The Effect of the Use of Technology to Explore Functions (3)
~The Development of Function Sense using Technology~

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Abstract: It is generally agreed that a guide for exploring-learning is needed (Yerushalmy, 1997). We consider the ability by which students discern the right direction of exploring function and press forward with the exploration as "function sense." Number sense (NCTM, 1989; Sowder, 1992), symbol sense (Fey, 1990; Arcavi, 1994), and graph sense (Freil, 2001) have already been studied, and in this paper function sense will be considered with reference to those studies. We will analyze what place and in what form such function sense appears in learning, and how it is further activated. From that viewpoint, the learning actually performed in the environment of integrated function learning (Kakihana, et al., 2000; Fukuda, et al., 2000) is analyzed. Results indicate various kinds of behaviors illustrating function sense.

1. Background

Understanding of function develops gradually as a result of exploring functions and visualizing them in a variety of contexts. From this, we have planned some classes focusing on integrated learning of functions. In some cases, it was found that students lost their direction of exploring activities. The state was as follows:

1. Students became overwhelmed by the variety of possibilities, while working experimentally.
2. They were unable to move beyond the 'trial and error' mode.
3. They were unable to read the results of exploring; for example, they viewed computer-generated representations only as pictures, unrelated to the mathematical content behind them.

It is generally agreed that a guide for such exploring-learning is needed, and teachers should set up class activities which develop students' abilities and make them conscious of ideas whose importance they had not before noticed. Moreover students themselves need the ability to discern the right direction for such exploring learning. According to function learning, we believe this ability to be function sense.

2. What is function sense?

According to the dictionary, the word "sense", has the following meanings:

> ability to appreciate something; the faculty whereby somebody appreciates a particular quality, intelligence; the ability to make intelligent decisions or sound judgments
>
> (Encarta World English Dictionary http://dictionary.msn.com/)

Number sense, symbol sense and graph sense have been already studied. Arcavi (ibid. p32) states that, "Symbol sense should become part of ourselves, ready to be brought into action almost at level of a reflex." And Friel (ibid. p145) remarks that, "Those senses can be considered as representing certain ways of thinking rather than as bodies of knowledge that can be transmitted to others." Similarly, function sense is one which grasps relations of functions and numerical relations of variables. Function sense is related with those senses, and especially is involved in symbol sense, but "function sense" emphasizes the conceptual aspect of numerical relationship and function. For example (fig.1), while each sense focuses on various sides of the sequence, 7, 10, 13, 16, …, at the same time they influence each other mutually, and consequently can be more thoroughly understood.

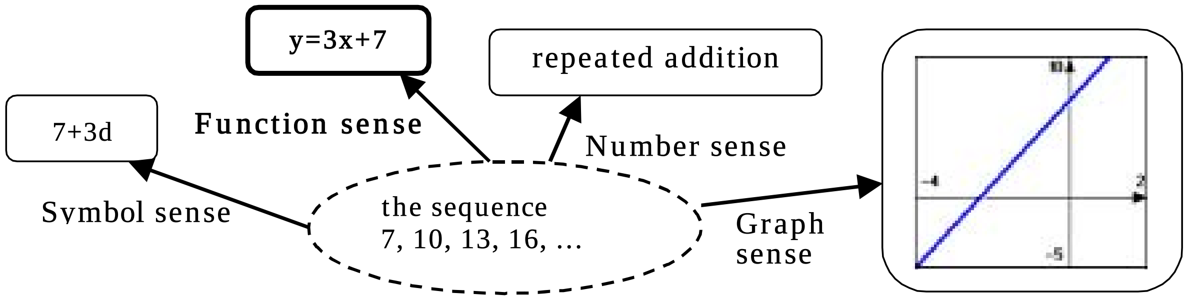

Fig.1 function sense and other three senses

Considering some aspects of function sense that are related to those senses, we can collate them into a table (table 1). In this table, we also collect behaviors that seem to demonstrate the presence of function sense and activities that seem to encourage it.

Table 1 Function sense, related behavior and activity

A part of function sense	Behaviors reflecting function sense	Activities to encourage function sense
Numerical sense of connecting numeric data with patterns of tables or graphs	· To recognize quantitative relationships in concrete situations · To identify some variables and arrange data by making a table or graph. · To find an algebraic expression displayed visually to fit numeric data of two variables	· Gather real-world data and determine the pattern. · Make a table of numeric data of two variables, draw a graph and evaluate the data. · Given a table of numeric data of two variables, find an algebraic expression to fit them.
Visual sense of connecting a graph with patterns of expressions or data	· To scan a graph and interpret a verbally stated condition or find an algebraic expression	· Categorize a table of algebraic expressions and their graph.
Algebraic sense of connecting expressions with patterns of graphs or data	· To make informed comparisons of orders of magnitude for functions · To inspect the result and judge the likelihood that it has been performed correctly.	· Make a comparison or categorize the following function $x^k, k^x (k=1,2,3,\cdots)$ · Estimate a new function not taught in class, by recognizing it visually as the sum or product of known functions.

Like Arcavi who did not define symbol sense directly but provided "catalogue" of its behaviors, at the first step towards describing function sense, we collect instances which raise our awareness of behaviors related to function sense.

3. Method

Through teacher observation, student worksheets and interviews, the developing ability of function sense was investigated.

view points

1. How do behaviors illustrating function sense appear in student learning activities?
2. How dose function sense interact between number sense, symbolic sense and graph sense in these activities?
3. How can the teacher direct activity towards collectively embarking on some sense-making activities?

Subject:

- Student1 (S1): A third-grader of lower secondary school, 15 years old.
- Student2 (S2): A second-grader of upper secondary school, 17 years old.
- Student3 (S3): A third-grader of upper secondary school, 18 years old.
- Student4 (S4): A junior university majoring in engineering, 21 years old.

Software: Calculus Unlimited developed in 1996 for exploring function visually by J. L. Schwartz and M. Yerushalmy (http://www.visual-math.com).

Worksheet: Three kinds of worksheets were prepared (Appendix). They were based on the materials of ATCM'98 tutorial[1] (Yerushalmy, M. 1998).

4. Result

The results of two of the cases are as follows: (T stands for the teacher's comments.).

$\boxed{\text{Case 1}}$: the case of Student2

A: The results of Sheet1: (Questions about two straight lines)

When two lines ($y = x + 5$ and $y = -x - 3$) intersect, she must obtain a line between the two. Trying to move the graph to find functions, she listed three conditions of this function:

S2: If the formulation for this function is $y = ax + b$

(1) It must be through the intersection.

(2) a must change in $-1 < a < 1$

(3) b must change in $-3 < b < 5$

T: Are you sure that you need all three conditions?

And then,

S2: If condition (1) and (2) are kept, I cannot produce b other than $-3 < b < 5$. If condition (1) and (3) are kept, a must be $-1 < a < 1$. So I only need condition (1) and either (2)

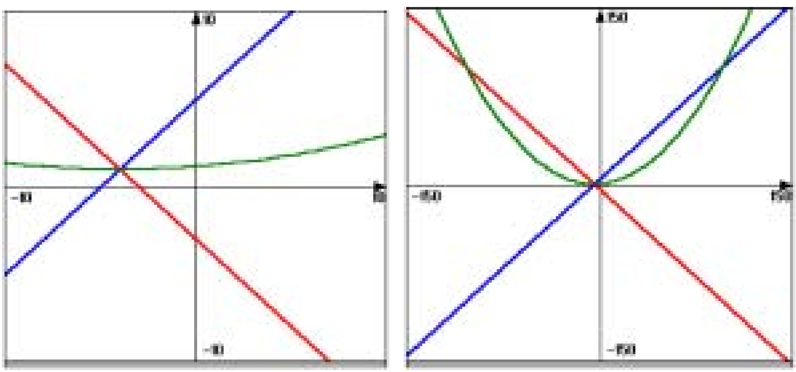

fig.2 Case1-B

[1] Yerushalmy, M. (1998) "On Tools and Classroom: New Approaches to Algebra and Calculus Using Innovative Software." (Tutorials of ATCM '98)

or (3).

B: What if parabola was completely inserted into two straight lines?

S2: If the parabola is very thin, can I insert the graph of it into those two lines?

She drew a graph of $y = 0.01x^2$, *and translated it (fig. 2)*

S2: OK? But what if I try to change the range of graph. Ah, that's exactly what I thought. Because $0.01 \cdot (1000)^2 = 10000$, it's a big number.

C: The results of Sheet 2: (Questions about two quadratic functions)

This task asked whether or not two parabolas ($y = x^2 - 2x + 2$ *and* $y = x^2 - 3x - 4$*) intersected.*

She input two expressions and displayed them.

S2: There is no intersection.

T: Try to change the range of graph.

S2: Ah! I found the intersection. It looks like a parallel around the vertex.

T: Why? Did you calculate the intersection?

S2: (-6,50). There is one intersection, just as I expected.

And then she tried to find the function between two quadratic functions.

S2: It must be through the intersection. And if I pick up another two points between them, for example (0,0) and (3,-2), I'll be able to get the parabola.

The parabola was displayed by software function "fitting" (fig. 3).

T: Um…, now, there are two intersections. How can you get one intersection?

S2: Um.…

T: What kind of function has only one intersection?

S2: It was the same type of parabola.

T: So you can use the congruent graph.

She started to move the graph of $y = x^2$

S2: I found it!

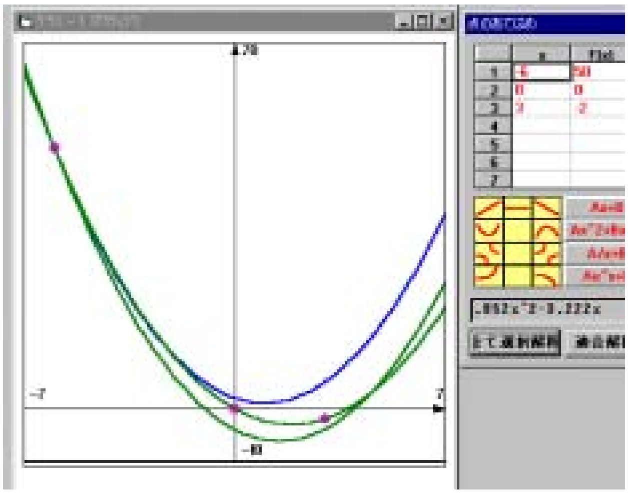

fig.3 Case1-C.

D: The results of Sheet 3: (Questions about two absolute functions)

S2: I'm not sure about absolute function.

T: This software draws an absolute function if you input an expression.

She displayed the two given functions, and then tried to draw the function between them.

S2: I cannot produce the expression for this function.

Case 2: the case of Student 4

A: The results of Sheet 1: (Questions about two straight lines)

Referring to linear functions between $y = x + 5$ *and* $y = -x - 3$, *he calculated algebraically.*

S4: As those lines pass at the intersection point (-4,1), the expression is necessarily $y = m(x+4)+1$. As the slope is more than –1 and less than 1, parameter m changes in the range of $-1 < m < 1$.

T: If you use “locus” - one of the software’s functions - you can display many graphs at once.

He input the equation of a family of linear functions $y = m(x+4)+1$ *and changed* m, *he obtained ten graphs (fig.4).*

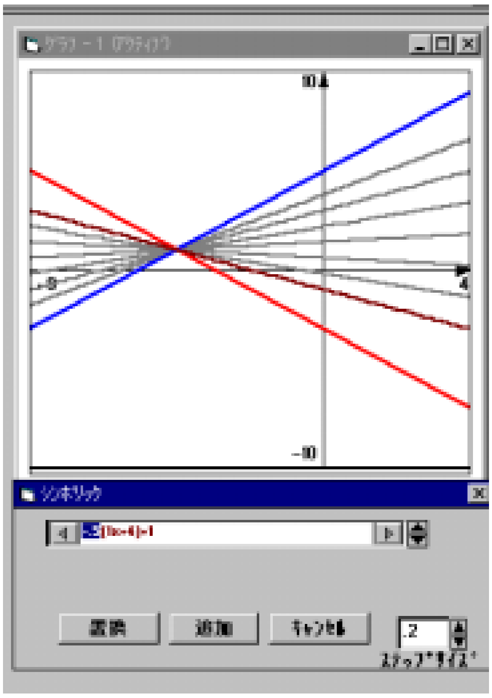

fig.4 Case2-A

<u>B: What function can you draw between them besides linear function?</u>

S4: If trigonometric functions, it could be inserted between two lines. The graph passes necessarily at the intersection point (-4,1).

He input the equation, $y = \sin(x+4)+1$ *(fig.5).*

T: This sin curve appears to be between the two lines, but can you prove it?

He zoomed up the graphs.

S4: The difference between the linear function and the sin function ($s = \{x+5\} - \{\sin(x+4)+1\}$) is not negative.

T: The idea of “function’s difference” is great, but is it really?

S4: Derivative of $s = \{x+5\} - \{\sin(x+4)+1\}$ is $s' = 1 - \cos(x+4)$. We can see s' is non-negative when $-4 < x$, because of $|\cos(x)| \le 1$.

S4: And s is also non negative when $-4 < x$, because s is increasing function and $s(-4) = 0$.

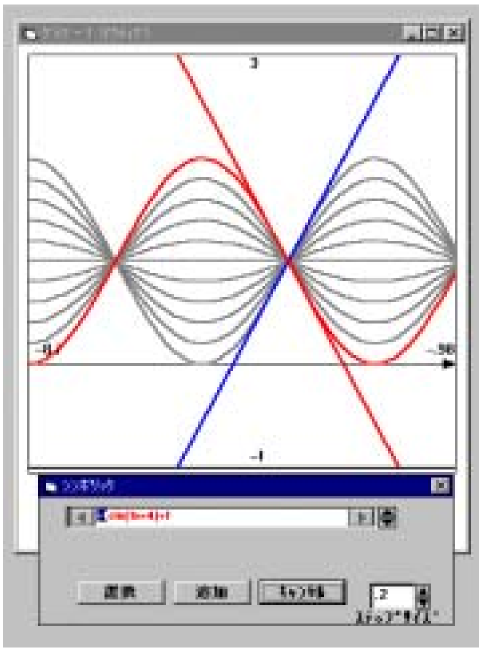

fig.5 Case2-B

<u>C: The results of Sheet 2: (Questions about two quadratic functions)</u>

Referring to the intersection of two congruent quadratic functions,

S4: Visually, because one of two congruent quadratic functions shifts from the other, there is one intersection point.

He manipulated those equations algebraically,

S4: When I solved the simultaneous equations, I got only one solution because two degrees terms were eliminated.

<u>D: The results of Sheet 3: (Questions about two absolute functions)</u>

At the beginning of this exploration, he did not know “abs()” one of this software’s functions.

S4: I have to separate x into $x < -4$ and $-4 \le x$.

He drew two lines and tried to restrict them to the required intervals (fig. 6)

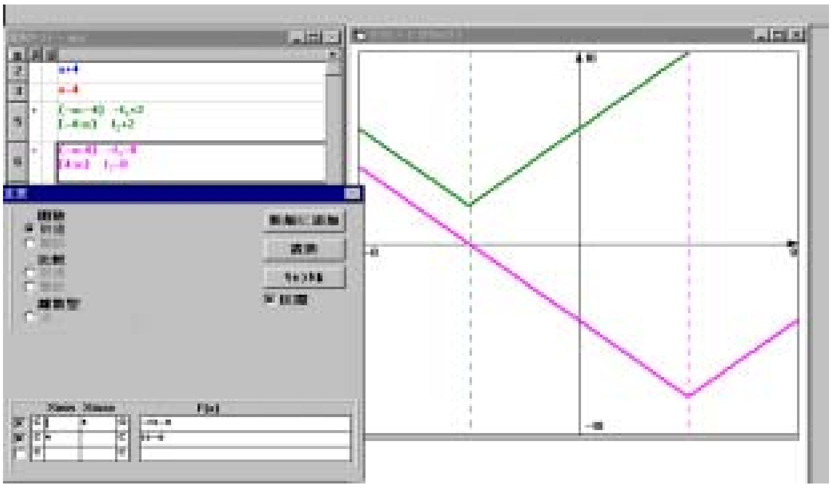

fig. 6 Case2-D

S4: It is easier to translate because I defined the piecewise-defined function as one function, one object.

He tried to find an absolute function which is between $y=|x+4|+2$ *and* $y=|x-4|-8$ *using this object.*

S4: When $y=|x-a|+b$, the sharp vertex (a,b) must be in a rectangle.

T: What rectangle?

S4: $a+6<b<a+12$ and $-a-4<b<-a-2$

E: About the number of intersection points of two absolute functions

($y=|x|$, $y=a+b|x+c|$.)

S4: This function is similar to a parabola with the vertex $(-c,a)$, the coefficient of x^2 b.

He got the idea from this image, catalyzed b into 6 conditions by moving the vertex of the function and made an excellent table.

F: Summarizing these, is there the function which passes between two arbitrary functions?.

Since he knew some of the software's functions calculating sums or difference of functions etc., he experimented with many options for a few minutes.

S4: How about the difference of functions $f1-f2$?

T: The position of this graph is strange.

S4: We had better slide it to $f2$, $(f1-f2)+f2$.

This graph moved on the graph $f1$.

S4:…

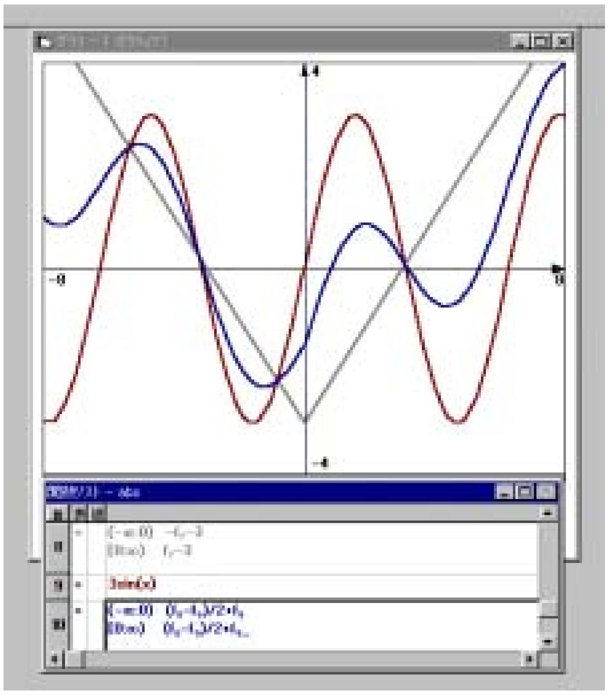

fig. 7 Case2-F

T: Can't you think of some good idea for the difference?

S4: Half of them? $(f1-f2)/2+f2$

He input this equation and succeeded (fig. 7)

T: We used an indirect method, simplified it $(f1+f2)/2$. In short, the average of $f1$ and $f2$. In other words, the middle point.

S4: An internally dividing point is also sufficient, and this method is generated to two arbitrary functions.

T: We have so far separately analyzed, liner, quadratic and absolute functions. What's your conclusion?

S4: If there are two arbitrary functions $f1$ and $f2$, then we can find the function between them. It's $f = p \cdot f1 + (1-p) \cdot f2$ (when $0 < p < 1$).

5. Discussion.

(1) The interaction of function sense and number sense.

In Case1-B, by imagining a concrete number whose absolute value became large, in this case 1000, number sense by which we recognize the difference between magnitudes x and x^2 worked. Furthermore, by imagining the graph, function sense by which we recognize the relation between x and x^2 was strengthened.

Case2-A obtained the hint of the question as to whether x was larger than $\sin x$ when $x > 0$, by visualizing the value of $\pi/6$ and $\sin(\pi/6)$ or $\pi/4$ and $\sin(\pi/4)$. In addition, number sense worked well, and it stimulated the solution of the function problem. Moreover, this activity was good example of the result of one activity urging on the subsequent activity with exploration being activated as a result.

(2) The interaction of function sense and graph sense.

The inquiry of case2-A is an example of how function sense and graph sense interact well. This software has a powerful exploring feature which creates a new function by changing the original function, and fits a function to some selected points. Students used this strategy repeatedly. Using graph sense, all of them found the plot type of a function, brought it close to the required function while considering the meaning of the parameter. Using graph sense, they gave the software some points that represented the feature of the required function to get a new function. The subsequent exploration of function determined what function they chose as the original function or which point information they gave the software. In contrast, in the activity of case1-B, at first the student stopped inquiring about function until the observer's comment helped her to extract the feature of graph (the coefficient of x^2 is 1) with the result that she restarted inquiring

In case2-C, the student anticipated from graph sense that if two congruent quadratic functions translated the parallel, the number of their intersections was one, and so he began the inquiry. Then, by zooming out of the screen or solving these simultaneous equations algebraically, he checked his anticipation. On the other hand, in Case1-C, that interaction did not work well, because she did not read the picture on the screen. By using software, although we could easily display the suitable graph, sometimes the semantic of graph was difficult to understand, and so the inquiry was not activated. Computer-generated representations were often viewed only as pictures, unrelated to the mathematical content behind them (Weigand, et, al., 2001, p109). The observer urged the student to see the graph formed visually by the search strategy (Weigand, ibid, p91) as the relation of two variables.

In case2-F's conclusion, graphs were categorized into 6 conditions by graph sense. If the student were to challenge algebraically (that is finding the number of the solution of the equation as $|x| = a + b|x + c|$), he would get a different table in the different viewpoint.

(3) The interaction of function sense and symbol sense.

Although all of the students thought that many linear functions were between two linear functions, three of them gave only a few examples. However, in Case2-A , he represented the family of the linear function by expression.

Through Case1-A, it was found that she switched back and forth between expressions and graphs on screen. If conditions determining a function were insufficient, a formula could not be specified, and the graph not drawn if they were in excess. By moving the function on a screen, this relation was discovered and confirmed by the algebraic expression. The interaction of function sense and symbol sense was recognized.

In case2-A, the student dealt with the function as an object (Yerushalmy, 1998). Thus, a piece-wise defined function could be treated as one, the operation of it performed, more advanced manipulation curried out, and the exploring function learning deepened further. In Case2-F, one formula, which summarizes all inquiries, was found. By guessing many examples behind this formula, the student completed it by himself with the result that his symbolization and abstraction could effectively be committed in everyday life. In this learning, we were able to see students' activities by which they were going to express the result of exploring learning symbolically or to check it by algebraic operation.

6. Conclusion.

Function sense, one of the core parts of symbol sense, sheds light on the importance of conceptual understanding of function. This works especially well in learning and using functions by using visualization and technology such as graphing software because number sense, symbol sense, graph sense and function sense can influence each other mutually.

As a result of analysis of students' activities to inquire about function, function sense was found as follows:

(1) Many behaviors which illustrate function sense were found in student's learning activities.
(2) Function sense interacted between number sense, symbolic sense and graph sense in these activities, and grew and changed with them.
(3) We recognized a teacher can direct students activities towards collectively embarking on some sense-making activities.

From our classes, we collected instances in which we found behaviors illustrate function sense. We can use this collection as a way to raise our awareness of many other behaviors related function sense. As Arcavi (ibid. p31) has said, we think the collection will be helpful as a first step towards describing function sense.

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Appendix

Worksheet 1

Q1. You are given two linear functions

$x+5$ And $x-3$

- One of these functions is everywhere larger than the other, i.e. at every value of x, the "larger" function's value is greater than the "smaller" function's value.
- Write a linear function which is everywhere smaller than the larger of these two functions and larger than the smaller of them.

Q2. You are given two linear functions

$x+5$ And $-x-3$

- What are the values of x at which they intersect?
- Write a linear function which, at each value of x, is smaller than the larger of these two functions at that value of x and larger than the smaller of them at that value of x.

Q3. Write a nonlinear function as above.

Worksheet 2

Q1. You are given two quadratic functions

x^2-2x+2 And x^2-3x-4

- One of these functions is everywhere larger than the other, i.e. at every value of x, the "larger" function's value is greater than the "smaller" function's value.
- Write a quadratic function which is everywhere smaller than the larger of these two functions and larger than the smaller of them.

Q2. You are given two quadratic functions

$5-x-x^2$ And x^2-x-4

- What are the values of x at which they intersect?
- Write a linear function which, at each value of x, is smaller than the larger of these two functions at that value of x and larger than the smaller of them at that value of x.
- Write a quadratic function as above.

Worksheet 3

Q1. You are given two absolute value functions

$|x+4|+2$ And $|x-4|-8$

- One of these functions is everywhere larger than the other, i.e. at every value of x, the "larger" function's value is greater than the "smaller" function's value.
- Write a absolute value function which is everywhere smaller than the larger of these two functions and larger than the smaller of them.

Q2. You are given two absolute value functions

$|x|$ And $a+b|x+c|$

What is true about a, b, and c if the graphs of these two functions do not intersect? Give an example.

- What is true about a, b, and c if intersect at one value of x? Give an example.
- What is true about a, b, and c if intersect at two values of x? Give an example.

Q3. You are given two arbitrary functions. Write a function which is everywhere smaller than the larger of these two functions and larger than the smaller of them.