How to teach Mathematics in showing all the hidden stages of a true research: examples with Cabri

Jean-Jacques DAHAN jjdahan@wanadoo.fr
IREM de Toulouse http://www.irem.ups-tlse.fr
(Institute of Research for the Teaching of Mathematics in Toulouse)

1. Abstract

How can we expect that our students could have an idea of the practicing of research of problems when we offer them, especially in Math books, only linear and dead demonstrations?

Only deductions from the hypothesis to the conclusion without any descriptions of the experiences we have done, the true and false conjectures we have imagined, the tests we have done to choose the good one.

Never an analogy, never a generalisation of a particularisation, never the reason of a conjecture, never inductive or plausible reasoning as Polya said.

I will show on several examples using Cabri how to present important results in Math lessons, detailing all the different phases of the research (all this phases, described by Polya, were and are used by mathematicians : Euler in the past and researchers using informatic actually):

How to present Pythagore theorem as Euclid would have like to do ?

How to discover and introduce the antiderivative functions using Euler's method ?

How to try to solve a special black box problem with an experimental reasoning leading to the logical reasoning of the proof ?

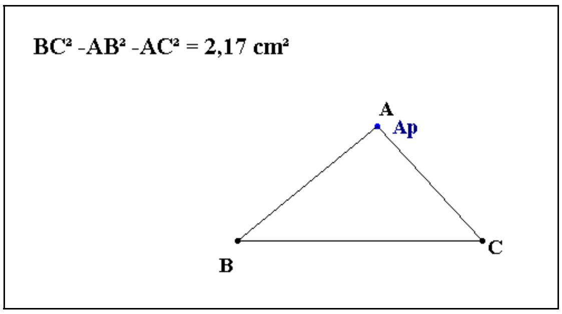
2. An experimental way leading to the demonstration of Pythagore theorem

2.1. Conjecturing a property (p)

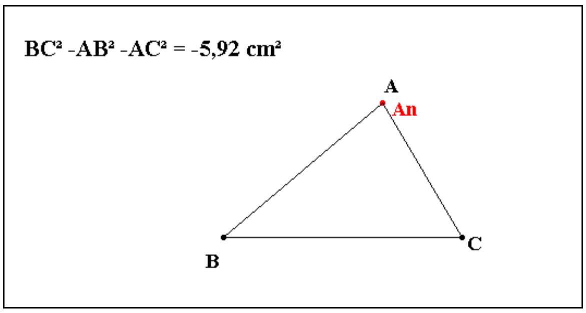
Problem: B and C are 2 given points: are we able to find **the positions of point A** on the plan so that $BC^2 = AB^2 + AC^2$?

Construction: we lead these constructions in order that:

A blue point Ap appears on point A when: $BC^2 - AB^2 - AC^2 \geq 0$



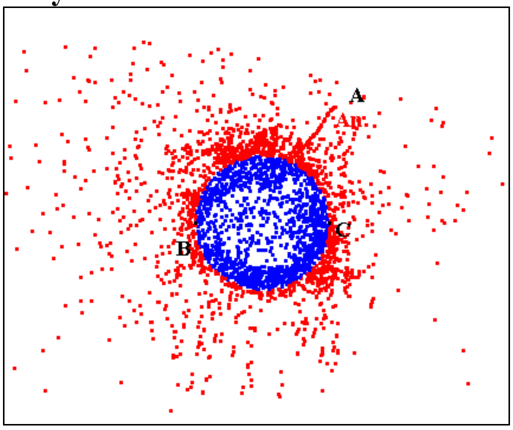
A red point An appears on point A when: $BC^2 - AB^2 - AC^2 \leq 0$



Experimentation:

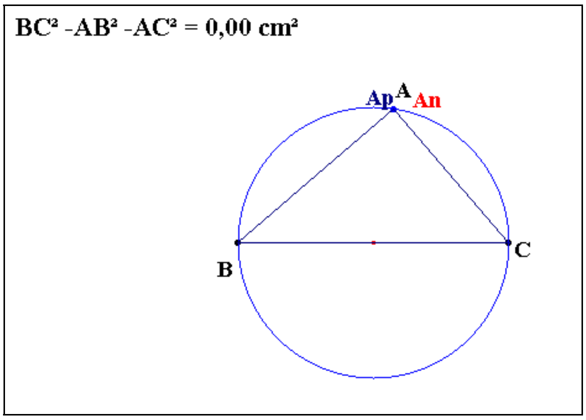
After that, we drag point A everywhere on the plan represented with Cabri after putting "ON" the traces of points Ap and An. We obtain this beautiful screen looking like the sun during an eclipse.

Analysis:



Points An **seem** to be located in the interior of a disc having BC as a diameter and points Ap **seem** to be located in the exterior of the same disc.

Validation: this observations continue to be done when we change the positions of points B and C and we experiment again.
Conjecture: it seems that the points we are searching for are the points on the circle having BC as a diameter, so that triangle ABC is a rectangle triangle having its right angle in A
Verification:
Let us redefine point A on this particular circle and Cabri says us the property $BC^2 = AB^2 + AC^2$ is true. We can remark that we are in the case where points An and Ap exist simultaneously.



About the answer we can give to our problem: we are not able to say that we know where are the points A on the plan so that $BC^2 = AB^2 + AC^2$ is true, but in the cabri microworld we have proven using analysis and synthesis reasoning that when the triangle ABC is rectangle in A this property is true and when this property is true, triangle ABC are rectangle in A (when we stay on the screen-page of Cabri (which represents a little part of the plan).
Cabri-proof and math-proof
This proof is a Cabri-proof : the way we followed is the experimental way of a searcher with each previous stage, but this demonstration is not a complete mathematic demonstration because we have treated this problem as a combinatory problem, a discrete problem or this problem must be treated on a non dicrete set : the plan of geometry!!!!!

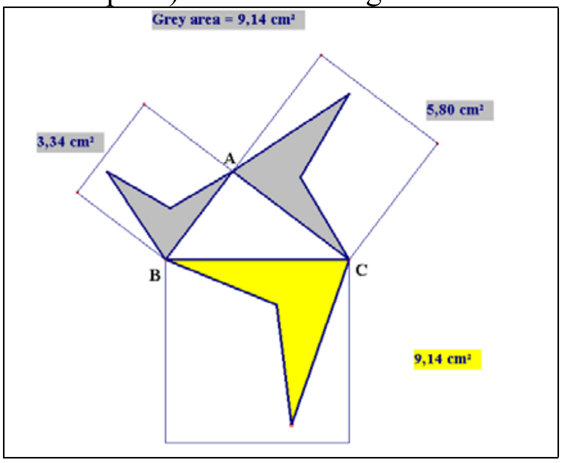
Now we will see, that from a very long time, mathematicians have found how to use an experimental way to math-prove this property. The story I will tell you is the translation of the way shown by Polya in his famous book "The plausible reasoning", fifty years ago. After this story told with the help of Cabri I will tell you who can be considered as the father of this solution.

2.2. Extending this conjecture for a new property (ep)

(Exploring possible generalisations and discovering them)

As we are not able immediately to give a correct demonstration, we will use a classical heuristic way to go ahead.

Is the theorem we want to prove, a particular case of a more general theorem; here, we can try to look on Cabri if the area of a polygon (not especially a square) having BC as one side is equal to the sum of the areas of the two similar polygons (having the same shape : squares when the first one is also a square) the firs thaving AB as one side and the second having AC as one side.



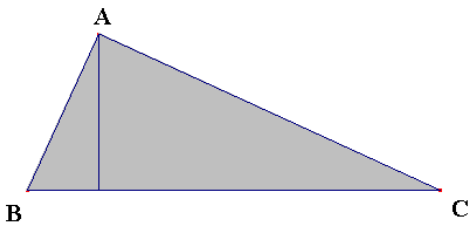
If we break the square having BC as one side, the yellow polygon we obtain below BC, lets us answer "YES" to this generalisation.

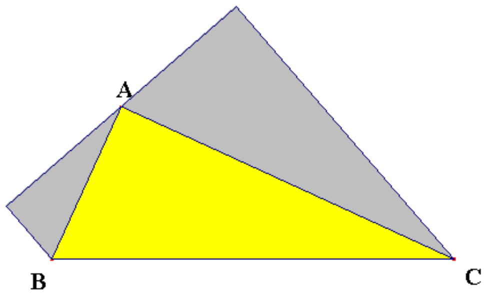
In reality:

We can say that, if the theorem with squares is true, it is **plausible** that the theorem with similar polygons is also true.

2.3. Remarking the truth of a particular case (ep1) of this general conjecture

Let us continue this analysis; if the theorem with similar polygons is true, it will be necessarely true in a particular case of similar polygons and also in the particular case of similar triangles.

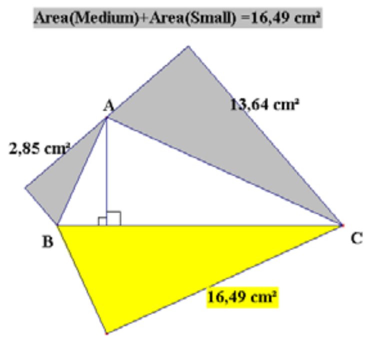




We will tranform each ot this 2 similar rectangle triangles with a reflection with respect to AB and AC to obtain the right figure

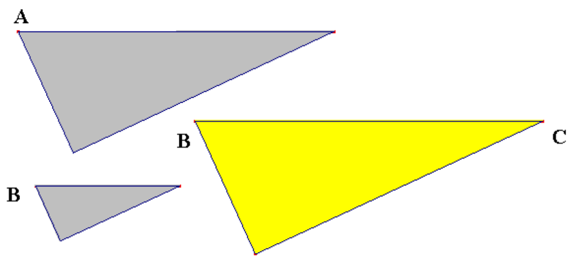
Now, we reflect the yellow one (triangle ABC) with respect to BC and we must verify that our second theorem is also **true in the cabri microworld**.

Demonstration : We are happy that this theorem is considered as exact for Cabri in this case because it is a case we are sure about the relation: in fact the proof is given when we remark that the area of the yellow triangle is evidently the sum of the areas of the grey ones.

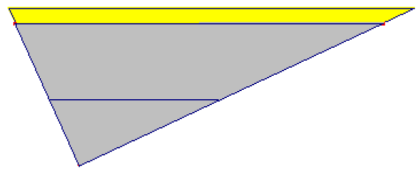


If we had a doubt about similarity of the 3 triangles, we can copy them and past them in order to drag them far from our initial figure and compare them.

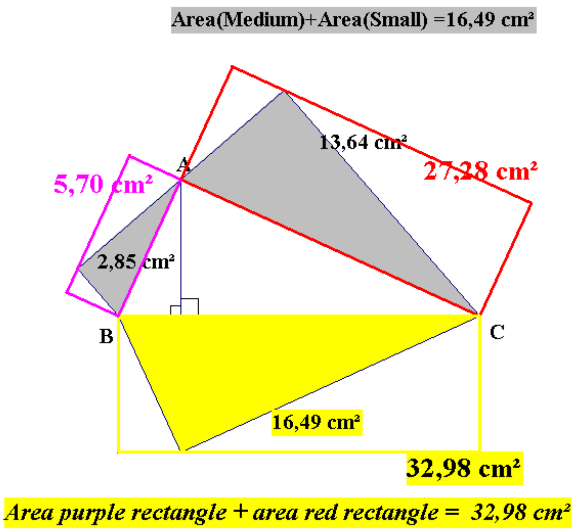
Here we use the unusal tool, “rotate” (and also the tool “reflection”), to put them in this position:



At last, we drag the grey triangles (the small and the medium) on the yellow one (the big one) to get this result that can be read as: “the 3 triangles are similar”!



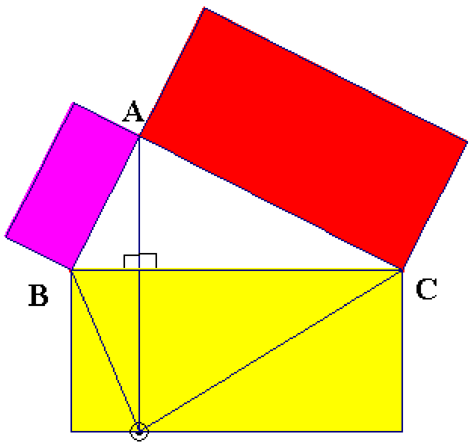
2.4. Prooving the truth of a second particular case (ep2) of this general conjecture starting from the truth of ep1

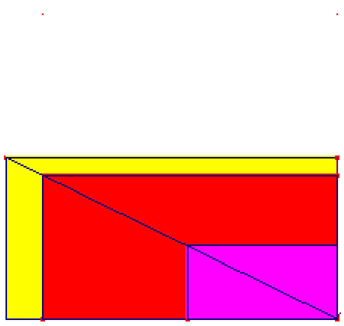


After the construction of these 3 rectangles, Cabri says us that the relation of our conjecture is verified for these 3 special polygons. We enjoy because:

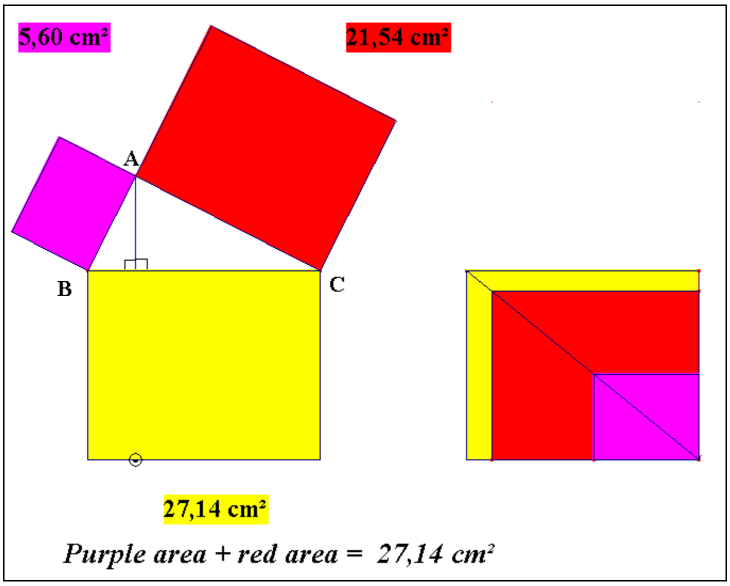
Demonstration: as each rectangle is the double of the triangle used to construct it, as the relation is true for the 3 rectangles, the relation will be true for these 3 rectangles (property ep2)

It is also possible to verify with Cabri that, in this case, the rectangles are really similar, as done above.



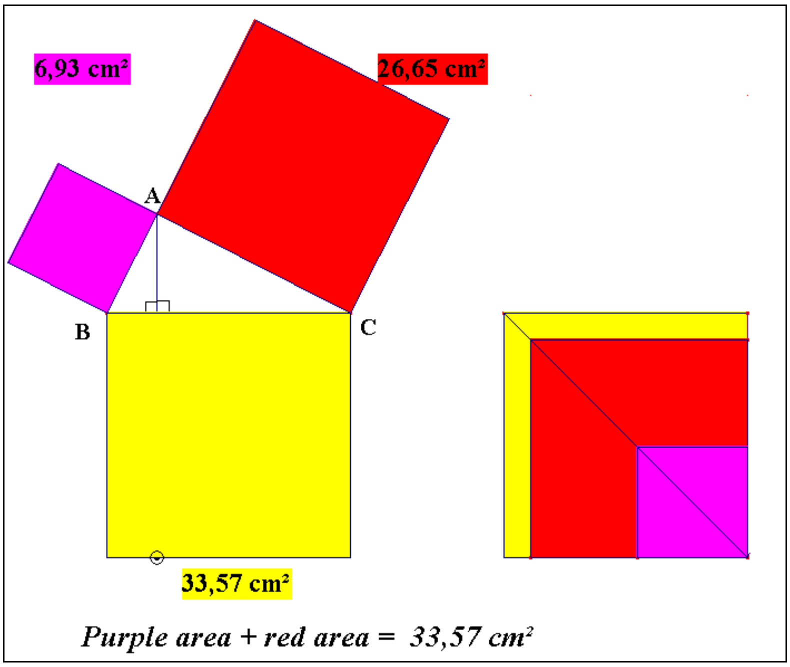


2.5. Prooving the truth of property (p) of this general conjecture starting from the truth of ep2



If we drag the black circled point in order to increase the width of the yellow rectangle (which is located below BC) so that the purple and the red ones (which are located above BC) become similar to the new yellow, Cabri says us that the conjecture continues to be verified.

We can drag this point until the yellow rectangle becomes a square: let us notice that at the same time the other rectangles become also squares. At this state, Cabri continues to tell us that the conjecture is verified:

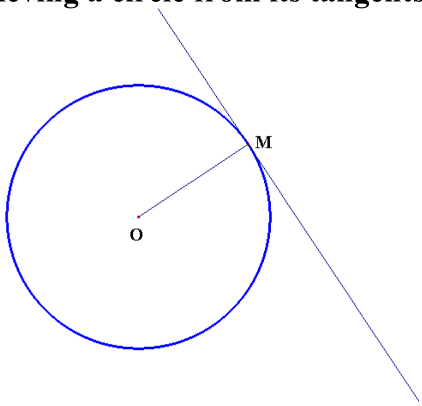


It is the one we have at the beginning of our reasoning but here there is a great difference, due to these following experiments we arrive to the demonstration we were searching for.

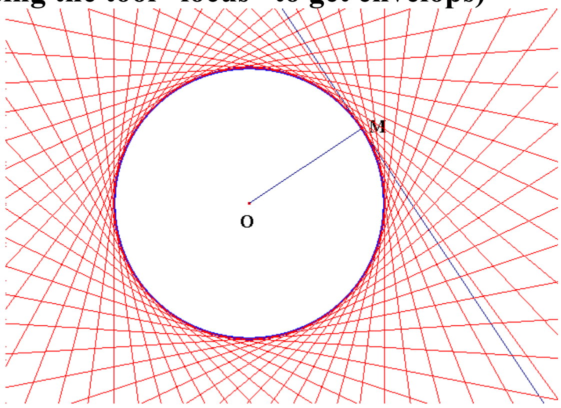
Demonstration: starting for the property (ep2), we are sure that our relation is verified for such rectangles; for each of these rectangles, we kept one dimension and we change the other; if the width of the modified rectangle is got in multiplying the initial one with a coefficient k, the other modified dimensions are obtained by the same operation so that the 3 areas are got from the first 3 areas by multiplying by this number k and the relation is always true even when the rectangles are squares so Pythagore theorem is at last proven

3. An informatic modelisation of the tangent line for a circle and a curve of a function

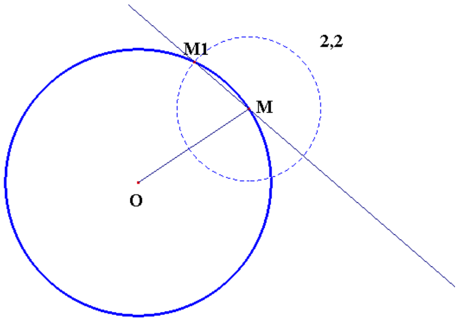
3.1. Retrieving a circle from its tangents lines (using the tool “locus” to get envelops)



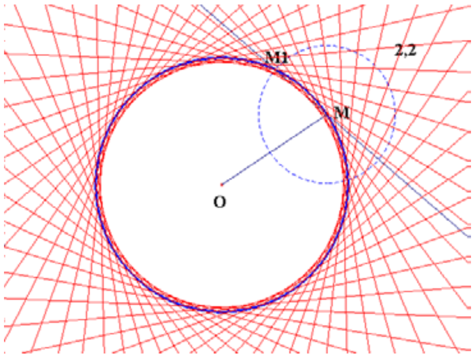
The tangent line on M to the blue circle is drawn using its geometric caracteristic property: this line is perpendicular in M to the segment [OM]



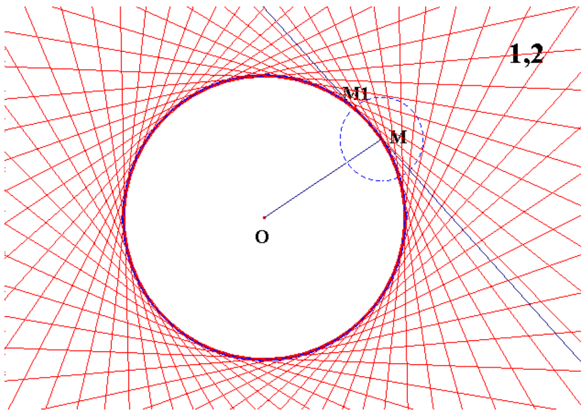
When we ask Cabri for giving us the locus ot this tangent line, we get a family of lines enveloping our circle



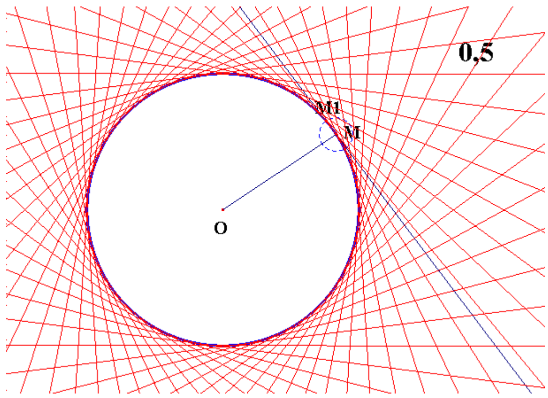
As we want to draw the tangent line in M without using neither point O nor any point out of the circle, we experiment to draw a line passing through M and another point M1 on the circle such as MM1 = 2.2



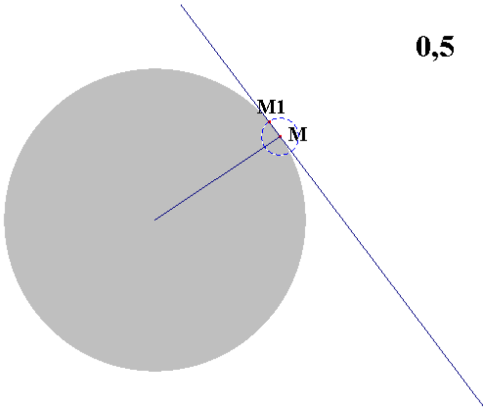
If we ask Cabri for giving us the locus ot this secant line, this locus seems to envelop a circle near our circle.



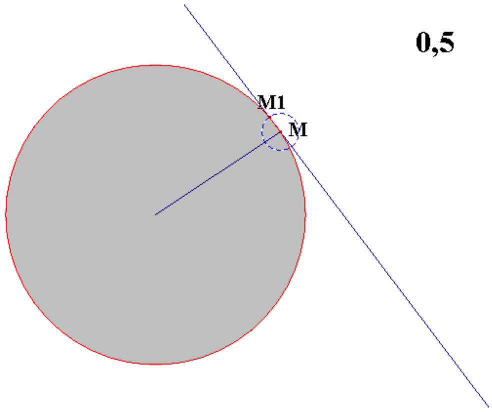
If we discrease the step-number (between M and M1), the envelop comes nearer from our circle.



Here for a step-number of 0.5, it seems we have coincidence between our circle and the envelop of this secant line.



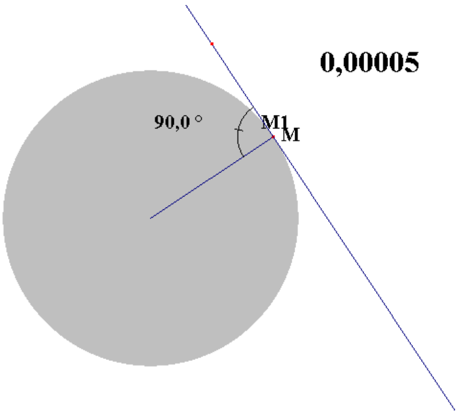
Our circle is filled in grey



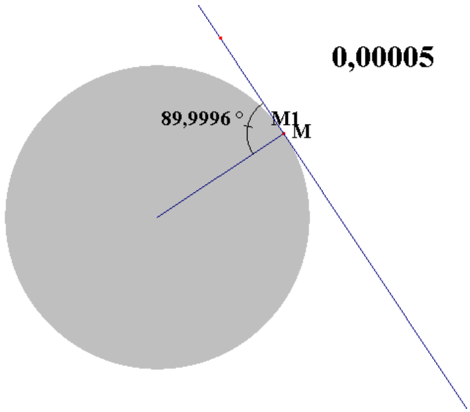
Here we have asked Cabri for giving us the locus of the secant line that appears in red and circling exactly our circle

3.2. Modelizing the tangent line with a special secant line

We will now choose as a model of drawing a tangent in a point M of a circle, **the secant line passing through the point M and another M1 of the circle with MM1 =** ε **(with a small value for** ε, here for example 0.00005)



Here is the tangent line got when using this model. It seems to be a suitable model because the angle between OM and this line is measured by Cabri with 90°.



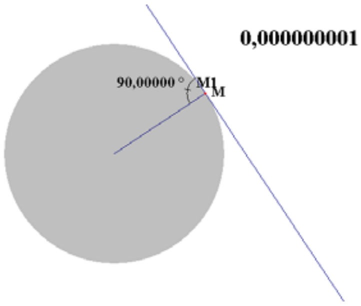
With a better accuracy of the measurement we can see that this angle is not exactly a square angle but almost a square one with a good accuracy

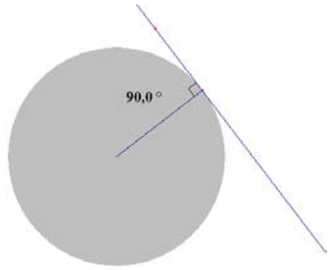
3.3. Definition of a tangent line in Cabri micro-world

We will decide to accept this model and will use it in the Cabri microworld as a definition

The tangent line on a point M of a curve is the secant line passing through the point M and another M1 of the circle with MM1 = ε **(with a small value for** ε**)**

To show that this definition is plausible for other curves, we will use this construction for several classical curves and remark that we do not reach an impossibility: having not the opportunity of refutation, we could continue to use this definition as long as we will not have any contradiction.



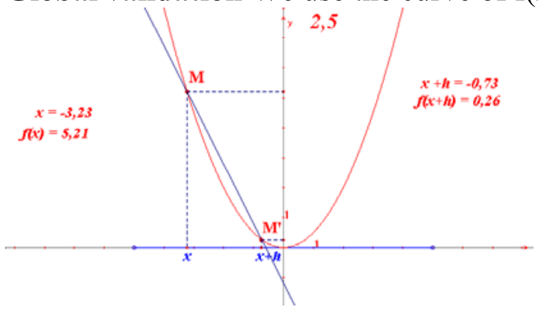


We can remark that the construction using this model gives here a good representation of the tangent line according to the geometric property

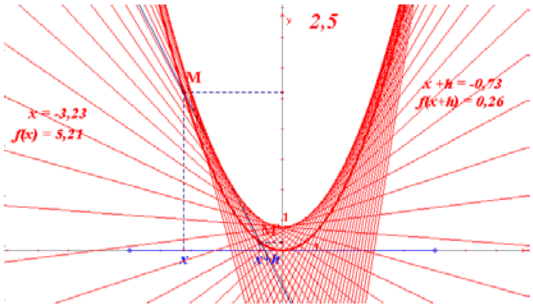
We cannot see a difference with the tangent line got with the theoric and exact model

3.4. Validation of this model with a parabola in Cabri micro-world

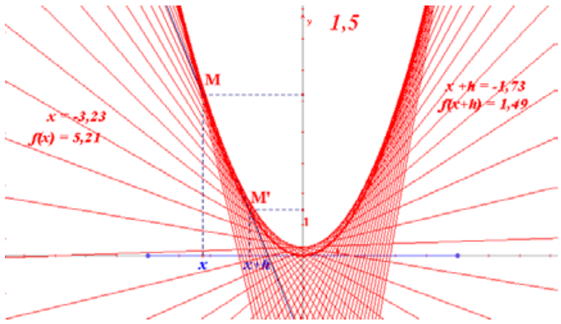
Global validation We use the curve of $f(x) = 0.5x^2$



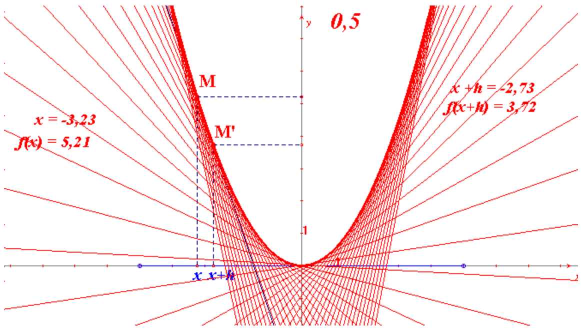
Here we can command the difference between the abscissa of M' and the abscissa of M (it is h equal here to 2.5; it is the h value)



It is normal to verify that MM' cannot be a good representation of the tangent line: the envelop of this line is not located near our curve.

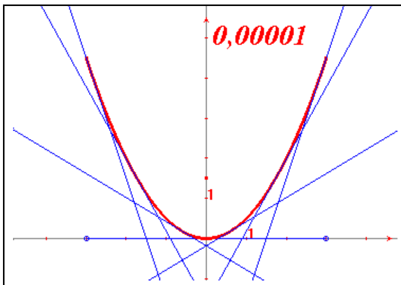
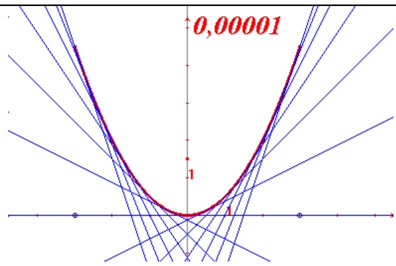
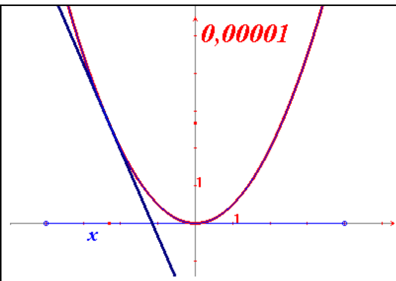


h = 1.5

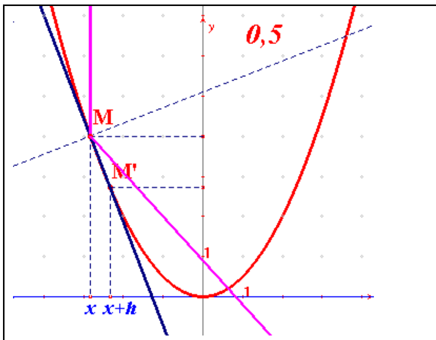


h = 0.5; the envelop already seems to be in coincidence with our parabola.

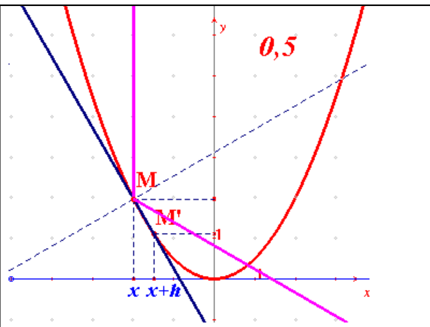
Local validation

 The secant lines seem to represent the tangent lines	 Here also	 So we get this drawing when we use the model 1

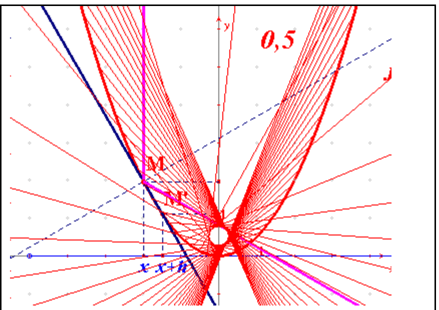
An other global validation using a geometric and optical property of the focus



Construction of the reflected ray of an incident ray parrallel to the ordinate axis

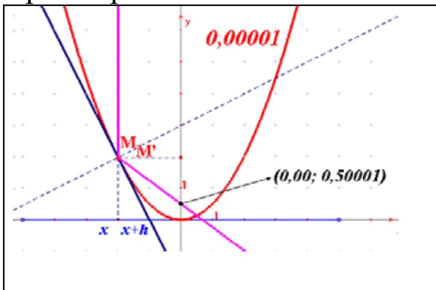


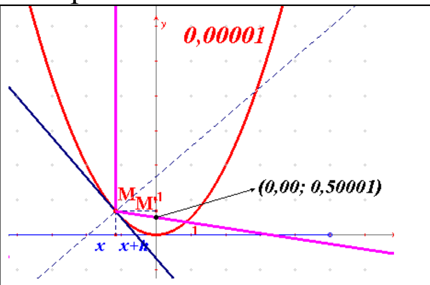
When we drag the point M on our parabola, the reflected ray doesn't intersect the ordinate axis at the same point.

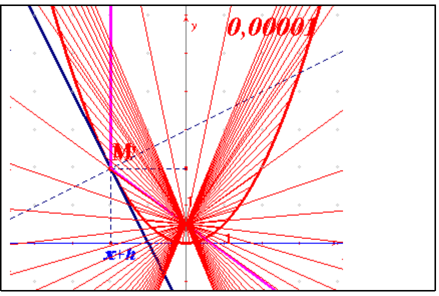


When we look at the locus of the reflected rays, we can confirm this remark.

The same experiences with a small value of ε, lead us to verify that these rays are convergent on a special point which is the focus of our parabola.







So from that moment we will accept the model 1 until we meet a contradiction in using it

4. A model for drawing curves of antiderivative functions

The problem, now is to create a model for drawing a curve, knowing on each of its point the tangent line and in fact the slope of this tangent line. We will use an analogy with the model for drawing tangent line. As done in the previous part, we will give an hypothesis we will verify on a circle, we will validate this hypothesis in an other particular case (antiderivative of the $x \longrightarrow 2x$ function)

4.1. From a model to another in using analogy

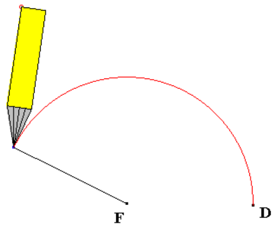
For the previous model, the **tangent line** on a point M of a curve is modelised with a **secant line** passing through M and a second point M' of the same curve such as MM' has a small value (10^{-6})

For this new model, we will test the hypothesis telling us that :

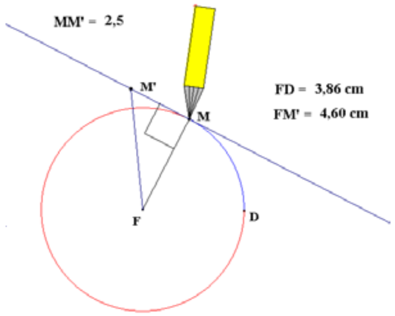
When we are searching for a curve knowing its tangent line in each of its point, if it exists such a curve passing through a point M, this curve will also pass through an other point M' of the tangent line on M with MM' having a small value (10^{-6})

4.2. An algorithm to draw a circle point after point (using the slope of the tangent line in each point)

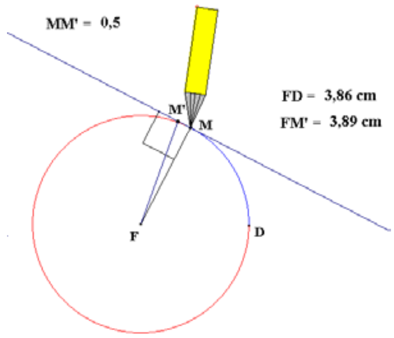
In the previous part we have used 2 points of a curve to draw a tangent line; using analogy we will try to draw the curve, point after point with the help of one point and the tangent line which will be given here:



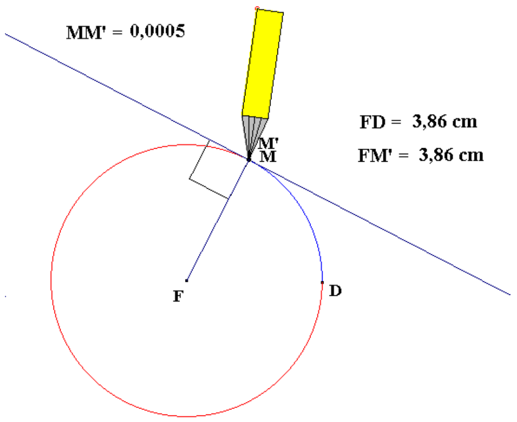
The classical way of drawing a circle with a compass using the center point



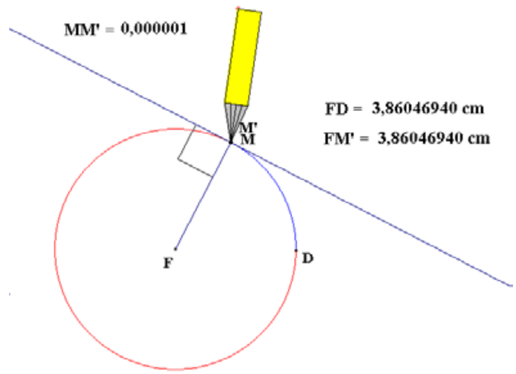
M’ is a point that can slide on the tangent line; The distance MM’ can be modified



When MM’ = 0.5, on this figure, M’ seems to belong to the circle; it is false but FM’ is close to FD

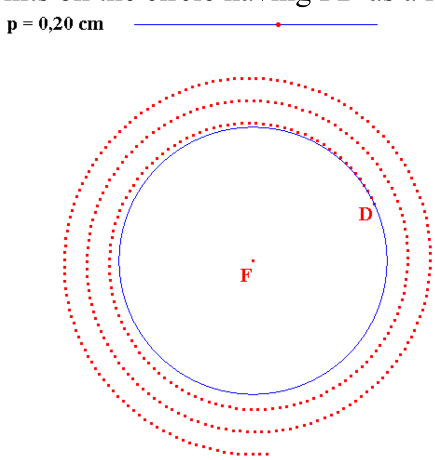


If we choose MM’ = 0.0005, Cabri gives us FD = FM’ and we can imagine than M’ can be considered as another point of the circle

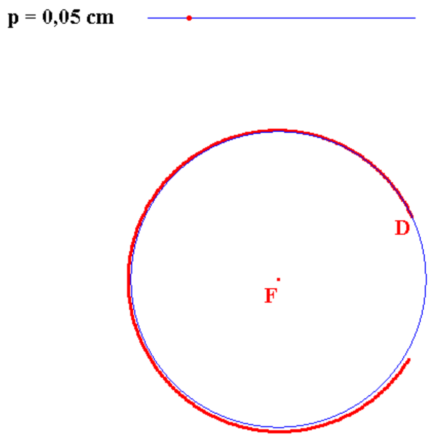


For MM’ = 0.000001, if we increase the accuracy of the measurement, Cabri continues to give us FD = FM’.

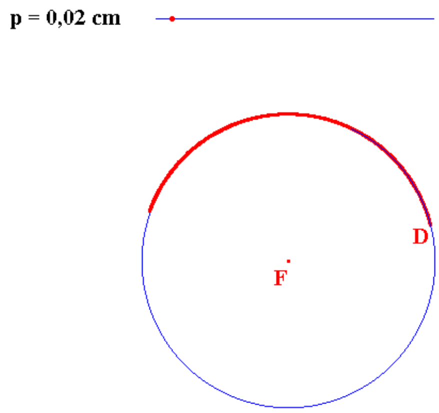
When we repeat this construction in an algorithmic way, starting from D do we obtain a sequence of points on the circle having FD as a ray?



For a step p = 0.20, this algorithm does not give the expected circle but a divergent spiral



When p is smaller the spiral approaches our circle.



Even this construction does not permit us to discrease the value of p as much as we would want, we can verify (conjecture…) that the sequence of points we have got seems to be superposed to the circle we wanted to draw. So we expect that with a p value of 10^{-6}, we would get a perfect representation of our circle with this model.

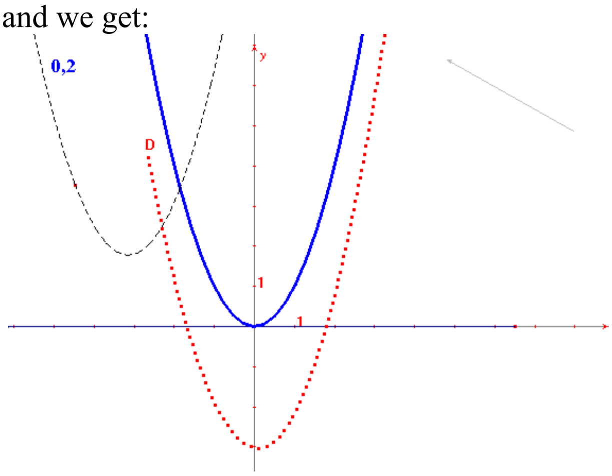
In this case, we have the impression that our process permits us to draw correctly our circle.

4.3. An algorithm to draw a curve of a function point after point knowing its derivative function (Euler method)

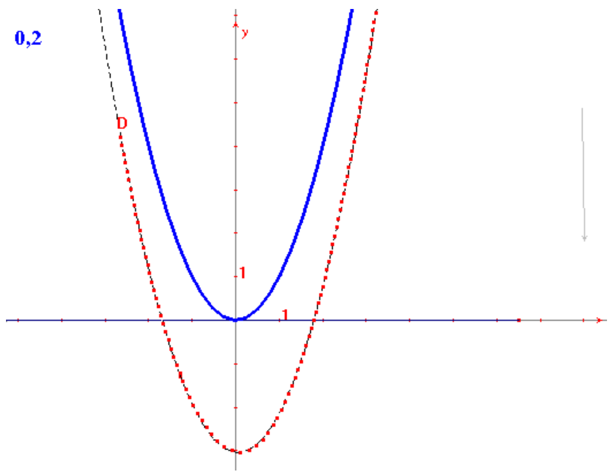
We will apply this model to the $x \longrightarrow 2x$ function and we get:

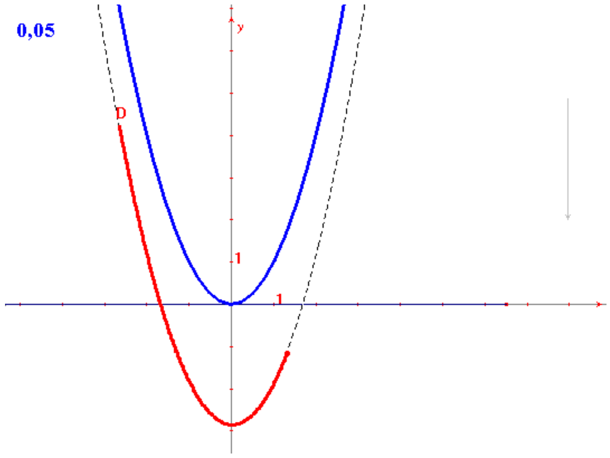
Starting for a point D, we have built a red chain, each point M' having being built from the previous one M such as the slope of line MM' is equal to 2.xM and MM' = 0.2.

It seems that the obtained curve looks like the curve of the square function. So we have drawn this square function in blue (the thick one) and the translated of this curve (the dotted one) with respect to the blue right vector (this vector can be modified).

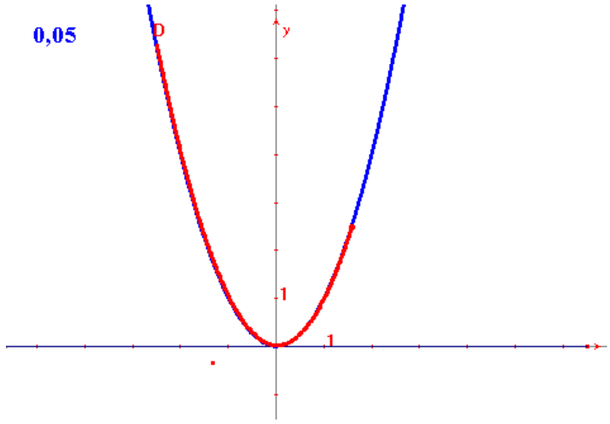


Here we have dragged the second extremity of our vector until the translated parabola recovers our chain: it is possible with a practically vertical vector. So, if our chain represents a sequence of points of the curve of one antiderivative function of the linear 2x function, this antiderivative function should be the square function plus a constant function (true because, even we drag D on the plan, the chain stays practically symetric with respect to the ordinate axis).





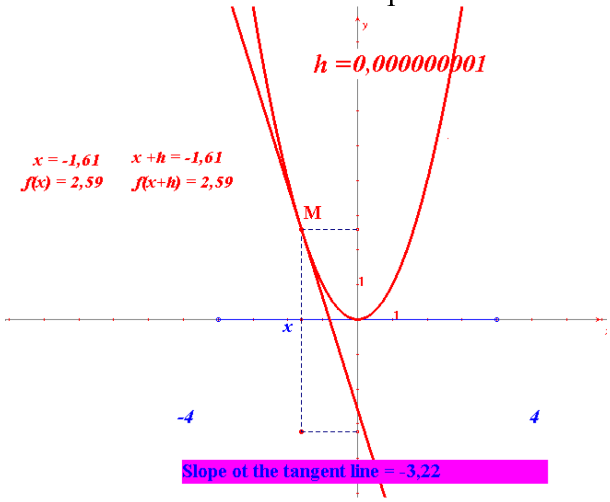
The coincidence of our chain with a x^2+k curve is better and better when the step of the chain discreases



Here we have verified that we can obtain the square function as a particular antiderivative function as we get the coincidence when D is lied on the square curve.

If this model for drawing was correct, the curves of antiderivative functions of f(x) = 2x might be the curves of F(x) = x² +k

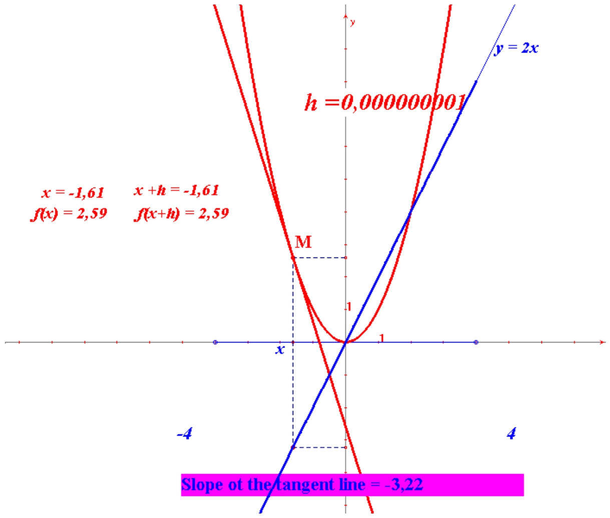
Verification: we will use the previous model to verify that the derivative function of x² is really 2x



We have used the model 1 to construct on a Cabri screen the tangent line (as a secant line between M and an other point M' of the same curve with MM'= 0.000000001) on a generic point of the curve of the square function s(x)= x².

We have measured the slope of this tangent line, that is to say f'(xM)

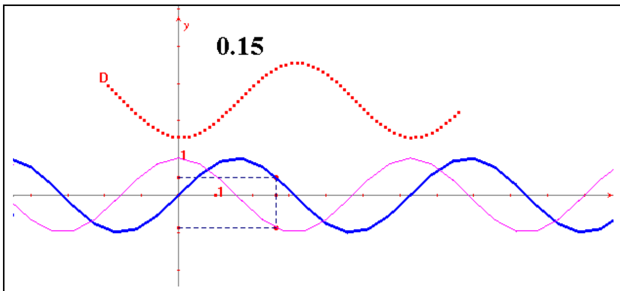
We have plot the point (xM ; f'(xM))



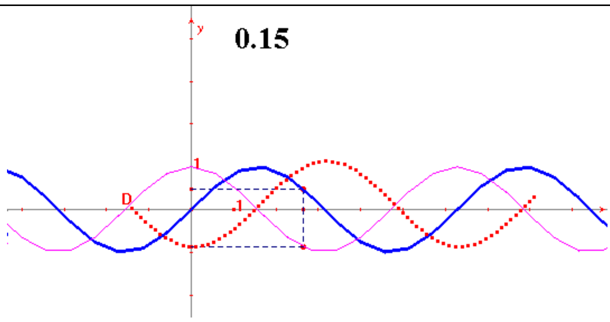
The locus of this point seems to be a segment as we have drawn the curve of the square function only on the interval [-4 ; 4]. To confirm that, we have drawn a line passing through 2 points of this locus, we have the coincidence and as Cabri gives us y = 2x for the equation of this line we can conclude that f'(x) = 2x, so that, we can not refute the validity of the model 2 we are testing.

4.3. Examples of other antiderivative functions built with this model 2

4.3.1. Antiderivative functions of the sine function



In this part, we have used this method, to determine antiderivative functions of the blue and thick sine function (in purple, we have drawn the cosine curve)

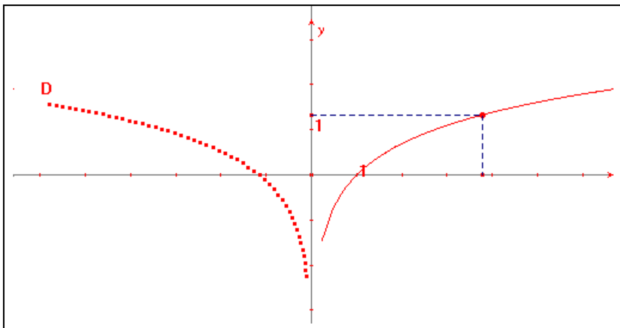


It is possible below to drag point D in order to conjecture that an antiderivative function of the sine function is minus cosine.

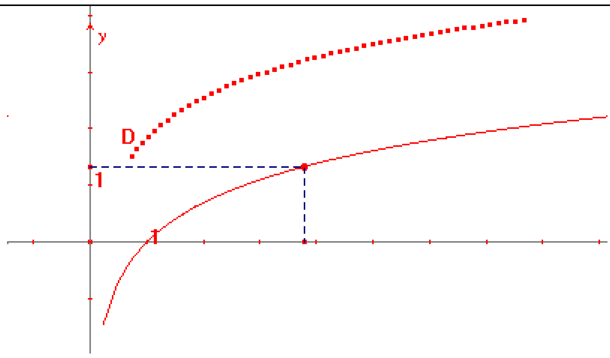
If we drag point D randomly, letting the trace of the red chain, we can conjecture that the antiderivative functions of sin(x) are –cos(x) + k.

4.3.2. Antiderivative functions of the inverse function

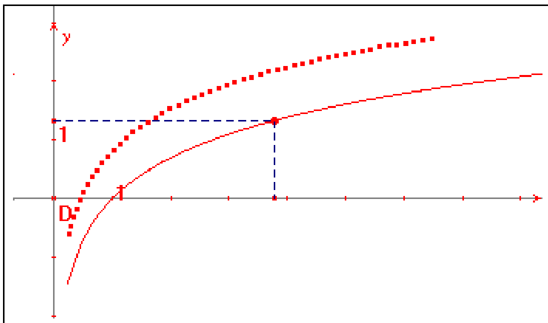
We have drawn the curve of the ln function in red



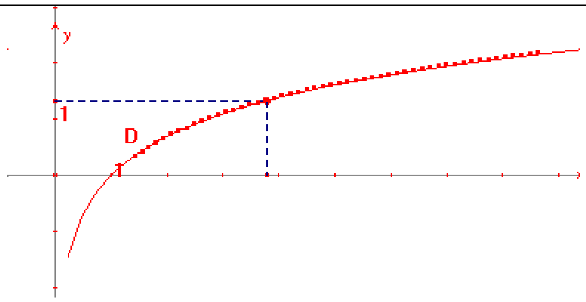
The chain starting at D represents a part of the curve of an antiderivative function of the inverse function on the interval]-∞ ; 0[



The chain starting at D represents a part of the curve of an antiderivative function of the inverse function on the interval] 0 ; ∞[



Here is another



And here is one that confirms our model is a good one because our chain can be superposed to the ln function.

Other experimentations can convince us that other ones we have found are given with the formulas ln(x) + k on] 0 ; ∞[or ln(-x) + k on]-∞ ; 0[.

5. An example of black box: how to solve it?

We have hidden a transformation that transforms point M in point M’ and we have to guess the nature of this transformation; we have not other possibility than experimenting.

Experimentation and conjecture

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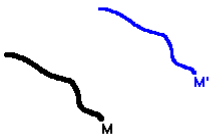
<tr>
<td>
</td>
<td>
</td>
<td>We have put “on” the traces of M and M’ and we have dragged M. The first figure makes us conjecture that it is probably a translation, that is confirmed with the second figure.</td>
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</table>

Plausibility of the conjecture

<table>
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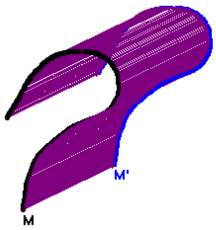
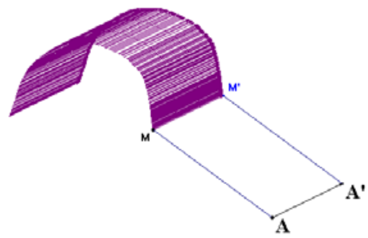
<td>

Another way to validate this conjecture</td>
<td>

Here, A is a given point and A’ its translated with respect to vector MM’. We can remark that even we move point M, vector AA’ does not change. So our conjecture becomes more plausible</td>
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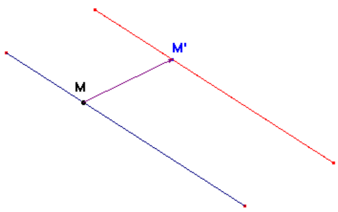
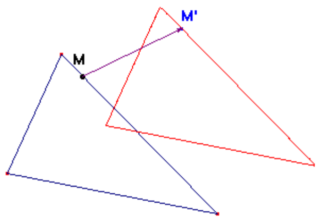
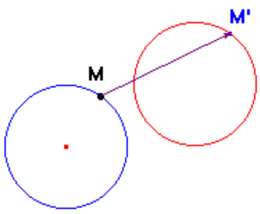
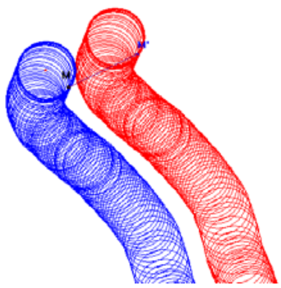
Analytic validation



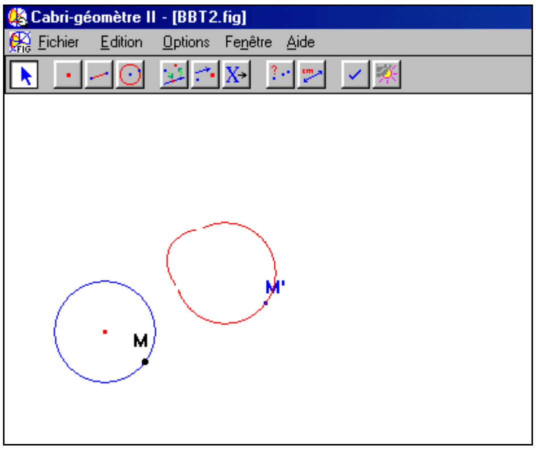


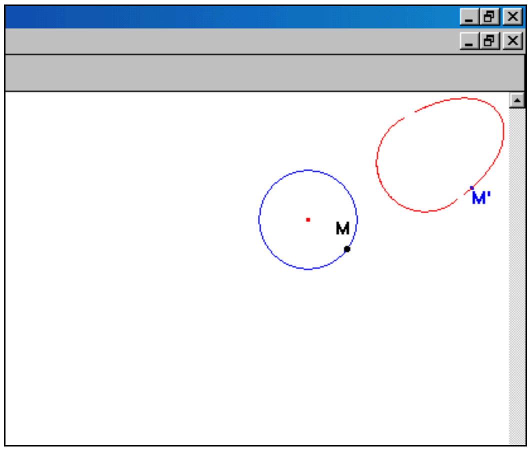


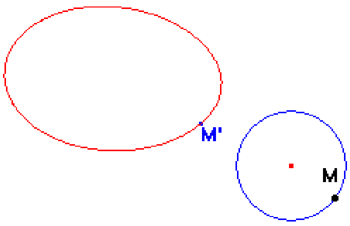


The images of classical configurations are, as we can remark above, as they must be with a translation.

Pay attention! We do not say that the proposed transformation is really a translation. But if we have to work with it, we will do as if it was such a transformation, until we observe what you can observe on the figures below:







If we explore another part of the plan we get this surprising image that obliges us after the 2 previous ones to refutate our conjecture. What is the nature of this transformation which was locally a translation?
I hope that after you read this paper you would want to experiment with Cabri to try and find a model nearer than the model of the translation we have found so easily and too quickly.

6. Conclusion

Starting from nothing to experiment with Cabri in order to solve a problem is not reasonable! Knowledges are necessary. In the Cabri microworld, we can create models, starting from our technical and mathematical knowledges : we experiment to test them, we experiment to confirm them, we experiment to validate them, we can also experiment to refutate them. We have seen that we can experiment to find demonstration of theorems conjectured with experiences also.
I think for the future that it will be necessary to introduce all these stages of math activity in our courses. Our students will get a conception of mathematics closer to the research activity than the formalised way they have used “classically”.