Automatically Solving Semi-algebraic Systems

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Abstract Three MAPLE-programs are demonstrated here for solving semi-algebraic systems at three levels, respectively, in fully automatic mode.

1 Introduction

By semi-algebraic system [2, 6], we mean a system consisting of polynomial equations, inequations and inequalities. In detail, we call

$$\begin{cases} p_1(x_1, x_2, ..., x_s) = 0, \\ p_2(x_1, x_2, ..., x_s) = 0, \\ \quad \cdots\cdots, \\ p_n(x_1, x_2, ..., x_s) = 0, \\ g_1(x_1, x_2, ..., x_s) \geq 0, \ \cdots, \ g_r(x_1, x_2, ..., x_s) \geq 0, \\ h_1(x_1, x_2, ..., x_s) > 0, \ \cdots, \ h_t(x_1, x_2, ..., x_s) > 0, \\ q_1(x_1, x_2, ..., x_s) \neq 0, \ \cdots, \ q_m(x_1, x_2, ..., x_s) \neq 0, \end{cases} \tag{1}$$

a *semi-algebraic system*, where $p_i (1 \leq i \leq n)$, $g_j (1 \leq j \leq r)$, $h_k (1 \leq k \leq t)$, $q_l (1 \leq l \leq m)$ are real polynomials in $x_1, ..., x_s$ with rational coefficients.

Usually, we "solve" semi-algebraic systems in the sense of three levels:

- In the case of $n \geq s$, it is expected that system (1) has finitely many solutions, and thus the numerical solutions are required. For example, find all the non-negative zeros of the following system:

$$\begin{cases} p_1 = x_1(2 - x_1 - y_1) + \dfrac{x_2}{2} - \dfrac{x_1}{2} = 0, \\ p_2 = x_2(2 - x_2 - y_2) + \dfrac{x_1}{2} - \dfrac{x_2}{2} = 0, \\ p_3 = y_1(5 - x_1 - 2\,y_1) + \dfrac{y_2}{2} - \dfrac{y_1}{2} = 0, \\ p_4 = y_2(\dfrac{3}{2} - x_2 - 2\,y_2) + \dfrac{y_2}{2} - \dfrac{y_1}{2} = 0. \end{cases} \tag{2}$$

- In the case of $n < s$, usually, system (1) has manifold solutions with positive dimensions in complex field, it is asked whether a real solution exists.

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- Furthermore, in the latter case, $n < s$, frequently we regard n elements as *dependent variables* and others as *parameters*, say, $x_1, x_2, ..., x_n$ and $u_1, u_2, ..., u_d$, respectively, write system (1) as

$$\begin{cases} p_1(u_1, ..., u_d,\ x_1, ..., x_n) = 0, \\ p_2(u_1, ..., u_d,\ x_1, ..., x_n) = 0, \\ \quad \cdots\cdots, \\ p_n(u_1, ..., u_d,\ x_1, ..., x_n) = 0, \\ g_1(u_1, ..., u_d,\ x_1, ..., x_n) \geq 0,\ \cdots,\ g_r(u_1, ..., u_d,\ x_1, ..., x_n) \geq 0, \\ h_1(u_1, ..., u_d,\ x_1, ..., x_n) > 0,\ \cdots,\ h_t(u_1, ..., u_d,\ x_1, ..., x_n) > 0, \\ q_1(u_1, ..., u_d,\ x_1, ..., x_n) \neq 0,\ \cdots,\ q_m(u_1, ..., u_d,\ x_1, ..., x_n) \neq 0, \end{cases} \tag{3}$$

  and ask to find the condition on parameters $u_1, ..., u_d$ such that system (3) has exactly a specified number of distinct solutions. For example, we may ask the condition on a, b such that the following system has exactly 1, 2, 3 or 4 distinct solutions.

$$\begin{cases} p_1 = x^2 + y^2 - xy - 1 = 0, \\ p_2 = y^2 + z^2 - yz - a^2 = 0, \\ p_3 = z^2 + x^2 - zx - b^2 = 0, \\ a - 1 \geq 0,\ b - a \geq 0, \\ x > 0,\ y > 0,\ z > 0. \end{cases} \tag{4}$$

Every items listed above is of importance for both theory and applications in various fields of science and technology. More and more problems arose which can be reduced to solving semi-algebraic systems at one of the three levels. For example, solution classification for "p-3-p" problem [24] which originates from computer vision, constructing limit cycles of planar algebraic differential systems [16, 20], automated theorem proving and discovering for algebraic and geometric inequalities [32, 31]. Moreover, many problems in geometry, topology and differential dynamical systems are expected to be solved by translating them into solving certain semi-algebraic systems [2].

2 Semi-algebraic System with Zero-dimensional Solutions

In this section we consider the issue at the first level: all the elements are dependent variables, no parameters exist. We call that a system *with constant coefficients*. It is often expected that such a system has only finitely many solutions, in other words, zero-dimensional solutions. It is well known that for a system of polynomial equations with zero-dimensional solutions, there exist many algorithms based on Ritt-Wu method, Gröbner basis or resultant computation, which can decompose the given system into triangular systems (see, for example, [25, 4, 21, 22, 14, 1, 35]), each has the form as follows,

$$\begin{cases} f_1(x_1) = 0, \\ f_2(x_1, x_2) = 0, \\ \quad \cdots\cdots, \\ f_s(x_1, x_2, ..., x_s) = 0, \\ g_1(x_1, x_2, ..., x_s) \geq 0,\ \cdots,\ g_r(x_1, x_2, ..., x_s) \geq 0, \\ h_1(x_1, x_2, ..., x_s) > 0,\ \cdots,\ h_t(x_1, x_2, ..., x_s) > 0, \\ q_1(x_1, x_2, ..., x_s) \neq 0,\ \cdots,\ q_m(x_1, x_2, ..., x_s) \neq 0. \end{cases} \tag{5}$$

A MAPLE-program `realzero` we demonstrate here is based on a zero-isolating algorithm [29], roughly speaking, which consists of three steps:

- Decompose the given system (with constant coefficients) into triangular systems.
- Isolate the real solutions for every triangular system obtained from step 1, get its *interval solutions* which are non-overlapped one another.
- Collect the interval solutions for all the triangular systems.

For more details, see [29].

The program `realzero` has three basic kinds of calling sequences for system (1) in the case of $n = s$:

$$\texttt{realzero}([p_1, \cdots, p_s], [g_1, \cdots, g_r], [h_1, \cdots, h_t], [q_1, \cdots, q_m], [x_1, \cdots, x_s]);$$
$$\texttt{realzero}([p_1, \cdots, p_s], [g_1, \cdots, g_r], [h_1, \cdots, h_t], [q_1, \cdots, q_m], [x_1, \cdots, x_s], width);$$
$$\texttt{realzero}([p_1, \cdots, p_s], [g_1, \cdots, q_r], [h_1, \cdots, h_t], [q_1, \cdots, q_m], [x_1, \cdots, x_s], [w_1, ..., w_s]); \ .$$

The command `realzero` returns a list of isolating intervals for all real solutions of the input system or reports the method does not work on some branches. If the 6-th parameter "width", a positive number, is given, the maximal size of the output intervals is less than or equal to the number. If the 6-th parameter is a list of positive numbers, $[w_1, ..., w_s]$, the maximal sizes of the output intervals on $x_1, ...$ and x_s are less than or equal to $w_1, ...$ and w_s, respectively. If the 6-th parameter is omitted, the most convenient width is used for each interval returned. In what follows, all the examples were computed on a Pentium/800 PC with 256 Mb RAM under MAPLE V.4.

Example 2.1 Find the isolating intervals of non-negative solutions for system (2). Call

$$\texttt{realzero}\,([p_1, p_2, p_3, p_4], [x_1, x_2, y_1, y_2], [\quad], [\quad], [x_1, x_2, y_1, y_2], 1/1000);$$

the output is (4.855 seconds),

$$\Bigg[\bigg[\Big[\frac{123699}{262144}, \frac{151}{320}\Big],$$
$$\Big[\frac{15604750193840633515355762525347641882989981}{15429603258688008185068797668747034522695597}, \frac{25646736065207290639}{25350470632055620751}\Big],$$
$$\Big[\frac{31940045261606640254 9}{15210282379233372450 6}, \frac{64807714054451707909444009190671657811201765}{30859206517376016370137595337494069045391194}\Big],$$
$$\Big[\frac{11766526981955972576 8}{16304965803039035040 1}, \frac{23867887436121200844755218097593146520662280}{33070540167780718023098481036025768815988257}\Big]\bigg],$$
$$\bigg[[0, 0], [0, 0], [0, 0], [0, 0]\bigg], \bigg[[0, 0], [0, 0], \Big[\frac{77397}{32768}, \frac{38699}{16384}\Big], \Big[\frac{283969593}{268435456}, \frac{71012665}{67108864}\Big]\bigg],$$
$$\bigg[[2, 2], [2, 2], [0, 0], [0, 0]\bigg]\Bigg],$$

which means the system has 4 non-negative solutions and obviously, only one of them is positive.

Example 2.2 [28] Given a system in b, c, d, e with constant coefficients,

$$\begin{cases} f_1(b) = 0,\ f_2(b,c) = 0,\ f_3(b,c,d) = 0,\ f_4(b,c,d,e) = 0, \\ b > 0,\ c > 0,\ d > 0,\ e > 0, \\ c - d \neq 0, \end{cases}$$

where

$$\begin{aligned}
f_1 = {} & 241538508382138075462768483549507937558926051383237186598921\backslash \\
& 35143477508299833761559265231377708635407176637146131171128509762\backslash \\
& 9761b^{32} + 635066713778840598710749498577504496793070850884097947974\backslash \\
& 38029219177777722424790935669882905230018966867662706346221816526\backslash \\
& 25273216640b^{30} - 61751672968559423134724687728230891908778934060236\backslash \\
& 33346079511963379997673499794946894262027603963333723121547282957\backslash \\
& 28824956115726848000b^{28} + 27390034646753639766624212069599001290967\backslash \\
& 59448312686194199639473757366983350460922339943178170551929762470\backslash \\
& 251477314187497028082105057280b^{26} - 1437145166237554579477639351890\backslash \\
& 59794915618143392460779148627234144024674310258895855296843282026\backslash \\
& 2735689676445367034239551743254142648320b^{24} - 298609258728339835915\backslash \\
& 11873209280400659942863793385889444751464527738926059859502184401\backslash \\
& 2436391877650836905308408943702288447254625779712b^{22} + 435447287511\backslash \\
& 29852462155896216013929270344442811835212525492551812771844033485\backslash \\
& 662489132077458388407791801673830767006425164301268418560b^{20} - 3414\backslash \\
& 88074367456093473004956003122708578333573667293973935929910141878\backslash \\
& 3540565919352395939247814785296729972490057003026109068312838144b^{18} + \\
& 82237565552698611657570566658152443771941213949595401920250155054\backslash \\
& 50490851490444645492294205808268848998735781764749955762521605406\backslash \\
& 72b^{16} - 18465863911534614222771407254727218540553077903981494060726\backslash \\
& 33493473284280274236822702569461113776661096178555364273916711096\backslash \\
& 2872320b^{14} + 210458398154301515264393872128757750183426243499830192\backslash \\
& 25945815801961543179698127213788008273551581371156105365020925245\backslash \\
& 8008412160b^{12} - 139786675574463317676421937828553960047493569539985\backslash \\
& 18250677206097934122810155232883104878564803527356387141413117228\backslash \\
& 18845540352b^{10} + 58518821530242525343370224841451531318336644453000\backslash \\
& 07497442033037334476891210594913793124432039758371062267351116039\backslash \\
& 560626176b^{8} - 1553901833784639522211865208589780740623802505099793\backslash \\
& 47778214149227003875939955374111373227667330769980827373188349301\backslash \\
& 88288b^{6} + 31399605401650712044647367132918454229000779662777456747\backslash \\
& 422632241786296296046865042734023650341502533877789531725365248b^{4} - \\
& 221479815284662082377510954691437636974999488557226201213514166702\backslash \\
& 1889911271014168057494169088077631899897509875554816b^{2} + 6072087665\backslash \\
& 34027611425076641314953202561482473671769904296105502296130677639\backslash \\
& 9525491814383795284511167695839821824
\end{aligned}$$

$$
\begin{aligned}
f_2 \;&= 2075\, b^{16}\, c^{12} + 284580\, b^{14}\, c^{10} + 357840\, b^{12}\, c^{9} + 10185588\, b^{12}\, c^{8} + 20167488\, b^{10}\, c^{7} \\
&+ (21285312\, b^{8} - 62355744\, b^{10})\, c^{6} - 99610560\, b^{8}\, c^{5} \\
&+ (-4855244976\, b^{8} - 361573632\, b^{6})\, c^{4} + (-37158912\, b^{4} - 3758980608\, b^{6})\, c^{3} \\
&+ (54181472832\, b^{6} + 429235200\, b^{4})\, c^{2} + (4897760256\, b^{4} + 488374272\, b^{2})\, c \\
&- 123974556480\, b^{4} + 18874368 + 9432723456\, b^{2},
\end{aligned}
$$

$$
\begin{aligned}
f_3 \;&= 9\, b^4\, d^3 + 45\, b^4\, c\, d^2 + (35\, b^4\, c^2 - 486\, b^2)\, d - 108\, b^2\, c - 264 + 10\, b^4\, c^3, \\
f_4 \;&= (36\, d^2\, b^2 - 8\, c^2\, b^2 - 28\, d\, b^2\, c)\, e + 15\, b^4\, c\, d^2 + 6\, b^4\, c^3 + 21\, b^4\, c^2\, d - 144\, b^2\, c \\
&+ 9\, b^4\, d^3 - 120 - 648\, b^2\, d.
\end{aligned}
$$

We call `realzero`$([f_1, f_2, f_3, f_4], [\quad], [b, c, d, e], [c - d], [b, c, d, e])$, and get six solutions,[1]

$$
\begin{aligned}
&\Bigg[\Bigg[\left[\frac{741}{2048}, \frac{1483}{4096}\right], \left[\frac{76905}{32768}, \frac{76995}{32768}\right], \left[17, \frac{35}{2}\right], \\
&\quad \left[\frac{1086192531934356577985472393 7}{2212779236770148942987919360 0}, \frac{16551194692093223298992446177 9}{33326217932928391846374355763 2}\right]\Bigg], \\
&\Bigg[\left[\frac{741}{2048}, \frac{1483}{4096}\right], \left[\frac{199727}{32768}, \frac{49971}{8192}\right], \left[\frac{21}{2}, 11\right], \\
&\quad \left[\frac{105012185099739815202157356 55}{49101355003146405025813626 88}, \frac{242456376002716641545680489 9}{105815387421059503299231744 0}\right]\Bigg], \\
&\Bigg[\left[\frac{1803}{2048}, \frac{3607}{4096}\right], \left[\frac{9}{4}, \frac{289}{128}\right], \left[\frac{17}{4}, \frac{9}{2}\right], \left[\frac{23829095983254931}{4908557229096960}, \frac{688086569774945108602 83}{1300007533417603268608 0}\right]\Bigg], \\
&\Bigg[\left[\frac{1803}{2048}, \frac{3607}{4096}\right], \left[\frac{311}{128}, \frac{39}{16}\right], \left[\frac{31}{8}, 4\right], \left[\frac{1393400289557972985919}{203852745321228533760}, \frac{624427174855568222 43}{85146316449744158 72}\right]\Bigg], \\
&\Bigg[\left[\frac{8177}{4096}, \frac{4089}{2048}\right], \left[\frac{1935}{2048}, \frac{121}{128}\right], \left[\frac{29}{16}, \frac{15}{8}\right], \\
&\quad \left[\frac{21563441393891116950382200 7}{18687908488874225251123200}, \frac{3132517750841677845229}{257952942173863280640}\right]\Bigg], \\
&\Bigg[\left[\frac{39343}{4096}, \frac{2459}{256}\right], \left[\frac{97}{512}, \frac{195}{1024}\right], \left[\frac{769}{2048}, \frac{1547}{4096}\right], \\
&\quad \left[\frac{5995545076788180708016364661}{1057593028450035414597632 00}, \frac{4555023581270481896273 7}{7895725093641508618 24}\right]\Bigg]\Bigg].
\end{aligned}
$$

Reference [28] gave the number of the solutions, 6, but no isolating intervals.

3 Verifying Consistency of Semi-algebraic Systems

In this section we consider the issue at the second level. A lot of problems can be converted to verifying whether a semi-algebraic system is consistent or not, i.e. whether the system has at least one solution. For example, let Φ be a system as (3), Φ_0 a polynomial equation, inequation or inequality. Prove or disprove that

$$\Phi \Rightarrow \Phi_0.$$

Obviously, the statement is true if system $\Phi \,\wedge\, \neg\Phi_0$ is not consistent.

Automated theorem proving in real algebra and real geometry is always considered a difficult topic in the area of automated reasoning. Relevant algorithms depend fundamentally on

[1] spend 202 seconds

consistency verification for a certain semi-algebraic system, and the computational complexity increases very quickly with the dimension, that is, the number of parameters. Besides the memory saving, the speed improvement in theorem proving sometimes is also of importance. When a problem requires verification of a batch of non-trivial propositions, an inefficient algorithm cannot handle it within the time allowed by human patience. For recent progress made in this aspect, see [9, 10, 5, 17, 26, 27, 31, 32]. Fortunately, the problem is easier for so-called *constructive geometry*, roughly speaking, a statement wherein the geometric elements (points, lines and circles) are constructed step by step with rule and compasses from ones previously constructed.

We demonstrate here a MAPLE-program `BOTTEMA`, an inequality prover, by which more than 1000 algebraic and geometric inequalities including hundreds open problems have been verified. The total CPU time spent for proving 100 basic inequalities[2] from Bottema et al's monograph [3] "*Geometric Inequalities*" on a Pentium III/450, was 10-odd seconds only.

On verifying an inequality with `BOTTEMA`, we need only type in a proving command, then the machine will do everything else. If the statement is true, the computer screen will show "*The inequality holds*", otherwise, show "*The inequality does not hold*" with a counter-example. There are three kinds of proving commands: *prove, xprove* and *yprove*.

prove – prove a geometric inequality on a triangle, or an equivalent algebraic inequality.

Calling Sequence:

prove(ineq);
prove(ineq, ineqs);

Parameters:

ineq – an inequality to be proven, which is encoded in the geometric invariants listed later.
ineqs – a list of inequalities as the hypothesis, which is encoded as well in the geometric invariants listed later.

Geometric Invariants on a Triangle: (extendable)

```
a,b,c,        lengths of sides of a triangle ABC
s,            s:=(a+b+c)/2, half the perimeter
x,y,z,        x:=s-a, y:=s-b, z:=s-c
S,            Area of the triangle
R,            circumradius
r,            inradius
ra,rb,rc,     radii of escribed circles
ha,hb,hc,     altitudes
ma,mb,mc,     medians
wa,wb,wc,     interior-angle-bisectors
p,            p:=4*r*(R-2*r)
q,            q:=s^2-16*R*r+5*r^2
HA,HB,HC,     distances from orthocenter to vertices
IA,IB,IC,     distances from incenter to vertices
```

[2] which include some classical results such as Euler's Inequality, Finsler-Hadwiger's Inequality, Gerretsen's Inequality, etc.

A, B, C,	interior angles
sin(A), sin(B), sin(C),	sines of the interior angles
cos(A), cos(B), cos(C),	cosines of the interior angles
tan(A), tan(B), tan(C),	tangents of the interior angles
cot(A), cot(B), cot(C),	cotangents of the interior angles
sec(A), sec(B), sec(C),	secants of the interior angles
csc(A), csc(B), csc(C),	cosecants of the interior angles
sin(A/2), sin(B/2), sin(C/2),	
cos(A/2), cos(B/2), cos(C/2),	
tan(A/2), tan(B/2), tan(C/2),	
cot(A/2), cot(B/2), cot(C/2),	
sec(A/2), sec(B/2), sec(C/2),	
csc(A/2), csc(B/2), csc(C/2),	

Examples:

```
> read bottema;
> prove(a^2+b^2+c^2>=4*sqrt(3)*S+(b-c)^2+(c-a)^2+(a-b)^2);
```

The theorem holds

```
> prove(A>=B,[a>=b]);
```

The theorem holds

```
> prove(2*ma+2*mb+2*mc<=2*a+2*b+(3*sqrt(3)-4)*c, [c<a,c<b]);
```

The theorem holds

xprove – prove an algebraic inequality with positive variables.

Calling Sequence:

xprove(ineq);
xprove(ineq, ineqs);

Parameters:

ineq – an algebraic inequality to be proven, with positive variables.
ineqs – a list of algebraic inequalities as the hypothesis, with positive variables.

Examples:

```
> read bottema;
> xprove(sqrt(u^2+v^2)+sqrt((1-u)^2+(1-v)^2)>=sqrt(2),[u<1,v<1]);
```

The theorem holds

```
> f:=(x+1)^(1/3)+sqrt(y-1)+x*y+1/x+1/y^2:
> xprove(f>=42496/10000,[y>1]);
```

The theorem holds

```
> xprove(f>=42497/10000,[y>1]);
```

$$\textit{with a counter example} \quad \left[x = \frac{29}{32},\ y = \frac{294117648}{294117647}\right]$$

The theorem does not hold

yprove – prove an algebraic inequality in general.

Calling Sequence:

yprove(ineq);
yprove(ineq, ineqs);

Parameters:

ineq – an algebraic inequality to be proven.
ineqs – a list of algebraic inequalities as the hypothesis.

Examples:

```
> read bottema;
> f:=x^6*y^6+6*x^6*y^5-6*x^5*y^6+15*x^6*y^4-36*x^5*y^5+15*x^4*y^6
    +20*x^6*y^3-90*x^5*y^4+90*x^4*y^5-20*x^3*y^6+15*x^6*y^2
    -120*x^5*y^3+225*x^4*y^4-120*x^3*y^5+15*x^2*y^6+6*x^6*y
    -90*x^5*y^2+300*x^4*y^3-300*x^3*y^4+90*x^2*y^5-6*x*y^6+x^6
    -36*x^5*y+225*x^4*y^2-400*x^3*y^3+225*x^2*y^4-36*x*y^5+y^6
    -6*x^5+90*x^4*y-300*x^3*y^2+300*x^2*y^3-90*x*y^4+6*y^5+15*x^4
    -120*x^3*y+225*x^2*y^2-120*x*y^3+15*y^4-20*x^3+90*x^2*y
    -90*x*y^2+20*y^3+16*x^2-36*x*y+16*y^2-6*x+6*y+1:
> yprove(f>=0);
```

The theorem holds

BOTTEMA is created according to a so-called *dimension-decreasing algorithm* which employs an incomplete *cell decomposition*. As for the proof of correctness of the algorithm, and more examples, see [33, 34].

4 When the System Has a Specified Number of Solutions

Now let us consider the issue at the third level. Given a semi-algebraic system with parameters $u_1, \cdots, u_d$ as (3), Find the condition on $u_1, \cdots, u_d$ such that the system has a specified number of solutions. Making use of the discriminant sequence for polynomials [30], WR algorithm [35], Wu's elimination and a partial cylindrical algebraic decomposition, we present here a practical algorithm for finding the condition required above. That is implemented as a MAPLE-program, DISCOVERER, which can help discover new inequalities automatically, without requiring us to put forward any conjectures beforehand, see [31, 32, 28]. That is complete for an extensive class of inequality-type theorems. Also the program is applicable to the classification of the real physical solutions of geometric constraint problems. Many inequalities with various background have been discovered or re-discovered in this way.

For the detailed description to the algorithm and the proof of correctness, see [32].

The program DISCOVERER has three basic kinds of calling sequences for system (3):

tofind($[p_1, \cdots, p_n], [g_1, \cdots, g_r], [h_1, \cdots, h_t], [q_1, \cdots, q_m], [x_1, \cdots, x_n], [u_1, \cdots, u_d],\ \mu$);
tofind($[p_1, \cdots, p_n], [g_1, \cdots, g_r], [h_1, \cdots, h_t], [q_1, \cdots, q_m], [x_1, \cdots, x_n], [u_1, \cdots, u_d],\ \mu \cdot\cdot \nu$);
tofind($[p_1, \cdots, p_n], [g_1, \cdots, q_r], [h_1, \cdots, h_t], [q_1, \cdots, q_m], [x_1, \cdots, x_n], [u_1, \cdots, u_d],\ \mu \cdot\cdot N$);

where

- μ is a non-negative integer that means the condition for system (3) to have exactly μ distinct solution(s),
- $\mu \cdot\cdot \nu$ is a range (μ and ν are non-negative integers, $\mu < \nu$) that means the condition for system (3) to have μ or $\mu + 1$ or $\cdots$ or ν distinct solutions,
- $\mu \cdot\cdot N$ is a range (μ is a non-negative integers, N a name) that means the condition for system (3) to have more than or equal to μ distinct solutions,

and the command tofind returns the required condition accordingly.

Example 4.1 System (4) originates from an open problem[11]: Which triangles can occur as sections of a regular tetrahedron by planes which separate one vertex from the other three?

If we let $1, a, b$ (assume $b \geq a \geq 1$) be the lengths of three sides of the triangle, and x, y, z the distances from the vertex to the three vertexes of the triangle, respectively, then, what we need is to find the necessary and sufficient condition that a, b should satisfy for the system to have at least one solution. We call

tofind($[p_1,\ p_2,\ p_3], [a-1], [b-a], [x, y, z], [\ ], [x, y, z], [a, b], 1 \cdot\cdot N$);

where $p_1 = x^2 + y^2 - xy - 1,\ p_2 = y^2 + z^2 - yz - a^2,\ p_3 = z^2 + x^2 - zx - b^2$.

Our DISCOVERER runs 12 seconds on a Pentium III/450 with MAPLE V.5.1, then returns

FINAL RESULT :

The system has required real solution(s) IF AND ONLY IF

$$[0 < R1,\ 0 < R2] \quad \text{or} \quad [0 < R1,\ R2 < 0,\ 0 < R3]$$

where

$$\begin{aligned}
R1 = &\ a^2 + a + 1 - b^2 \\
R2 = &\ a^2 - 1 + b - b^2 \\
R3 = &\ 1 - \frac{8}{3}a^2 - \frac{8}{3}b^2 + \frac{16}{9}a^8 - \frac{68}{27}b^6a^2 + \frac{241}{81}b^4a^4 - \frac{68}{27}b^2a^6 - \frac{68}{27}b^4a^2 - \frac{68}{27}b^2a^4 - \frac{2}{9}b^6 \\
&+ \frac{16}{9}b^8 - \frac{2}{9}a^6 + \frac{46}{9}b^2a^2 + \frac{16}{9}b^4 + \frac{16}{9}a^4 + \frac{46}{9}b^2a^8 + \frac{46}{9}b^8a^2 - \frac{68}{27}b^6a^4 - \frac{68}{27}b^4a^6 \\
&+ \frac{16}{9}b^4a^8 - \frac{8}{3}b^{10}a^2 + \frac{16}{9}b^8a^4 - \frac{2}{9}b^6a^6 - \frac{8}{3}b^2a^{10} - \frac{8}{3}b^{10} + b^{12} - \frac{8}{3}a^{10} + a^{12}.
\end{aligned}$$

The article [11] gave a sufficient condition that any triangle with two angles $> 60°$ is a possible section. It is easy to see that this condition is equivalent to $[R1 > 0,\ R2 > 0]$.

Example 4.2 Find a necessary and sufficient condition for a $p \times q$ rug to fit on an $a \times b$ floor. When $p \le a$, obviously the required condition is $q \le b$. The only non-trivial case is $p \ge a \ge b \ge q$ which leads (by an argument omitted here) to the following system:

$$\begin{cases} h_1 = x^2 + y^2 - z\,q^2 = 0, \\ h_2 = (a-x)^2 + (b-y)^2 - z\,p^2 = 0, \\ h_3 = (a-x)x - (b-y)y = 0, \\ p \ge a,\ a \ge b,\ b \ge q,\ z \ge 1, \\ x > 0,\ y > 0,\ q > 0. \end{cases} \tag{6}$$

We call tofind($[h_1,\ h_2,\ h_3], [p-a, a-b, b-q, z-1], [x, y, q], [\,], [x, y, z], [p, q, a, b],\ 1 \cdot\cdot N$); the program runs 25 seconds on a Pentium III/450 with MAPLE V.5.1, then returns

FINAL RESULT :

The system has required real solution(s) IF AND ONLY IF

$$[0 < R1]$$

where

$$R1 = -4\,b\,q\,a\,p + q^2\,a^2 + a^2\,p^2 + p^2\,b^2 + q^2\,b^2 - p^4 + 2\,p^2\,q^2 - q^4.$$

This condition coincides with that given in [23].

Example 4.3 Find the condition on a, b, c, d such that

$$(\forall x > 0)\ x^5 + ax^3 + bx^2 + cx + d > 0.$$

This is equivalent to find the sufficient and necessary condition such that system

$$f(x) = x^5 + ax^3 + bx^2 + cx + d = 0, \quad x > 0$$

has no solution. We call tofind($[f], [\,], [x], [\,], [x], [a, b, c, d],\ 0$); then the program runs 2803 seconds on a Pentium III/450 with MAPLE V.5.1, then returns

FINAL RESULT :

The system has required real solution(s) IF AND ONLY IF

$[R1 < 0,\ 0 < R2,\ R3 < 0,\ 0 < R4,\ 0 < R5,\ 0 < R6,\ 0 < R7]$ or

$[0 < R2,\ 0 < R3,\ R6 < 0,\ 0 < R7]$ or

$[R1 < 0,\ 0 < R2,\ R3 < 0,\ 0 < R4,\ 0 < R6,\ R7 < 0]$ or

$[0 < R1,\ 0 < R2,\ R3 < 0,\ R5 < 0,\ R6 < 0,\ 0 < R7]$ or

$[0 < R2,\ 0 < R3,\ 0 < R4,\ R5 < 0,\ 0 < R6,\ 0 < R7]$ or

$[R1 < 0,\ 0 < R2,\ R3 < 0,\ 0 < R5,\ R6 < 0,\ 0 < R7]$ or

$[0 < R2,\ 0 < R3,\ 0 < R4,\ 0 < R5,\ 0 < R6]$

where

$$R1 = a$$

$$R2 = d$$

$$R3 = -\frac{8}{9}\,c\,a + \frac{4}{15}\,a^3 + b^2$$

$$R4 = \frac{20}{27}\,d\,a^2 + \frac{4}{27}\,a^3\,b - \frac{16}{9}\,b\,a\,c + b^3$$

$$R5 = \frac{88}{27}\,a^2\,c^2 - \frac{13}{3}\,a\,c\,b^2 - \frac{4}{9}\,a^4\,c + \frac{4}{27}\,a^3\,b^2 + \frac{40}{27}\,b\,a^2\,d - \frac{125}{27}\,a\,d^2 + b^4 + \frac{100}{9}\,b\,c\,d - \frac{160}{27}\,c^3$$

$$R6 = -\frac{28}{9}\,b\,c\,a^2\,d + \frac{20}{9}\,a^3\,d^2 + \frac{4}{27}\,a^3\,c\,b^2 + \frac{4}{9}\,b\,a^4\,d - \frac{16}{27}\,a^4\,c^2 - \frac{16}{3}\,b^2\,a\,c^2 + \frac{128}{27}\,a^2\,c^3 - \frac{400}{27}\,c\,a\,d^2 + c\,b^4 + 3\,b^3\,a\,d - \frac{25}{3}\,d^2\,b^2 + \frac{80}{3}\,b\,d\,c^2 - \frac{256}{27}\,c^4$$

$$R7 = a^5\,d^2 + \frac{4}{27}\,a^3\,b^3\,d - \frac{1}{4}\,b^4\,c^2 + b^5\,d - \frac{1}{27}\,a^3\,b^2\,c^2 - \frac{25}{3}\,a^3\,c\,d^2 + \frac{500}{27}\,a\,c^2\,d^2 - \frac{625}{18}\,a\,b\,d^3 + \frac{275}{36}\,a^2\,b^2\,d^2 + \frac{140}{27}\,a^2\,c^2\,b\,d + \frac{4}{3}\,a\,c^3\,b^2 + \frac{125}{6}\,b^2\,d^2\,c - \frac{400}{27}\,b\,d\,c^3 - \frac{2}{3}\,a^4\,c\,d\,b - \frac{35}{6}\,a\,c\,d\,b^3 + \frac{4}{27}\,a^4\,c^3 - \frac{32}{27}\,a^2\,c^4 + \frac{64}{27}\,c^5 + \frac{3125}{108}\,d^4.$$

It can be seen from the last example that the problem we discuss in this section is non-trivial even for a system of a single equation.

Remark. For the system to have a specified number of solutions, the condition obtained by calling `tofind` is absolutely sufficient, and that is also necessary except for possibly some *boundaries* which are defined by polynomial equations in the parameters. Adding some of the equations to the original system and applying the program to the extended system accordingly, step by step, we can obtain at last the required condition without any exception.

References

[1] P. Aubry, D. Lazard and M. Moreno Maza, On the theories of triangular sets, *J. Symb. Comput.* **28** 105-124 (1999).

[2] L. Blum, F. Cucker, M. Shub and S. Smale, *Complexity and Real Computation*, Springer-Verlag, New York, 1997.

[3] O. Bottema, R. Z. Dordevic, R. R. Janic, D. S. Mitrinovic & P. M. Vasic, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.

[4] B. Buchberger, Gröbner bases: An algorithmic method in polynomial ideal theory, In *Multidimensional Systems Theory*, (Edited by N.K. Bose), pp. 184-232, D. Reidel, Dordrecht, (1985).

[5] S. C. Chou, X. S. Gao & D. S. Arnon, On the mechanical proof of geometry theorems involving inequalities, *Advances in Computing Research*, **6**, JAI Press Inc., pp. 139–181, 1992.

[6] G. E. Collins, Quantifier elimination for real closed fields by cylindrical algebraic decomposition, in *Lecture Notes in Computer Science* **33** 134-183 (1975).

[7] G. E. Collins and R. Loos, Real Zeros of Polynomials, In *Computer Algebra: Symbolic and Algebraic Computation*, (Edited by B. Buchberger *et al.*), pp. 83-94, Springer-Verlag, New York, (1983).

[8] D. Cox, J. Little, and D. O'Shea, *Using Algebraic Geometry*, Springer-Verlag, New York (1998).

[9] A. Dolzmann, T. Sturm, & V. Weispfenning, A new approach for automatic theorem proving in real geometry, *Journal of Automated Reasoning*, **21**(3), 357–380, 1998.

[10] A. Dolzmann, T. Sturm & V. Weispfenning, Real quantifier elimination in practice, *Algorithmic Algebra and Number Theory*, B. H. Matzat, G.-M. Greuel & G. Hiss (eds.), Springer, Berlin, pp. 221–247, 1998.

[11] E. Folke, Which triangles are plane sections of regular tetrahedra? *American Mathematics Monthly,* Oct., 1994, 788-789.

[12] D. Grigoriev and N. Vorobjov, Solving systems of polynomial inequalities in subexponential time, *J. of Symbolic Computation* **5** 37-64 (1988).

[13] L. Gonzalez-Vega, F. Rouillier, M.-F. Roy and G. Trujillo, Symbolic Recipes for Real Solutions, In *Some Tapas of Computer Algebra*, (Edited by A.M. Cohen *et al.*), pp. 121-167, Springer-Verlag, New York, (1999).

[14] D. Lazard, A new method for solving algebraic systems of positive dimension, *Discrete Appl. Math.* **33** 147-160 (1991).

[15] R. Loos, Generalized Polynomial Remainder Sequences, In *Computer Algebra: Symbolic and Algebraic Computation*, (Edited by B. Buchberger *et al.*), pp. 115-137, Springer-Verlag, New York, (1983).

[16] S. L. Ma and Z. M. Zheng, On a cubic system with eight small-amplitude limit cycles, *Appl. Math. Lett.*, vol. 7, (4)1994, 23-27.

[17] N. McPhee, S. C. Chou & X. S. Gao, A combination of Ritt-Wu's method and Collins' method, *Proc. CADE-12*, Springer-Verlag, pp. 401–415, 1994.

[18] S. Smale, The fundamental theorem of algebra and complexity theory, *Bulletin of the Amer. Math. Soc.*, **4** 1-36 (1981).

[19] A. Tarski, *A Decision Method for Elementary Algebra and Geometry*, University of California Press, 1951.

[20] D. M. Wang, A class of cubic differential systems with 6-tuple focus, *J. Diff. Eqns.*, **87** 305-315 (1990).

[21] D. M. Wang, An elimination method for polynomial systems, *J. Symb. Comput.* **16** 83-114 (1993).

[22] D. M. Wang, Decomposing polynomial systems into simple systems, *J. Symb. Comput.* **25** 295-314 (1998).

[23] J. E. Wetzel, Rectangles in rectangles, *Mathematics Magazine*, **73**(3), 204-211 (2000).

[24] W. J. Wolfe, D. Mathis, C. Weber and M. Magee, The Perspective View of Three Points, *IEEE Transaction on Pattern Analysis and Machine Intelligence*, **13**(1), 66-73, January 1991.

[25] W.-t. Wu, On zeros of algebraic equations–An application of Ritt principle, *Kexue Tongbao* **31** 1-5 (1986).

[26] W.-t. Wu, On a finiteness theorem about problem involving inequalities, *Sys. Sci. & Math. Scis.*, **7**, 193–200, 1994.

[27] W.-t. Wu, On global-optimization problems, *Proc. ASCM '98*, Lanzhou University Press, Lanzhou, pp. 135–138, 1998.

[28] B. C. Xia, DISCOVERER: A tool for solving problems involving polynomial inequalities, *ATCM 2000*, pp. 472-481, ATCM, Inc. (2000).

[29] B. C. Xia and L. Yang, An effective algorithm for isolating the real solutions of semi-algebraic systems and its applications, *Proceedings of the Asian Symposium in Computer Mathematics*, September, 2001, Japan.

[30] L. Yang, Recent advances on determining the number of real roots of parametric polynomials, *Journal of Symbolic Computation*, **28**(1999), 225-242.

[31] L. Yang, X. R. Hou and B. C. Xia, Automated Discovering and Proving for Geometric Inequalities, In *Automated Deduction in Geometry*, (Edited by X.-S. Gao *et al.*), *LNAI 1669*, pp. 30-46, Springer-Verlag, New York, (1999).

[32] L. Yang, X. R. Hou and B. C. Xia, A Complete Algorithm for Automated Discovering of a Class of Inequality-Type Theorems, *Science in China*, Series **F**, Vol. 44, 33-49, 2001.

[33] L. Yang and S. H. Xia, An inequality-proving program applied to global optimization, *ATCM 2000*, pp. 40-51, ATCM, Inc. (2000).

[34] L. Yang and J. Zhang, A Practical program of automated proving for a class of geometric inequalities, *Automated Deduction in Geometry*, Lecture Notes in Artificial Intelligence **2061**, pp. 40-56, Springer Verlag, (2001).

[35] L. Yang, J. Z. Zhang and X. R. Hou, An Efficient Decomposition Algorithm for Geometry Theorem Proving Without Factorization, In *Proceedings of Asian Symposium on Computer Mathematics 1995*, (Edited by H. Shi *et al.*), pp.33-41, Scientists Incorporated, Japan, (1995).