Visualizing Cohomology Aspects of 3D Objects

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Abstract

We propose a method for visualizing homology, cohomology, cup products and cup–i products of three–dimensional objects represented as simplicial complexes obtained from cubical decompositions. In order to compute the (co)homology of a given object, we first thin the simplicial complex topologically. We then compute the invariants using the resulting skeleton. Finally, we present a small prototype for visualizing this method and some examples concerning topological thinning and the computation of the cohomology ring and the cochain operations cup–i products.

1 Notions from Algebraic Topology

We give a brief summary of concepts and notations used in the following sections. Our terminology follows Munkres [Mun84].

In $\mathbf{R}^n$, a *q–simplex* (where $q \leq n$) is the convex hull of $q+1$ affinely independent points. A 0–simplex is a vertex, a 1–simplex is an edge, a 2–simplex is a triangle, a 3–simplex is a tetrahedron, and so on. A simplex σ with vertices $\{v_0, \dots, v_q\}$ is denoted by $(v_0, \dots, v_q)$. An *orientation* is a class of vertex orderings, which have the property that any two orderings in the class differ by an even number of transpositions. The *boundary* of σ is the formal sum:

$$\partial_q \sigma = \sum_{i=0}^{q} (-1)^i (v_0, v_1, \dots, \hat{v}_i, \dots, v_q)$$

where the hat means that v_i is omitted. The simplices $(v_0, v_1, \dots, \hat{v}_i, \dots, v_q)$ are the $(q-1)$–faces of σ. In general, if $i < q$, an *i–face* of σ is a simplex of dimension i whose vertices are in

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the set $\{v_0, \ldots, v_q\}$. A simplex is *shared* if it is a face of more than one simplex. Otherwise the simplex is *free* if it belongs to one higher–dimensional simplex, and *maximal* if it does not belong to any. A *simplicial complex* K is a collection of simplices in which the intersection of any two is also a simplex of K or empty. Then, a simplicial complex K can be given by the set of its maximal simplices. The set of all the q–simplices of K is denoted by K_q. The largest dimension of any simplex in K is the *dimension of* K. The number of simplices in K is denoted by $|K|$. A subset $K' \subseteq K$ is a *subcomplex* of K if it is a simplicial complex itself. All simplices in this paper have finite dimension and all simplicial complexes are finite collections. From now on, K denotes a finite simplicial complex.

A *q–chain* a is a formal sum of q–simplices

$$\lambda_1\sigma_1 + \lambda_2\sigma_2 + \cdots + \lambda_m\sigma_m$$

where each λ_i is chosen from some abelian group G. The q–chains form a group with respect to the component–wise addition; this group is the qth *chain group* of K, denoted by $C_q(K)$. By linearity, the boundary operator ∂_q can be extended to q–chains, where it is a homomorphism

$$\partial : \; C_q(K) \to C_{q-1}(K)\,.$$

It can be readily shown that for any chain a, $\partial\partial a = 0$. A q–chain a is called a *q–cycle* if $\partial a = 0$. If $a = \partial a'$ for some $a' \in C_{q+1}(K)$ then a is called a *q–boundary.* We denote the groups of q–cycles and q–boundaries by $Z_q(K)$ and $B_q(K)$ respectively, and define $Z_0(K) = C_0(K)$. Since $B_q(K) \subseteq Z_q(K)$, we can define the *qth homology group* to be the quotient group $Z_q(K)/B_q(K)$, denoted by $H_q(K)$. Given that elements of this group are cosets of the form $a + B_q(K)$, where $a \in Z_q(K)$, we say that the coset $a + B_q(K)$ is the *homology class* in $H_q(K)$ determined by a. We denote this class by $[a]$.

Dual concepts to the previous ones can also be defined. The *cochain complex* of K, denoted by $C^*(K)$, is defined in each dimension q by

$$C^q(K) = \mathrm{Hom}(C_q(K); G) = \{c : C_q(K) \to G : \; c \text{ is a homomorphism } \}$$

and $\delta^q : C^q(K) \to C^{q+1}(K)$ called *codifferential* is given by

$$\delta^q(c)(a) = c(\partial_{q+1}a)$$

where $c \in C^q(K)$ and $a \in C_{q+1}(K)$. Note that a q–cochain can be defined on K_q and it is naturally extended by linearity on $C_q(K)$. $Z^q(K)$ and $B^q(K)$ are the kernel of δ^q and the image of δ^{q-1}, respectively. The elements in $Z^q(K)$ are called *q–cocycles* and those in $B^q(K)$ are called *q–coboundaries.* It is also true that $\delta^q\delta^{q-1} = 0$, so the qth *cohomology group*

$$H^q(K) = Z^q(K)/B^q(K)$$

can be defined for each integer q.

In this paper, we use a special type of homotopy equivalence between two simplicial complexes K and K'. A *contraction* [EM53] r from a simplicial complex K onto another simplicial complex K' is a set of three homomorphisms (f, g, ϕ), where $f : C^q(K) \to C^q(K')$ (projection) and $g : C^q(K') \to C^q(K)$ (inclusion) have good behaviour with respect to the codifferentials and satisfy that $fg = 1$; and $\phi : C^q(K) \to C^{q+1}(K)$ (homotopy operator) satisfies

$$1 - gf = \phi\delta + \delta\phi\,.$$

Moreover, $\phi g = 0$, $f\phi = 0$ and $\phi\phi = 0$. An important poperty is that if there exists such a contraction, then $|K'| \leq |K|$, and the homology and the cohomology ring of K are isomorphic to those of K'.

2 Representation of 3D objects

To begin with, we need an appropriate discrete representation of 3D objects. Typical types of representation for 3D digital images are the voxel representation [Ros81], the tetrahedral representation [Boi88], and the surface representation [KRW91], corresponding to the partition of the 3D space into unit cubes, tetrahedra, and closed and bounded surfaces, respectively. We use a representation which is defined by a special finite simplicial complex. The key idea is the following: first, let us consider a 3D cube grid. Next, let us decompose each cube into six tetrahedra as in Figure 1. The tetrahedra in the figure may be listed using their vertices:

$$\{(1,2,4,6),(4,5,6,8),(3,5,7,8),(1,3,4,5),(3,4,5,8),(1,4,5,6)\}.$$

In this way we can see a cube as a simplicial complex. Two adjacent cubes present in their shared boundary vertices, edges and triangles that are simplices in both. Now, the initial 3D cubic grid is converted into a 3D simplicial complex. A *simplicial representation* is a finite subcomplex of this infinite complex obtained from the cube grid. It is clear that every 3D object represented by voxels also has a simplicial representation. On the other hand, our simplicial representation in which only tetrahedra appear can be seen as a special tetrahedral representation. We use simplicial representations as they are adequate for computing the cohomology ring and cup–i products and, at the same, they lend themselves to convenient visualization of 3D objects.

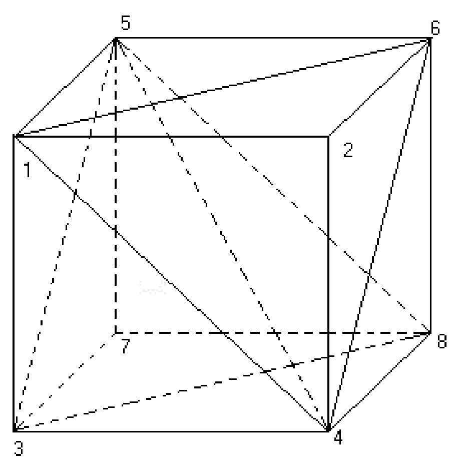

Figure 1: A cube decomposed in six tetrahedra.

3 Topological Thinning

Topological thinning is an important preprocessing operation in image processing [DP81]. The aim is to shrink a digital image to a smaller, simpler image which retains a lot of the significant information of the original. Then, further processing or analysis can be performed on the shrunken image.

In our approach, a given 3D digital image is converted into a simplicial complex using a simplicial representation. There is a process for thinning a simplicial complexes using simplicial collapses [Bjo95]. Suppose K is a simplicial complex, $\sigma \in K$ is a maximal q–simplex and σ' is a free $(q-1)$–face of σ. Then, K *simplicially collapses* onto $K - \{\sigma, \sigma'\}$. More generally, a *simplicial collapse* is any sequence of such operations (see Figure 2). An important property of this process is that there exists an explicit contraction from $C^*(K)$ onto $C^*(K - \{\sigma, \sigma'\})$ [For99]. A *thinned* simplicial complex $M_{\text{top}}(K)$ is a subcomplex of K with the condition that all the faces of the maximal simplices of $M_{\text{top}}(K)$ are shared. Then, it is obvious that it is no longer possible to collapse. There is also an explicit contraction from $C^*(K)$ onto $C^*(M_{\text{top}}(K))$. Therefore, we can apply our process to compute topological invariants on the thinned simplicial complex $M_{\text{top}}(K)$, and the results can be easily visualized in the original simplicial complex K. The following algorithm computes $M_{\text{top}}(K)$. Suppose that K is given by the set of its maximal simplices.

```
Initially, M_top(K) = K
while there exists a maximal simplex σ with a free face σ' do
      M_top(K) = M_top(K) − {σ, σ'}
end while
```

We have implemented an alternate algorithm based in the results given in [SDR98]. That paper deals with the tetrahedral representation and the thinning consists in deleting tetrahedra if they do not change the topology of their neighbors. This procedure can be seen as a particular case of that of Forman's, as a tetrahedron σ is simplicially collapsed if the result is a collection of shared faces of σ.

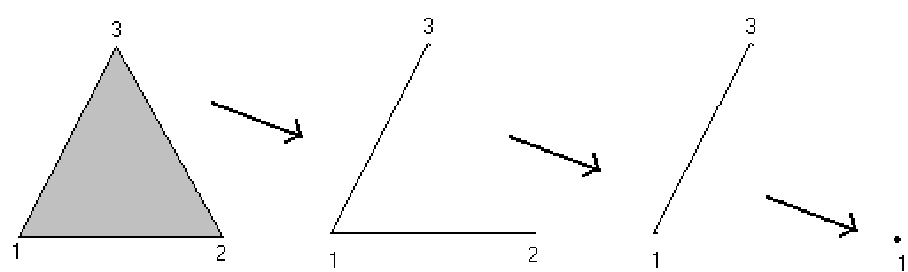

Figure 2: Simplicial collapse of a 2–simplex onto a vertex.

4 Algebraic Thinning

Having obtained the thinned complex $M_{\text{top}}(K)$, we next compute the (co)homology. The computation of a contraction (f, g, ϕ) from $C^*(M_{\text{top}}(K))$ onto its cohomology is essential for us because we use it in order to compute the cohomology ring and the cup–i products. By interpreting the well-known classical "reduction" algorithm for computing (co)homology groups [Mun84], in terms of homotopy equivalences, we construct an algebraic minimal model of a given finite simplicial complex K, inductively. That is, we compute a *minimal cochain complex* $M_{\text{alg}}(K)$ (a finitely generated free abelian group such that the Smith normal form of the matrix corresponding to the codifferential in each degree has all the elements different to one) together with a contraction from $C^*(K)$ onto $M_{\text{alg}}(K)$. Indeed, there is an algebraic minimal model for every finite simplicial complex K, and any two algebraic minimal models of K are isomorphic. Whenever G is a field or the cohomology of K is free, $M_{\text{alg}}(K)$ is isomorphic to the cohomology of K and, therefore, we can construct a contraction from $C^*(K)$ onto $H^*(K)$. All this procedure can be interpreted as an algebraic thinning of the cochain complex of K. We compose the contraction from $C^*(K)$ onto $C^*(M_{\text{top}}(K))$, described in the last section, with that from $C^*(M_{\text{top}}(K))$ onto $H^*(M_{\text{top}}(K))$. Then, we have a description of the generators of the cohomology groups of K in terms of cochains. Details can be found in [GR].

5 Cohomology ring and Cup–i Products

After applying topological and algebraic thinning to the starting simplicial complex K in order, we are able to compute the cohomology ring. Actually, we will first calculate the cohomology ring on $M_{\text{top}}(K)$. We then translate the results for K via the first contraction from $C^*(K)$ onto $C^*(M_{\text{top}}(K))$.

If G is a ring, then the cohomology of K is a ring with the *cup product*

$$\smile : H^i(K) \otimes H^j(K) \to H^{i+j}(K)$$

defined at a cocycle level by $c \smile c'(\sigma) = \mu(c(v_0, \dots, v_i) \otimes c'(v_i, \dots, v_{i+j}))$, where c and c' are an i–cocycle and a j–cocycle, respectively, $\sigma = (v_0, v_1, \dots, v_{i+j}) \in K_{i+j}$ is such that $v_0 < v_1 < \cdots < v_{i+j}$, and μ is the product on G.

Using the contraction (f, g, ϕ) from $C^*(M_{\text{top}}(K))$ onto $H^*(K)$, we can compute the cohomology ring of K in the following way:

```
Take α and β, cohomology classes of K
Compute f(g(α) ⌣ g(β))
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The resulting cohomology class is determined by the cocycle $c = g(\alpha) \smile g(\beta)$. If the cohomology class is zero, then c is also a coboundary. In order to compute a cochain c' such that $c = \delta c'$, we use the relation

$$c - gfc = \phi\delta c + \delta\phi c\,.$$

Since c is a cocycle, then $\delta c = 0$, and it is also a coboundary, then $f(c) = 0$. Therefore, we have

$$c = \delta\,\phi c\,.$$

More complicated examples can be done with the cochain operations cup–i products [Ste47]

$$\smile_i : C^p(K) \otimes C^q(K) \to C^{p+q-i}(K)$$

using the combinatorial formulae obtained in [GR99]. We give an example of these operations, in the next section.

6 Graphical User Interface

In this section we show a small prototype developed for manipulating higher abstract notion such as cohomology of simplicial complexes.

We use the free 3D graphic library Morfit (www.morfit.com) for development. This is a free program for building 3D worlds. In our case, a world is a 3D simplicial complex. In order to create a new world, the user introduces the number of rows, columns and "levels" of depths. For example, in Figure 3 we create a 3D world, called Prueba, with 4 rows, 3 columns, and 2 levels.

Figure 3: Creating a 3D simplicial complex.

Initially, the world is empty. There are several processes in the program to create a simplicial representation in the world. One method is to manually select the simplices using "Editar matriz" command, as shown in Figure 4. On the left, there is a menu in which we select the simplices that we want to render. The selected simplices are shown in the window above. The simplicial representation that we are creating in the world **Prueba** appears in the window on the right. The yellow square in the window represents the selected cube. By selecting "Movimiento" command and using the cursors, we can move in any direction. Another method is to create a simplicial complex by hand in a cube, and afterwards to use "Repetir patrón" command. With this command, the rendered simplices in the cube are rendered in the rest of the cubes of the world. In Figure 5, we can observe that only maximal simplices are rendered in the tessellation. Finally, it is possible to create simplicial representations randomly using "Generar aleatoria" command.

A way to distinguish the different maximal simplices of the simplicial complex associated with a simplicial representation is by using different colors: red for tetrahedra, green for triangles, blue for edges, and black for vertices. In this way, we can distinguish a tetrahedron from four triangles joined by their edges. There is a red voxel into each rendered tetrahedron indicating whether we are inside or outside the tetrahedron.

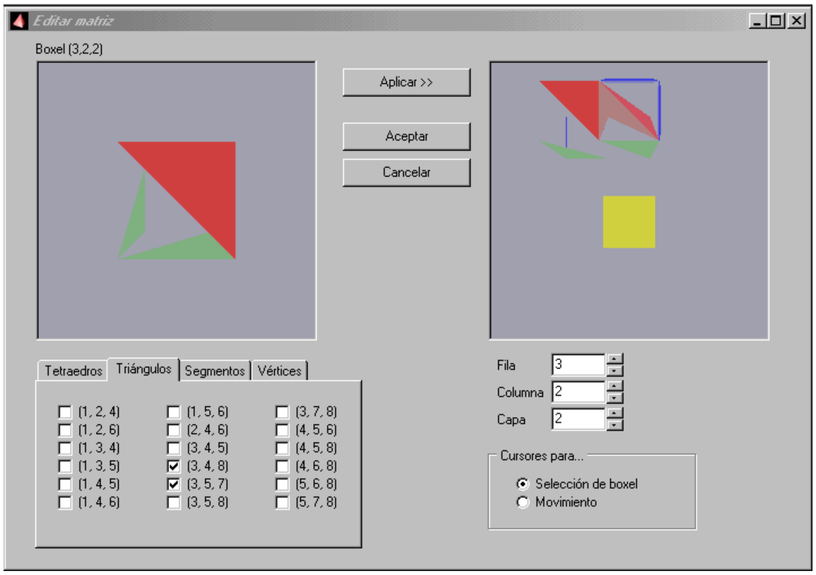

Figure 4: Creating a simplicial representation by hand.

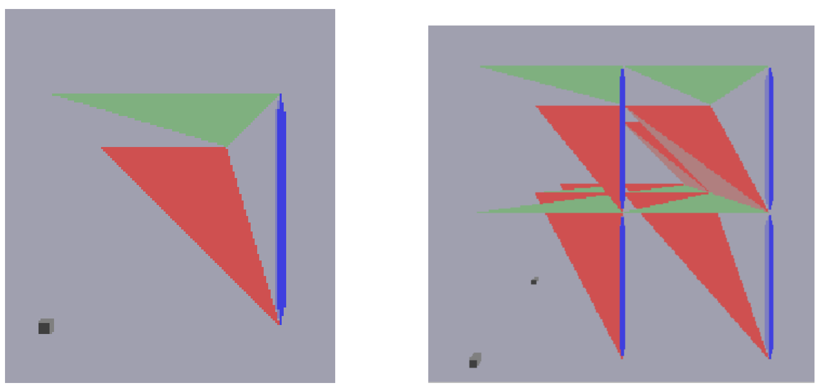

Figure 5: Rendered simplices in a cube and a tessellated $(2 \times 2 \times 2)$–world with it.

"Adelgazar" command thins a simplicial complex, following the procedure given in [SDR98]. We show an example of this process in Figure 6.

For visualizing cochains, the simplices on which a given cochain is non–null, are lighted in yellow color. As an example, using the combinatorial description of $M_{\text{top}}(K)$ from Figure 7, let us define on $M_{\text{top}}(K)$ the cochains $c : C_2(M_{\text{top}}(K)) \to \mathbf{Z}$ and $c' : C_3(M_{\text{top}}(K)) \to \mathbf{Z}$, where

$$\begin{array}{lcl} c(6,7,22) = -10 & & c'(7,8,12,28) = 3 \\ c(5,6,10) = 5 & & c'(5,6,10,26) = 2 \\ c(11,12,32) = -2 & \text{and} & c'(11,12,16,32) = -11 \\ c(16,31,32) = 1 & & c'(16,31,32,36) = -3 \\ c(16,31,36) = 3 & & \end{array}$$

and they are zero over the rest of the simplices. Then we obtain that

$$c \smile_2 c' : C_3(M_{\text{top}}(K)) \to \mathbf{Z}$$

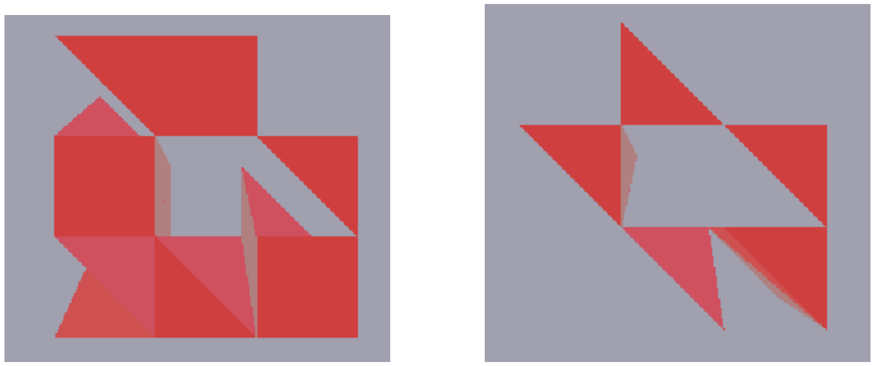

Figure 6: A simplicial complex K and $M_{\text{top}}(K)$ using [SD98].

is determined by

$$\begin{aligned} c \smile_2 c'(5,6,10,26) &= 10 \\ c \smile_2 c'(11,12,16,32) &= -22 \\ c \smile_2 c'(16,31,32,36) &= 6 \end{aligned}$$

and it is zero elsewhere.

Let us consider now the following simplicial complexes: the torus (the space X) and the wedge of two topological circles and a topological 2-sphere (the space Y). Simplicial representations of these spaces are shown in Figure 7.

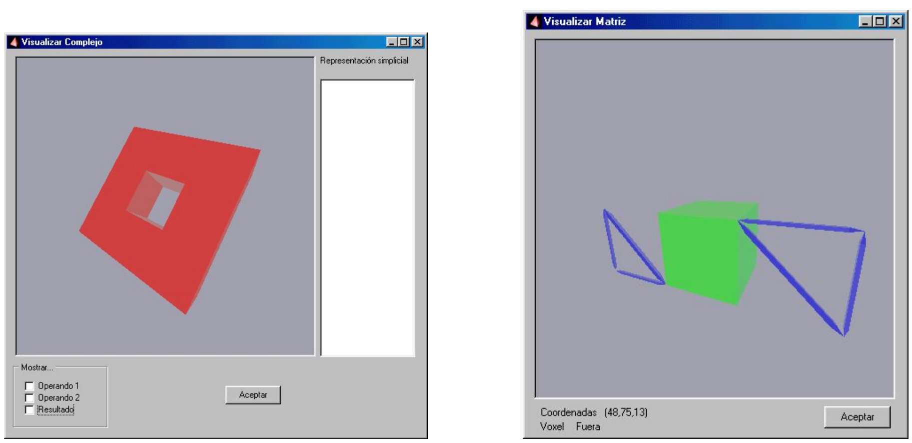

Figure 7: Our simplicial description of the spaces X and Y

We will distinguish these spaces using cup products (this example is showed in [Mun84], pag. 295-297). It is clear that the cohomology groups of the torus are isomorphic to those of Y. They are $\mathbf{Z}$, $\mathbf{Z} \oplus \mathbf{Z}$ and $\mathbf{Z}$ in dimension 0,1 and 2, respectively. However, the cup product on cohomology allows us to distinguish these simplicial complexes.

In Figure 8 we show two representative cocycles $u, v : C_1(X) \to \mathbf{Z}$ generating $H^1(X) = \mathbf{Z} \oplus \mathbf{Z}$, obtained using the process explained in Section 4. The cup product $u \smile v$ is a 2-cocycle that is non-null only for a 2-simplex and it is a representative cocycle w of $H^2(X)$ (see Figure 9). Let us recall that our algorithm for computing cohomology gives us not only the cohomology groups but also an explicit contraction from $C^*(X)$ to $H^*(X)$ (the minimal cochain complex in this case), allowing us to determine both a representative cocycle for each cohomology class and the cohomology class for each cocycle. An analogous computation shows that $[u] \smile [u] = [v] \smile [v] = 0$ and $[v] \smile [u] = -[w]$ (the cup product is anticommutative in cohomology).

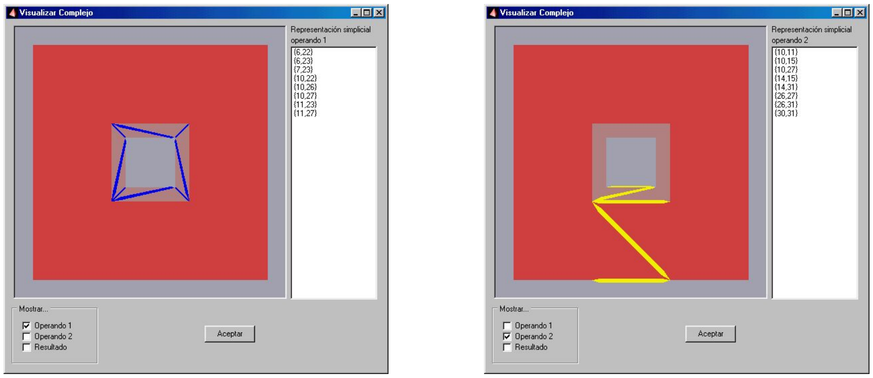

Figure 8: The visualization of the cocycles u and v

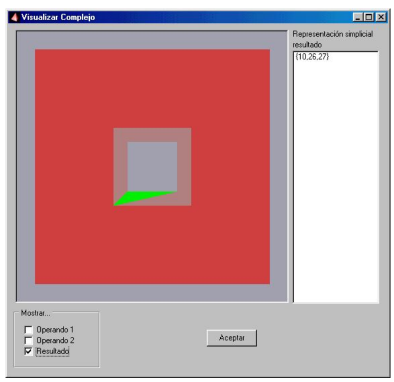

Figure 9: The cup product $u \smile v$ (in green).

If we consider now two representative cocycles u' and v' generating $H^1(Y)$, the result of $u' \smile v'$ is null. Analogous computations show that $[u'] \smile [u'] = [v'] \smile [v'] = [v'] \smile [u'] = 0$. Therefore, it is immediate to conclude that X and Y are not homeomorphic.

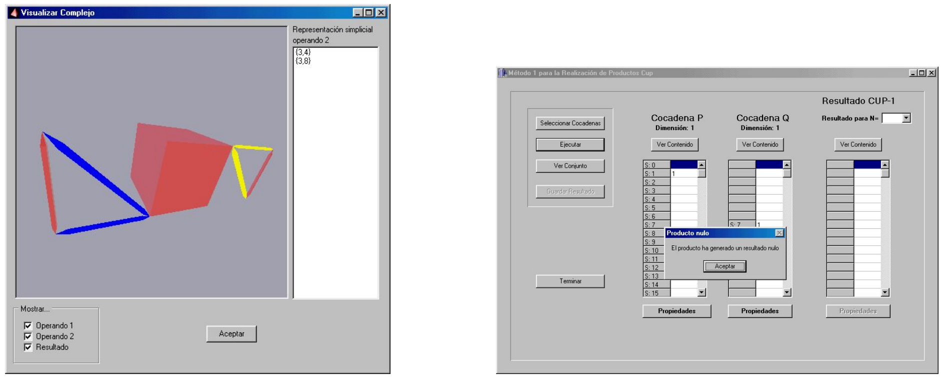

Figure 10: On the left, the visualization of the cocycles u' (in blue) and v' (in yellow), and, on the right, the answer the cup product $u' \smile v'$ is null.

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