THE MODELING OF RDB AND THE IMPROVEMENT OF MILLER'S CONCLUSION

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ABSTRACT. This paper is dealt with the RDB(Restricted Difference Basis) problem. Let K_n be a complete graph of order n and k_n be the number of the edges of K_n. Miller proved that for n large enough the maximal number $f(n)$ of edges that can be marked successively in K_n is about one third of k_n. In this paper we establish an "x link" marks distribution model of ruler. By using the model the following results are proved:

$$f(n) \geq k_{n+1}/2 \quad \text{for} \quad n \geq 3;$$
$$f(n) \geq k_{n+2}/2 \quad \text{for} \quad n \geq 9;$$
$$f(n) \geq k_{n+3}/2 \quad \text{for} \quad n \geq 33;$$
$$f(n) \geq k_{n+4}/2 \quad \text{for} \quad n \geq 409.$$

Thus the Miller's conclusion is improved.

KEYWORDS. complete graph, graceful label, RDB(Restricted Difference Basis), model.

1. INTRODUCTION

Let K_n be a labelled complete graph of order n. Then the mark of an edge in K_n is defined as the difference of the marks of the two vertices incident with it. It is well known that when $n > 4$ there does not exist any graceful labelling in a complete graph K_n. In other words, we are not able to label the vertices in such a way that the marks of edges just take 1 through $\binom{n}{2}$ without repetition. However, in certain practical applications, e.g. in test technology and network design, the request of labelling is weakened. It means that the edges are marked with successive integers from 1 to some $f(n, \sigma) < \binom{n}{2}$ while the repetition of marks is allowed. Here $f(n, \sigma)$ denotes the number of edges can be successively marked, where σ denotes the labelling. For fix n if there exists a labelling σ_0 such that

$$f(n) = f(n, \sigma_0) = \max_{\sigma}\{f(n, \sigma)\},$$

we call the set of the marks of the vertices of K_n, corresponding to σ_0, an **RDB** (**restricted difference basis**). We observe that $f(n)$ is the maximal number of edges that can be marked successively in K_n.

Miller has proved [3] that for n large enough the number $f(n)$ is about one third of k_n, the number of edges of K_n. In this paper our main purpose is to improve the Miller's conclusion. In Section 2 we establish an "x link" marks distribution model of ruler. It is the basic tool for realizing our purpose. Our main result is stated and proved in Section 3. A few numerical results are given in Section 4.

2. "X LINK" MARKS DISTRIBUTION MODEL

As in [2] and [1], we adopt the ruler model in considering the RDB problem: The vertex marked as d in K_n corresponds to a mark on the ruler and the distance between the mark and the started end of the ruler is d. The distance between any two marks of the ruler is just the mark of the edge incident with the two corresponding vertices in K_n. Assume that the ruler's length is L and there are $n(n > 3)$ marks on the ruler, including the two marks of its ends. If any integer length between 1 and L can be measured by the ruler we call the ruler **perfect**. Clearly, a perfect ruler with n marks corresponds to a labelled graph K_n whose edges can be marked successively (repetition allowed). The set of all perfect rulers with n marks is denoted by P_n. For fixed n if the length of a ruler in P_n is not less than the length of any other ruler in P_n we call it an **optimal ruler** or an **RDB ruler**. Obviously, the length of an RDB ruler in P_n is equal to $f(n)$ defined in Section 1.

To construct a perfect ruler we introduce a few terminologies. The distance between any two adjacent marks of the ruler is called a **segment**. If the length of a segment is x units (x is integer) we call it an **"x" segment** and denote it by "x". Successive u "x" segments is called an **"x" block** denoted by $D(x, u)$. Especially, $D(x, 1)$ implies "x" segment and $D(x, 0)$ means null.

A huge numerical experience with computer inspires us to discover the following perfect ruler.

Lemma 2.1. *Let a ruler be divided into successive segments and blocks as below*

$$D(x,u)+D(1,x-1)+m\{"t"+D(1,x-1)\}+"r"+D(1,x-1), \tag{2.1}$$

Where "+" denotes "succeeded by " and $m\{"t"+D(1,x-1)\}$ *denotes* m *successive* $\{"t"+D(1,x-1)\}$*'s. Assume that* x,u,t *and* r *satisfy*

$$r=(u+1)x=t-2x+1. \tag{2.2}$$

Then the ruler is perfect.

Remark. For convenience we call (2.1) an **"x link" marks distribution**. Clearly, $t>r$ and $t>x$. So we call "t" a **maximal segment** and call "r" a **splice segment** of the "x link" marks distribution.

Proof. In the "x link" marks distribution there are m maximal segments and one splice segment. At the left side and the right side of each "t" segment, there is one $D(1,x-1)$ block, respectively. The total number of $D(1,x-1)$ blocks is $m+2$, including the $D(1,x-1)$ block located at one end of the ruler. Under the condition (2.2), by using the appropriate "x link" marks distribution we can measure the following integer length s, respectively:

(1) $1\le s\le ux+x-1=r-1$: by using $D(x,u)+D(1,x-1)$;

(2) $r\le s\le r+2(x-1)=t-1$: by using $D(1,x-1)\}+"r"+D(1,x-1)$;

(3) $t\le s\le t+x-1+r-1$: by using $D(x,u)+D(1,x-1)+"t"+D(1,x-1)$;

$t+x-1+r\le s\le t+x-1+t-1$: by using $D(1,x-1)+"t"+D(1,x-1)+"r"+D(1,x-1)$;

(4) $(c-1)(t+x-1)+t\le s\le c(t+x-1)+r-1$ ($c=1,2,\ldots,m$): by using $D(x,u)+D(1,x-1)+c\{"t"+D(1,x-1)\}$;

$c(t+x-1)+r\le s\le c(t+x-1)+t-1$: by using $D(1,x-1)+c\{"t"+D(1,x-1)\}+"r"+D(1,x-1)$;

(5) $m(t+x-1)+t\le s\le L$: by using $D(x,u)+D(1,x-1)+m\{"t"+D(1,x-1)\}+"r"+D(1,x-1)$.

Hence the ruler is perfect. □

3. MAIN RESULT AND PROOF

Theorem 3.1. *Let $f(n)$ be the maximal number of edges which can be marked successively in K_n under the assumption of RDB. Let k_n be the number of the edges of K_n. Then*

$$f(n) \geq k_{n+1}/2 \quad \textit{for} \quad n \geq 3; \tag{3.1}$$

$$f(n) \geq k_{n+2}/2 \quad \textit{for} \quad n \geq 9; \tag{3.2}$$

$$f(n) \geq k_{n+3}/2 \quad \textit{for} \quad n \geq 33; \tag{3.3}$$

$$f(n) \geq k_{n+4}/2 \quad \textit{for} \quad n \geq 409. \tag{3.4}$$

Proof. We apply the "x link" marks distribution model of ruler to prove the theorem. From (2.2) we have

$$t = (u+3)x - 1. \tag{3.5}$$

The ruler is divided into $n-1$ parts by the n marks. The expression (2.1) implies $u + m + (m+2)(x-1) + 1 = n - 1$. That is

$$u = n - (m+2)x. \tag{3.6}$$

Clearly

$$L = ux + mt + r + (m+2)(x-1).$$

From (2.2), (3.5) and (3.6), eliminating u, t and r, we obtain

$$L = (m+2)(n+4-2/x-mx-2x)x - 5x + 2. \tag{3.7}$$

(1) Letting $x = 1$("1 link" marks distribution) in (3.7) we get

$$L = (m+2)(n-m) - 3.$$

In the last expression, for fixed n, L is the quadratic function of m. Then for even n, L has a maximum at $m = (n-2)/2$

$$L_{\max} = \left(\frac{n+2}{2}\right)^2 - 3 = \frac{n^2}{4} + n - 2,$$

and for odd n, L has a maximum at $m = (n-1)/2$ or $m = (n-3)/2$

$$L_{\max} = \frac{n^2}{4} + n - \frac{9}{4}.$$

Hence we conclude that the maximal length $L_{\max}$ of the ruler constructed by "1 link" marks distribution satisfies

$$L_{\max} \geq \frac{n^2}{4} + n - \frac{9}{4}.$$

To make $f(n) \geq k_{n+1}/2$ we need only to make

$$\frac{n^2}{4} + n - \frac{9}{4} \geq \frac{n(n+1)}{4}.$$

Whence $n \geq 3$. Thus (3.1) is verified.

(2) In (3.7), letting $x = 2$ and noticing $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$, we obtain

$$L = (2m + 4)(n - 1 - 2m) - 8.$$

Then for $n \equiv 3 \pmod 4$, L has a maximum at $m = (n-3)/4$,

$$L_{\max} = \frac{n^2}{4} + \frac{3n}{2} - \frac{27}{4}. \tag{3.8}$$

For each $n \equiv 0, 1, 2 \pmod 4$ we can find the maximum of L greater than the right hand side of (3.8). Therefore the maximal length L_{max} of the ruler constructed by "2 link marks distribution" satisfies

$$L_{\max} \geq \frac{n^2}{4} + \frac{3n}{2} - \frac{27}{4}.$$

To make $f(n) \geq k_{n+2}/2$ we need only to make

$$\frac{n^2}{4} + \frac{3n}{2} - \frac{27}{4} \geq \frac{(n+2)(n+1)}{4}.$$

Whence $n \geq 10$. On the other hand, when $n = 9$, by using the "2 link marks distribution" $\{2, 2, 2, 1, 11, 1, 8, 1\}$, we can construct a ruler with length $L = 28 > \frac{k_{11}}{2} = 27.5$. Thus (3.2) is proved.

(3) From (3.8) and

$$\frac{n^2}{4} + \frac{3n}{2} - \frac{27}{4} \geq \frac{(n+3)(n+2)}{4}$$

we get $n \geq 33$. Thus (3.3) is proved.

(4) Taking $x = 7$ in (3.7) we get

$$L = (7m + 14)(n - 7m - \frac{72}{7}) - 33 = \left(\frac{n + \frac{26}{7}}{2}\right)^2 - \left(\frac{n - 14(m+2) + \frac{26}{7}}{2}\right)^2 - 33.$$

Then for $n > 44$ and $n \equiv 3 \pmod{14}$ L has a maximum at $m = (n-31)/14$

$$L_{\max} = \frac{n^2}{4} + \frac{13n}{7} - \frac{1143}{28}. \tag{3.9}$$

For each $n \not\equiv 3 \pmod{14}$ We can find the maximum of L greater than the right hand side of (3.9). Therefore the maximal length $L_{\max}$ of the ruler constructed by "7 link marks distribution" satisfies

$$L_{\max} \geq \frac{n^2}{4} + \frac{13n}{7} - \frac{1143}{28}.$$

Let

$$\frac{n^2}{4} + \frac{13n}{7} - \frac{1143}{28} \geq \frac{(n+4)(n+3)}{4}.$$

We get $n \geq 409$. Thus (3.4) holds.

And the proof of the theorem is finished. □

Remark: There is a reasonable problem: for larger n whether we can derive

$$f(n) \geq \frac{k_{n+5}}{2}?$$

The answer is negative. Really, from (3.7) we can infer

$$L = \left(\frac{n}{2} + 2 - \frac{1}{x}\right)^2 - x^2\left(m - \frac{nx + 4x - 2}{2x^2} + 2\right)^2 - 5x + 2. \tag{3.10}$$

Whence for any positive integers x and m we have

$$L < \left(\frac{n}{2} + 2\right)^2 < \frac{k_{n+5}}{2} = \frac{(n+5)(n+4)}{4}. \tag{3.11}$$

4. NUMERICAL RESULTS

In the following table we give a few typical lengths of rulers with n marks generated by "x link" marks distributions. In the third column of the table we list the data from Miller's conclusion to contrast with the data from our results.

n	L	("x link")	$k_n/3$ (Miller)	$k_{n+1}/2$	$k_{n+2}/2$	$k_{n+3}/2$	$k_{n+4}/2$
3	3	("1")	1	3			
9	28	("2")	12		28		
33	316	("2")	176			315	
409	42539	("7")	27812				42539

The computer implementation of these numerical results is simpler, so we omit it.

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