GRAPHS BEFORE ALGEBRA: AN EXERCISE IN IMAGINARY CURRICULUM

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Abstract

The technology that we use to present a topic to our students, and the technology that we expect our students to use in class, can have profound effects on what is taught and the way in which it is taught. Sometimes this does not happen for a number of reasons, including concerns as to how much the students will "understand" using the new technology, and the development of new curriculum and new methods can be slowed or stopped altogether. One way to work through these difficulties is to present blocks of "imaginary curriculum" as a basis for discussion among teachers and authorities, working through the sorts of understanding we would expect students to develop and identifying points where something might be gained or lost. A specimen topic is developed in this paper.

In Australia in the middle years of secondary school students typically learn to work algebraically with quadratic functions, first multiplying out products of terms and then reversing the process to factorize quadratics. Later this is extended into completing the square and solving quadratic equations by factorizing, by completing the square or by use of the standard formula. Later again students use these algebraic skills in constructing graphs of quadratic functions.
In this paper the author demonstrates how the process can be completely reversed, using modern hand-held graphing calculators, by studying graphs first and then deducing all the standard algebraic processes. Examination of the underlying processes suggests that this method is at least as easy to understand, if not clearer; that students will "understand" at least as much as they do by learning the traditional processes; and that they will be more able to extend their skills into more advanced problems. In an ideal curriculum students would probably learn both approaches and the connections between them, to enable them to check their work and reinforce their understanding in multiple ways.

Introduction

Over the last three decades there has been a technological revolution in the teaching and learning of mathematics centred on computers and various hand-held calculators. Typically the arrival of new technologies has been driven by the manufacturers somewhat in advance of the bulk of the teaching community, which has then had to react to each new object retrospectively. Often this leads to situations of conflict between those keen to exploit new possibilities and those anxious to preserve existing standards and practices. The most common outcome is that old curriculum is preserved either intact or modified slightly, with the uses of new technology limited by being retro-fitted to the old requirements (Jones, 2000).

In Australia and elsewhere much current discussion is centred on the new possibilities of algebraic processing. Computer algebra systems have been available for more than a decade but have made very little impression on the school curriculum, rather like the earliest computer-based graphing packages. More recently hand-held devices that provide full algebraic processing have been available, and a number of initiatives are under way that attempt to address the issues that follow. How should such devices be used in teaching? What changes should be made to the curriculum? What are the key skills and knowledge that we want students to acquire? Should the device be available to students without restriction, or should its use be limited in some way? What should our students be able to do with the technology, and what without it? Good current examples with extensive further bibliographies can be found at Stacey et al (2000) and Kutzler (1999); and a recent local discussion focussing on alleged "understanding" is reported in Tobin (1999).

One element of the discussion has been individuals or groups listing items of curriculum or test items according to whether they should be done with or without technology (example: Herget et al (2000)). It is not always clear what methodology is being applied here, or how the essential skills are to be identified, except by the expression of opinion of one or more writers making essentially ad hoc decisions item by item.

To take an example, "should" a senior secondary student be able to factorize the trinomial $x^2 + 5x + 6$ without technology? Consider the following imaginary students:
A is well-trained in traditional methods, and immediately knows "I need two factors of 6 that add up to 5"; quickly chooses 2 and 3, and writes $(x + 2)(x + 3)$.
B is well-trained in graphical methods: he or she takes a graphical calculator, graphs the function, observes the roots at –2 and –3 and then deduces the factorization.
C takes an algebraic processor and uses the factorization command to obtain the answer directly.
Which students "understand" the process? Which "understands" it best? To say that A exhibits the greatest "understanding" simply because he or she has produced the answer without technology is far too simplistic: a trick has been performed, but what meaning the student attributes to it has not been established. Student B, I would argue, clearly "understands" the relationship between factors and roots, and the nature of quadratic functions; what has not been established is whether the student could have found the roots unassisted by technology (e.g. by trial substitutions and calculation). What student C understands is impossible to say!

In the real world curricula are established by fairly cumbersome processes and become very difficult to change. When a new technology appears we seem to be trapped in an inevitable cycle of initial suspicion or rejection (apart from a few enthusiasts), followed by gradual acceptance in limited contexts, followed by eventual concession, often without any real rethinking of what is being taught or why; the new technology is limited by the old and inflexible curriculum. Radical experimentation is rarely possible except in very limited contexts, so that we rarely see the full potential of the new technology to transform and enrich the material we teach.
As a way forward, I want to suggest that one way of furthering the debate is to produce specimens of **imaginary curriculum**, in which we imagine how a new or traditional topic might be taught using available technology and try to follow through the understandings and thought processes that students so taught would be expected to develop. Informed discussion can then follow as to what might be gained and what lost, and value judgements made as to the worth or otherwise of the new approach versus the old. If consensus emerges, the imaginary curriculum can then be trialed on real students and tested thoroughly. In the balance of this paper I give a specimen of this approach, in the middle secondary topic of the factorization and algebraic manipulations of quadratic functions.

While other authors and teachers have certainly used a graphical approach to illuminate the standard algebra (such as Brown, 1998), I am proposing a rather more radical reworking.

QUADRATIC FUNCTIONS: A TRADITIONAL APPROACH

1. Local students typically begin their study of this topic with the algebraic task of multiplying out products of the forms $(x + a)(x + b)$, $(ax + c)(bx + d)$ etc, and collecting like terms to obtain a trinomial. This is taught initially as a purely mechanical process, but students may begin to observe patterns in their results.
A special study is usually made of perfect squares.

2. A little later students meet the reverse process of factorization:
(a) All positive terms $x^2 + ax + b = (x + A)(x + B)$
Students must find two factors A and B of b adding to a. Results are only expected for integers a, b with two integer factors A, B.
(b) Negative middle term $x^2 - ax + b = (x - A)(x - B)$
Essentially this is the same process.
(c) Negative constant term $x^2 \pm ax - b = (x + A)(x - B)$
Students must find two factors of b with difference a, and then work out which term in the factors has which sign. Generally this is found to be "harder", but only expected with simple integers.
(d) Recognition of special type: perfect squares.
(e) Recognition of special type: difference of two squares.
Here it is customary to introduce a new complication of irrational roots, e.g.

$$x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$$

(f) Hardest types: non-monic $(a \neq 1)$ $ax^2 + bx + c = (Ax + B)(Cx + D)$
These are normally solved by repeated trial-and-error through possible combinations of (integer) factors of a and c balancing to give a middle term of b. Good students pick up the "trick" fairly easily but are not often able to explain it well.

Throughout this sequence of lessons the objects of the study are treated as abstract algebraic forms. The only "checking" possible is for the student to remultiply the terms obtained to recover the original expression, which is unlikely to clarify misunderstandings for struggling students. It is rare for the expressions to be related to functions (e.g. by substituting values for x and comparing values in factored and unfactored forms), and practically unknown for them to be graphed — a much later part of the curriculum.

3. The process of completing the square is a separate study, with a quite unrelated process ("trick"). Students who master the idea simply follow the rules; many find it a perpetual puzzle.

4. Solving quadratic equations $ax^2 + bx + c = 0$:
The order of precedence here has hardly changed in decades. First, the student attempts a factorization and then reads off the roots by setting each term to zero. If factors cannot be found an attempt will be made to solve by completing the square (in many cases with little success or understanding); then the quadratic equation formula will be developed and presented as an alternative. Students will rapidly fall into the habit of first attempting a factorization, and then, if that does not succeed, using the formula. In the state of Victoria the formula has been provided in

most examinations since 1970. If the formula leads to integer or simple rational solutions some students will then be able to reconstruct a factorization; in other problems the formula leads to irrational expressions which most students evaluate on electronic calculators. In Victoria it is unusual for emphasis to be placed on the difference between the exact (irrational) and approximate (numerical) values obtained, which raises the question as to why numerical values are not found by other methods such as the calculator's solving functions.

5. Graphing quadratic functions $y = ax^2 + bx + c$:
In Australia graphing comes relatively late, and all of the above algebra is used to assist it. Once the general parabolic shape is established students predict the steepness and orientation from the value of a, the y-intercept from c, the x-intercepts (if any) from the solutions of the equation y = 0, and the location of the vertex (A, B) by completing the square into $y = a(x - A)^2 + B$. The graph is then drawn to incorporate all these expectations.

SOME QUESTIONS CONCERNING PRECEDENCE OF IDEAS

The first question I would like to pose is why detailed graphing is postponed as long as it is in the traditional syllabus. Students meet the idea of graphing a simple function near the beginning of secondary education, but rarely go beyond linear functions. They then learn a great deal of algebra and even calculus which is ultimately used as the basis for drawing detailed graphs of more complicated functions: always the algebra or calculus precedes the graphing and is used to justify or clarify it. Why?

There is nothing intellectually difficult about the idea of graphing a function, once the actual calculation of values is possible: we calculate paired values of x and y and plot the points, and join them into a smooth curve. In theory any points of confusion can be clarified by calculating further points, at least in the case of the functions graphed in the traditional secondary syllabus. For example, what would a student unfamiliar with the traditional algebra just described make of the function $y = 3x^2 - 8x + 1$?

Before modern technology was available we first calculated (mentally) some integer values:

x	0	1	2	3	4
y	1	–4	–3	4	17

and then plotted these on graph paper (an old technology). We can see the y-intercept exactly, we can see there must be x-intercepts between 0 and 1 and between 2 and 3, and with some experience of the general shape we can see roughly where the vertex will come.

To refine these details all we need is to calculate further values, but it is here that we really start to appreciate modern calculators! With a lot of patience we could explore the lower root by calculating:

x	y	x	y	x	y
0.0	1	.10	.2300	.130	.010700
0.1	0.23	.11	.1563	.131	.003483
0.2	–0.48	.12	.0832	.132	–.003728
		.13	.0107		
		.14	–.0612		

and so far we have almost found the root to three decimal places, but at the cost of no little calculating time. *Under these circumstances* it would be greatly more efficient to use the formula to obtain $x = \frac{8 - \sqrt{52}}{6} = \frac{4 - \sqrt{13}}{3}$, which requires only a simple table of square roots to give us a numerical value of 0.1315. Similarly, to find the vertex by calculating alone would require more messy and boring work:

x	y	x	y	x	y
1.0	–4	1.30	–4.3300	1.330	–4.333300
1.1	–4.17	1.31	–4.3317	1.331	–4.333317
1.2	–4.28	1.32	–4.3328	1.332	–4.333328
1.3	–4.33	1.33	–4.3333	1.333	–4.333333
1.4	–4.32	1.34	–4.3332	1.334	–4.333332

It is *much* easier to "know" that the vertex occurs at $x = -\frac{b}{2a} = \frac{4}{3}$!

So in the old technological context, where calculating beyond mental and integer arithmetic is cumbersome and tedious, algebra and (later) calculus are powerful weapons for reaching important information precisely and efficiently. But there have been not one but two important advances since those days: simple machines to do computations vastly more efficiently (calculators, spreadsheets), and slightly more advanced machines or programs to graph the results (graphical calculators and packages). In this not exactly new circumstance, is it not time to ask whether the algebraic and calculus approaches have not lost some of their importance?

Of course I produced all of the above calculations on a machine: in fact using the tabulation functions of a graphical calculator. Students could have been doing this with ordinary pocket calculators, or with spreadsheets on a computer, for more than twenty years, and in many cases have been, but only as an adjunct to the traditional approach. For the last ten years or more we have had the additional option of producing instant graphs, which allows the function to be visualized in detail as never before. All these machines or packages are doing is bulk calculation followed by a joining-the-dots process: there is no higher mathematics involved, and no circular reasoning if we use this technology to develop and explain the algebra or calculus. This is essentially what I am proposing in the specimen of imaginary curriculum that follows.

A GRAPHICAL APPROACH TO QUADRATIC FUNCTIONS

Assume a group of students comfortable with graphing calculators and beginning to explore the world of functions and their graphs. We introduce the squaring function $y = x^2$ and note the new shape, and then explore the effect of altering it, adding or multiplying by constants, or adding a linear function, noting how robust the fundamental shape is.

1. We propose a group of quadratic functions of the form $y = ax^2 + bx + c$.
We note the "obvious" features of the graphs: all are vertically symmetric parabolas, with the orientation determined by the coefficient a; all cut the y-axis exactly once (at y = c); all have a vertex; there may be one, two or no x-axis intercepts. It should be clear and understood that the function is pairing values of x and y, with the graph somehow illustrating the connections.

2. We note the effect on the shape of some simple transformations:
(a) $y = ax^2$ changes the apparent steepness and the orientation, depending on the value of a. [In fact the shape does not change, only the portion of it viewed.]
(b) $y = ax^2 + c$ moves the shape vertically.
(c) $y = a(x - B)^2$ moves the shape horizontally.

In some graphics-based courses the concepts of horizontal and vertical translation are introduced earlier on, with linear functions, but unfortunately we cannot distinguish visually between vertical and horizontal movements with lines. Parabolas provide the simplest examples where these concepts can be clearly illustrated and distinguished.

3. From this we deduce that a parabola with vertex (A, B) must have the equation

$$y = a\,(x - A)^2 + B$$

for some value of a.

Completing the square can thus be accomplished purely graphically: given the form $y = ax^2 + bx + c$, we obtain the graph, locate the vertex using the calculator's minimum or maximum functions, and write out the squared form, using the same value of a. The reverse process should probably be done algebraically by multiplying out, but can instantly be checked by comparing the graphs of the two forms.

[Students who appreciate the relative inaccuracy of the maximum and minimum functions can exploit the symmetry properties of the parabola. If there are two x-axis intercepts they can be found from the graph to high accuracy, and the average of these two x-values is the x-value of the vertex. If there are no x-axis intercepts we can draw some appropriate line y = d that cuts the parabola twice and use the two intersection points instead.]

4. A parabola that cuts the x-axis twice at x = A, B must have the equation

$$y = a\,(x - A)\,(x - B)$$

for some value of a.

Factorizing, when possible, can thus be accomplished purely graphically: given the form $y = ax^2 + bx + c$, we obtain the graph, locate the intercepts using the calculator's zero or root functions, and write out the factorized form, using the same value of a. The reverse process can again be done algebraically by multiplying out, and can instantly be checked by comparing the graphs of the two forms.

When this is understood it is a simple extension to note that a parabola that touches the axis at x = A must have the equation $y = a\,(x - A)^2$, and there is no particular mystery as to why some quadratics have no real roots at all, and hence no real factors.

5. Students who are used to the idea of the quadratic function (rather than the rather more abstract algebraic trinomial) can then exhibit and check the factorization in a sort of spreadsheet, for example:

$$x^2 + 5x + 6 = (x + 2)(x + 3).$$

x	x^2	5x	$x^2 + 5x$	$x^2 + 5x + 6$	x + 2	x + 3	(x + 2) (x + 3)
0	0	0	0	6	2	3	6
1	1	5	6	12	3	4	12
2	4	10	14	20	4	5	20
3	9	15	24	30	5	6	30
4	16	20	36	42	6	7	42

This provides a sometimes fascinating alternative to the other possibilities of graphical and algebraic checking.

6. The process of solving quadratic equations is essentially the same as factorizing, when done graphically. Given an equation to solve, a student might go directly to the calculator's (numerical) equation solver, or read the numerical roots from the graph. Some calculators at least can recover

non-integer rational roots in exact rather than decimal notation (such as with the Texas Instruments fraction converter). Given some experience and some algebraic insight, advanced students can also recover the exact form of irrational roots for equations with rational coefficients if desired. The algebraic form of the quadratic equation formula remains, of course, a piece of algebraic work, but it can be easily illustrated and more readily digested when the graphical approach is understood.

Essentially that covers the standard secondary syllabus. In addition there are a number of purely graphical "tricks" which can be developed if desired. For instance, it is possible to identify the equation of a quadratic function entirely from its graph, if the graph can be interrogated: one can find intercepts (if they exist) and deduce the form $a(x – A)(x – B)$; or one can find the vertex and deduce the form $a(x – A)^2 + B$. In either case the coefficient "a" is to be determined, probably most simply by stepping one unit away from a root or from the vertex and observing the value of y. Students can then convert to standard form $y = ax^2 + bx + c$ by algebra and check their work by making sure the graphs coincide.
Another "trick" can find the complex roots of a quadratic with negative discriminant. Locate the vertex of the graph at (A, B) and draw the line $y = 2B$; then find the intersection points of the parabola and the line. If half the horizontal difference between them is b, the complex roots are $A \pm ib$.

EXTENSIONS AND OBJECTIONS

One of the really exciting features of the graphical approach is the way it generalizes easily to polynomials of higher degree. A student who has mastered the above points will have no difficulty in working with polynomials of degree n that have n distinct roots, because the factorization will be immediately apparent from the graph. Repeated roots will be identifiable by turning points on the axis, and complex root pairs by insufficient real roots. The process of division to find outstanding roots (or quadratic factors) can be done graphically by dividing the equation by the so-far known factors; if the result is then a parabola the student can work out its equation from the graph as explained above.
Of course there are occasions when a purely graphical approach will be inadequate. All graphing packages and devices work on the basis of a fairly small number of calculations and pixels, and anything that falls into the gaps between the pixels is effectively invisible. It is very difficult to tell the difference between a quadratic that "just" misses the axis and one that touches it, or one that has two extremely close roots, but of course these examples are not seen in the conventional syllabus either, where typically all coefficients are small integers. More complicated misleading examples can be constructed with polynomials of higher degree, but again this is unfair given that in the standard syllabus students deal only with cubics, for instance, that have one small integer root (and then deal with the balance by factoring to get a quadratic quotient). The calculator is not concerned if the coefficients of a polynomial are integers or fractions, small or large numbers, as long as students gain the skill of selecting an appropriate window each time. The set of problems that can be handled effectively with graphing technology seems to be much more extensive than those covered in the current syllabuses.
I am not proposing an abolition of algebra! I believe that students understand best when more than one approach is used, and the connections are made explicit, and students are encouraged to use technology to explore problems and check their work (Barling & Jones, 2000). What I particularly want to challenge is the current primacy of algebra in this basket of topics. I want to suggest that some algebraic topics were introduced initially to facilitate graphing, and that with modern

technology some might well be obsolete, or more usefully studied via graphing rather than the reverse.

In addition I believe that, whatever sense we end up making of that much invoked concept "understanding", an average student trained through the graphical approach will exhibit a deeper and more mature level of it than an average student trained in the traditional way. The former I would expect to have some notion of the polynomial as a shorthand for a function as well as an object of pure algebra, and be able to relate this function as well to its graph and to a table of selected values; he or she will also have a direct knowledge of the relationship between factors of the polynomial, roots of the related equation and intercepts of the matching graph. The latter, all too often, has mastered a series of "tricks" that each belong to a different situation with few obvious connections, and it is precisely in the connection between related concepts that the heart of "understanding" seems to lie. While the graphically trained student might also develop this sort of segmentation, it seems much less likely. For this paper I pass over the obvious next stage of what happens when the student has access to a computer algebra device that can do all of the required tasks directly.

It may well be that in the current reconsiderations of curriculum we end up requiring students to perform certain sorts of bookwork without the use of any electronic technology, to demonstrate traditional "understanding". But even when studying topics and techniques so labelled, students should also be permitted and encouraged, outside the very artificial situation of technology-free assessment exercises, to use technology to check answers and reinforce their understanding, if not also to make preliminary explorations.

In conclusion, I would like to restate my belief that most current secondary algebra can be done equally effectively using graphing technology, and in many cases the graphing technology makes the task much simpler for students who understand its use. I suspect that students who use this technology well will be found to exhibit at least as much "understanding" as those adept at the current box-of-tricks procedures. Graphing technology enables students to fulfil all current expectations of the secondary syllabus, and generalizes much more readily to harder or "impossible" situations on which the current syllabus is effectively silent.

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