function could not be inverted, random behavior and strange phenomena occurring in differential equations or approximations. I got in the middle of the comprehensive New Math reform that led to an axiomatic approach without computation.
Much later, with the introduction of handheld calculators, the curriculum reform was reversed once more into a real world approach. Problems are practical now, but most of the phenomena studied are still linear or quadratic.
Introduction of CAS did not change the motivation of educators who still offer traditional content using new media. Even today, a lot of teaching content addresses the very smooth results found in situations of first degree (linear algebra, first order differential equations) or second degree (quadratic forms, convexity and curvature, second order differential equations with constant coefficients) in which the number of roots to be considered is either one or two, at worst two complex conjugates. CAS can address new important real world problems exhibiting highly nonlinear behavior. Introduction of CAS therefore necessitates curriculum changes in secondary and tertiary education. It has to include new results and graphics objects that are obtained in more generality. These can be offered early in the curriculum since the approach is highly visual and requires few prerequisites in theoretical knowledge.

What follows indicates one of many possible hiking journeys over a variety of domains through the beautiful scenery of polynomial objects. Our first concern is about accidents that may happen on slopes.

Catastrophes.

Linear functions have constant slope and a ball at rest cannot stay on it unless the slope is zero. Quadratic functions are well known too. We know that a ball can either stay at the stable single minimum if the parabola curves upward, or unstably at the maximum if the parabola curves downward. Just by plotting polynomial functions we can see surprising behavior. A simple example is the following: unless it was sitting precisely at a local maximum, a bead will slide along the plot of a fourth degree polynomial (with positive highest-order coefficient) until it rests at a local minimum. Changing slightly the lower degree terms of the fourth degree polynomial can influence the number of stable minima. The derivative polynomial has roots that can be complex numbers. This can be visualized by plotting in the complex plane.
Such a phenomenon is called a catastrophe in mechanics, and these catastrophes are completely classified. They correspond to physical phenomena and play a role in the qualitative study of differential equations [Thom].

Discrete mathematics

During the last decades many curriculum bodies have emphasized discrete mathematics. Counting and describing discrete phenomena should be in the curriculum of all students. Since the classical treatise of G. Polya on counting [Polya] we know that many counting techniques and solution methods in discrete mathematics involve the construction of generating polynomials. Easy examples are the counting of possible colorings of a map, partitions of a number of objects, derangements where no object stays in its place, and counting the number of possible words in a finite alphabet subject to some rules. The higher-degree generating polynomials solving these problems allow to address much more complicated problems than simple combinatorial techniques can handle. All problems can easily be translated in finding coefficients of a polynomial and the computer performs such computations instantly. The generating polynomials are special cases of power series solutions and the transition to power series is easier if students are already familiar with arbitrary degree polynomials.

Multivariate polynomials and implicit functions.

By Descartes' folium example we know that implicitly defined curves may have singular points and self-intersections that appear or disappear in a fashion similar to the catastrophes [Griffiths]. To see the beauty of these objects it is necessary to have powerful techniques for implicit plotting, and these have improved in recent versions of CAS. Some of these techniques use information about the nature of the critical points. If the implicit functions are level curves of a polynomial in two variables, flooding the surface as is done in Morse theory gives all required information.

Skewed logistic curve and nonlinear statistics.

Many articles discuss the qualitative behavior of the solution of the logistic equation with standard right-hand side $k\, x\, (c - x)$ for different values of k while the capacity of the system, denoted c, is one (one hundred percent). Replacing this parabola by a higher degree curve still zero at the value c

and the origin may introduce different behavior, and the instability in the discrete system occurs at different branching points.
This may be important since marketing models may differ slightly from the classical logistic model. The sales statistics for mobile phones are an important example for explaining the telecom crisis. Suppose sales of such items increase such that the differences over time intervals, divided by the total number sold, is almost linear. Then it will satisfy the logistic equation and the total capacity can be computed from the linear approximation. If however initial sales after dividing do not correspond to a linear model, then we have a polynomial right hand side that is not quadratic. Making reliable sales projections is vital for all manufacturers and not all economic theories can be modeled by polynomials of the second degree.
More generally, strange behavior is spotted in many nonlinear differential equations. These are much easier to visualize using a CAS and patterns in the behavior can be conjectured from these observations.

Approximating data.

Every CAS can approximate a list of numerical data by (a combination) of given functions, and linear regression is the simplest example of this technique. This approximation converts graphical input of data into a tractable formula. In most cases one wants to obtain a polynomial of given degree. It takes some practice to guess what the required degree will be to have a good approximation. Some distributions of values will not allow good approximations by polynomials: distributions looking periodic will have to be approximated by trigonometric polynomials and humps will require exponentials with polynomials in the exponent. It takes some manipulation to achieve such approximations but the technique is more reliable than built-in statistical methods.

Approximating curves.

Approximating curves joining points in the plane is traditionally done by Bézier curves. These are already studied in some curricula and implemented in most software packages for drawing. Their behavior may be unstable in the sense that slightly moving some of the points may create unwanted cusps. Using higher order polynomials approximating simultaneously more points reduces the number of cusps in a drawing. Again, sets of points are compressed into a single polynomial expression that can next be used in computations.
Replacing the polar coordinates of a set of points in a plane by pitch and volume describes sounds. Approximating by smooth curves results in more melodious sounds.

Complex mappings.

A special case of the Riemann mapping theorem states that a closed nonintersecting smooth curve in the (complex) plane is always the image of the unit circle by an analytic function on the unit disk. Since the proof of this classical theorem is not constructive, it is not easy to compute the function that defines the right map. Another approach would be to try and find which curves are approximated by a polynomial (or a rational function with only a fixed power of the variable in the denominator). The variety of shapes is much larger than one expects [Cnop].
This can be applied to building tunnels. Engineering tradition makes tunnel sections circular or (part of a) conic. Some prefabricated sections have the shape of an ovoid, and connecting tunnel sections takes up a lot of space and material, without allowing smooth flows. A variety of cross-sections and three-dimensional layouts needs to be studied to optimize tunnel design. It is possible to approximate arbitrary cross-sections by (complex) polynomials and use these in stress analysis.

Tori and knots.

Tubular layouts [Weisstein] are pleasant to the eye and intertwined tubes are therefore used in advertising for their sheer beauty. Knots have been classified by their polynomial invariants [Adams] and this resulted in mathematical theories yielding applications in physics and other sciences.
It is interesting to see which tubes can be obtained replacing circles by polynomials maps of circles in the equation of a torus. Experimenting reveals surprising new designs. Some of these show cusps and the journey ends where we started.

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