An Inequality-Proving Program Applied to Global Optimization

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Abstract It is illustrated how to use an efficient inequality-proving program to solve global optimization problems whereof the objective function and the constraints all are polynomials. Some examples are as benchmarks taken from relevant literature.

1. Inequalities with radicals

It was introduced a practical algorithm and accordingly a generic program BOTTEMA for automated reasoning on inequalities. A sketch about the algorithm can be found in [12, 13, 14], and a more detail exposition (for that may also see [17] which is aimed at geometric inequalities) will be given here in the next section.

The main purpose of this article is to demonstrate how to apply our algorithm and program to a class of problems for global optimization. The basic strategy we take is to convert the optimal value finding to finitely many inequality verifying. The argument based on a very simple fact: if $f \le b$ is true but $f \le a$ is false, then $a \le f_{\max} \le b$ is true where $f_{\max}$ stands for the greatest value of f.

Let us begin with an inequality proposition.

Proposition 1. The following inequality holds,

$$x_1 x_2 x_3 x_4 x_5 \le 3, \tag{1}$$

provided

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 10, \tag{2}$$
$$x_2 x_3 = 5\, x_4 x_5, \tag{3}$$
$$x_1^3 = x_2^3 + 1, \tag{4}$$
$$x_1 > 0,\, x_2 > 0,\, x_3 > 0,\, x_4 > 0,\, x_5 > 0, \tag{5}$$
$$x_1 \le \frac{23}{10},\, x_2 \le \frac{23}{10},\, x_3 \le \frac{32}{10},\, x_4 \le \frac{32}{10},\, x_5 \le \frac{32}{10}. \tag{6}$$

To verify this proposition, we may employ a decomposition algorithm such as cylindrical algebraic decomposition [3, 4]. The basic idea is: decompose the space of parameters $x_1, \cdots, x_5$ into a finite number of parts, i.e. some cells with different dimensions, pick out all the parts where the hypothesis of the proposition holds, and then check whether the conclusion $x_1 x_2 x_3 x_4 x_5 \le 3$ holds over the parts picked out. If so, Proposition 1 is true; otherwise, it is false.

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We need deal with a problem of 5-dimensional space if we do not take any dimension-decreasing measure. Generally speaking, it would be impossible to implement a non-trivial algebraic decomposition for a 5-dimensional space by means of the current computer softwares and hardwares. In fact, it was said that such a decomposition is very difficult even for spaces of dimensions more than 3. So, we had better take some measures to keep the dimension as low as possible.

Noting that the hypothesis includes 3 equations, we thus eliminate 3 variables (x_1, x_4, x_5) by solving equation system (2-4) and convert Proposition 1 to a form encoded in x_2, x_3.

Proposition 2. The following inequality holds,

$$\frac{1}{5}(x_2^3+1)^{\frac{1}{3}}x_2^2\,x_3^2 \le 3, \tag{7}$$

provided

$$x_2 > 0,\ x_3 > 0, \tag{8}$$

$$x_2^3 + 1 \le \frac{12167}{1000}, \tag{9}$$

$$x_3 \le \frac{32}{10}, \tag{10}$$

$$10 - (x_2^3+1)^{\frac{2}{3}} - x_2^2 - x_3^2 - \frac{2}{5}x_2x_3 \ge 0, \tag{11}$$

$$\sqrt{250 - 25\,(x_2^3+1)^{\frac{2}{3}} - 25\,x_2^2 - 25\,x_3^2 + 10\,x_2x_3} + \sqrt{250 - 25\,(x_2^3+1)^{\frac{2}{3}} - 25\,x_2^2 - 25\,x_3^2 - 10\,x_2x_3} \le 32. \tag{12}$$

In comparison with Proposition 1, the number of variables reduces to 2, but some radicals occur in both hypothesis and conclusion. To eliminate these radicals, the conventional means is to introduce new variables, that way the problem returns to 5-dimensional again. So, we had better employ an efficient algorithm not only eliminating the radicals but also keeping the dimensions non-increasing. It is what the dimension-decreasing algorithm does.

we proved Proposition 2 (hence Proposition 1) using program BOTTEMA on a Pentium 3/450 with time about 6 seconds. One more example seems simpler than Proposition 1 but it took us more time.

Proposition 3. The following inequality holds,

$$\frac{2455}{100}x_1 + \frac{2675}{100}x_2 + 39\,x_3 + \frac{405}{10}x_4 \ge 30, \tag{13}$$

provided

$$x_1 + x_2 + x_3 + x_4 = 1, \tag{14}$$

$$0 < x_1 \le 1,\ 0 < x_2 \le 1,\ 0 < x_3 \le 1,\ 0 < x_4 \le 1, \tag{15}$$

$$\frac{23}{10}x_1 + \frac{56}{10}x_2 + \frac{111}{10}x_3 + \frac{13}{10}x_4 \ge 5, \tag{16}$$

$$12\,x_1 + \frac{119}{10}x_2 + \frac{418}{10}x_3 + \frac{521}{10}x_4 - 21 - \frac{1645}{1000}\sqrt{\frac{28}{100}x_1^2 + \frac{19}{100}x_2^2 + \frac{205}{10}x_3^2 + \frac{62}{100}x_4^2} \ge 0. \tag{17}$$

The hypothesis includes one equation, so we eliminate x_4 by solving equation (14) and convert Proposition 3 to a form encoded in x_1, x_2, x_3.

Proposition 4. The following inequality holds,

$$-\frac{319}{20}x_1 - \frac{55}{4}x_2 - \frac{3}{2}x_3 + \frac{81}{2} \geq 30, \tag{18}$$

provided

$$x_1 > 0,\ x_2 > 0,\ x_3 > 0, \tag{19}$$

$$x_1 + x_2 + x_3 \leq 1, \tag{20}$$

$$10\,x_1 + 43\,x_2 + 98\,x_3 \geq 37, \tag{21}$$

$$401\,x_1 + 402\,x_2 + 103\,x_3 + \frac{329}{200}(90\,x_1^2 + 81\,x_2^2 + 2112\,x_3^2 + 62 - 124\,(x_1 + x_2 + x_3 - x_1x_2 - x_2x_3 - x_1x_3))^{\frac{1}{2}} \leq 311. \tag{22}$$

This proposition is false. We disproved it (hence Proposition 3) on the same machine with time 68 seconds.

Automated theorem proving on inequalities is always considered as a difficult topic in the area of automated reasoning. The relevant algorithms depend fundamentally on real algebra and real geometry, and the computational complexity increases very quickly with the dimension, that is, the number of parameters. Some well-known algorithms are complete theoretically but inefficient in practice, which cannot verify non-trivial propositions in batches. For progress made in this aspect recent years, see [5, 6, 1, 2, 10, 11, 15, 16].

2. Dimension-Decreasing Algorithm

Before we describe the so-called dimension-decreasing algorithm, some definitions should be introduced and illustrated.

Definition 1. Assume $l(x,y,z,\cdots)$ and $r(x,y,z,\cdots)$ are continuous algebraic functions of $x,y,z,\cdots$. We call

$$l(x,y,z,\cdots) \leq r(x,y,z,\cdots) \quad \text{or} \quad l(x,y,z,\cdots) < r(x,y,z,\cdots)$$

an algebraic inequality in $x,y,z\cdots$, and $l(x,y,z,\cdots) = r(x,y,z,\cdots)$ an algebraic equality in $x,y,z,\cdots$.

Definition 2. Assume Φ is an algebraic inequality (or equality) in $x,y,z,\cdots$. $L(T)$ is called a *left polynomial* of Φ, provided

- $L(T)$ is a polynomial in T, its coefficients are polynomials in $x,y,z,\cdots$ with rational coefficients.
- The left hand side of Φ is a zero of $L(T)$.

The additional item following is unnecessary for this definition, but it would be helpful to reducing the computational complexity in the process later.

- Amongst all the polynomials satisfying the two items above, $L(T)$ is what has the lowest degree in T.

The *right polynomial* of Φ, namely, $R(T)$, can be defined analogously.

Definition 3. Assume Φ is an algebraic inequality (or equality) in $x, y, \cdots$ etc., $L(T)$ and $R(T)$ are the left and right polynomials of Φ, respectively. By $P(x, y, \cdots)$ denote the resultant of $L(T)$ and $R(T)$ with respect to T, and call it the critical polynomial of Φ, and the surface defined by $P(x, y, \cdots) = 0$ *the critical surface of* Φ, respectively.

For example, let us go to compute the left, right and critical polynomials of inequality (12), i.e.

$$\sqrt{250 - 25\,(x_2^3+1)^{\frac{2}{3}} - 25\,x_2^2 - 25\,x_3^2 + 10\,x_2x_3} \\ +\sqrt{250 - 25\,(x_2^3+1)^{\frac{2}{3}} - 25\,x_2^2 - 25\,x_3^2 - 10\,x_2x_3} \le 32.$$

We first set auxiliary variables,

$$\begin{aligned} y_1 &= (x_2^3+1)^{\frac{1}{3}}, \\ y_2 &= \sqrt{250 - 25\,y_1^2 - 25\,x_2^2 - 25\,x_3^2 + 10\,x_2x_3}, \\ y_3 &= \sqrt{250 - 25\,y_1^2 - 25\,x_2^2 - 25\,x_3^2 + 10\,x_2x_3}, \end{aligned}$$

and rationalize these radical equalities to obtain polynomial equations, namely, $f_1 = 0,\ f_2 = 0,\ f_3 = 0,$ where

$$\begin{aligned} f_1 &= y_1^3 - x_2^3 - 1, \\ f_2 &= y_2^2 - (250 - 25\,y_1^2 - 25\,x_2^2 - 25\,x_3^2 + 10\,x_2x_3), \\ f_3 &= y_3^2 - (250 - 25\,y_1^2 - 25\,x_2^2 - 25\,x_3^2 - 10\,x_2x_3). \end{aligned}$$

Then, eliminate y_1, y_2, y_3 from $y_2 + y_3 - T$ successively by resultant computation: Let

$$\begin{aligned} \tau_2 &= \text{resultant}(y_2 + y_3 - T,\ f_3,\ y_3), \\ \tau_1 &= \text{resultant}(\tau_2,\ f_2,\ y_2), \\ \tau_0 &= \text{resultant}(\tau_1,\ f_1,\ y_1). \end{aligned}$$

Obviously, τ_0 is a left polynomial of (12). Print it in detail:

$$\begin{aligned} &T^{12} + 300\,(x_3^2 - 10 + x_2^2)\,T^{10} \\ &\quad + 1200\,(-500\,x_2^2 + 25\,x_3^4 + 25\,x_2^4 + 2500 + 51\,x_2^2\,x_3^2 - 500\,x_3^2)\,T^8 + \\ &\quad 40000(-1560\,x_2^2\,x_3^2 + 81\,x_3^4\,x_2^2 + 81\,x_2^4\,x_3^2 + 50\,x_2^6 + 50\,x_2^3 - 24975 \\ &\quad + 7500\,x_3^2 - 750\,x_2^4 - 750\,x_3^4 + 25\,x_3^6 + 7500\,x_2^2)T^6 + 480000 \\ &\quad x_2^2\,x_3^2\,(-500\,x_2^2 + 25\,x_3^4 + 25\,x_2^4 + 2500 + 51\,x_2^2\,x_3^2 - 500\,x_3^2)\,T^4 \\ &\quad + 48000000\,x_2^4\,x_3^4\,(x_3^2 - 10 + x_2^2)\,T^2 + 64000000\,x_2^6\,x_3^6. \end{aligned}$$

It is trivial to find a right polynomial for this inequality because the right hand side contains no radicals. We simply take $T-32$. Computing the resultant of the left and right polynomials with respect to T, we have the critical polynomial,

$$\begin{aligned}&15625\,x_2^6\,x_3^6+12000000\,x_2^6\,x_3^4+12000000\,x_2^4\,x_3^6\\&\quad+3072000000\,x_2^6\,x_3^2+6146880000\,x_2^4\,x_3^4+3072000000\,x_2^2\,x_3^6\\&\quad+524288000000\,x_2^6+787906560000\,x_2^4\,x_3^2+787906560000\,x_2^2\,x_3^4\\&\quad+262144000000\,x_3^6+188743680000\,x_2^4+377664307200\,x_2^2\,x_3^2\\&\quad+188743680000\,x_3^4+524288000000\,x_2^3+45298483200\,x_2^2\\&\quad+45298483200\,x_3^2+265767878656\end{aligned}$$

A proposition which our algorithm is applicable to should take the following form:

$$\Phi_1\wedge\Phi_2\wedge\cdots\wedge\Phi_s\Rightarrow\Phi_0,$$

where $\Phi_0,\ \Phi_1,\ \cdots,\ \Phi_s$ are algebraic inequalities in $x,y,z,\cdots$ etc., the hypothesis $\Phi_1\wedge\Phi_2\wedge\cdots\wedge\Phi_s$ defines either an open set[2] or an open set with whole/partial boundary.

We take the following procedures when the conclusion Φ_0 is of type $\le$. (As for Φ_0 of type $<$, what we need do in additional is to verify if the equation $l_0(x,y,\cdots)-r_0(x,y,\cdots)=0$ has no real solutions under the hypothesis, where $l_0(x,y,\cdots)$ and $r_0(x,y,\cdots)$ denote the left and right hand sides of Φ_0, respectively.)

1. Find the critical surfaces of the inequalities $\Phi_0,\ \Phi_1,\ \cdots,\ \Phi_s$.

2. These critical surfaces decompose the parametric space into a finite number of cells. Among them we just take all the connected open sets, $D_1,D_2,\cdots,D_k$, and discard the lower dimensional cells. Choose at least one test point in every connected open set, say, $(x_\nu,y_\nu,\cdots)\in D_\nu,\ \nu=0,1,\ldots,k$. This step can be done by an incomplete cylindrical algebraic decomposition which is much easier than the complete one since all the lower dimensional cells were discarded. Furthermore, we can make every test point a rational point because it is chosen in an open set.

3. We need only check the proposition for such a finite number of test points, $(x_1,y_1,\cdots)$, $\cdots$, $(x_k,y_k,\cdots)$. The statement is true if and only if it holds over these test values.

The proof of the correctness of the method is sketched as follows.

By $l_\mu(x,y,\cdots)$, $r_\mu(x,y,\cdots)$ and $P_\mu(x,y,\cdots)=0$ denote the left, right hand sides and critical surface of Φ_μ, respectively, and

$$\delta_\mu(x,y,\cdots)\stackrel{\text{def}}{=}l_\mu(x,y,\cdots)-r_\mu(x,y,\cdots),$$

for $\mu=0,\ldots,s$.

The set of real zeros of all the $\delta_\mu(x,y,\cdots)$ is a closed set, so its complementary set, say Δ, is an open set. On other hand, the set

$$D\stackrel{\text{def}}{=}D_1\cup\cdots\cup D_k$$

[2]may be disconnected

is exactly the complementary set of real zeros of all the $P_\mu(x, y, \cdots)$.

We have $D \subset \Delta$ since any zero of $\delta_\mu(x, y, \cdots)$ must be a zero of $P_\mu(x, y, \cdots)$. By $\Delta_1, \cdots, \Delta_t$ denote all the connected components of Δ, so each one is a connected open set. Every Δ_λ must contain a point of D for an open set cannot be filled with the real zeros of all the $P_\mu(x, y, \cdots)$. Assume Δ_λ contains a point of D_i, some connected component of D. Then, $D_i \subset \Delta_\lambda$ because it is impossible that two different components of Δ both intersect D_i. By step 2, D_i contains a test point $(x_i, y_i, \cdots)$. So, every Δ_λ contains at least one test point obtained from step 2.

Thus, $\delta_\mu(x, y, \cdots)$ keeps the same sign over Δ_λ as that of $\delta_\mu(x_{i_\lambda}, y_{i_\lambda}, \cdots)$ where $(x_{i_\lambda}, y_{i_\lambda} \cdots)$ is a test point in Δ_λ, for $\lambda = 1, \ldots, t$; $\mu = 0, \ldots, s$. Otherwise, if there is some point $(x', y', \cdots) \in \Delta_\lambda$ that $\delta_\mu(x', y', \cdots)$ has the opposite sign to $\delta_\mu(x_{i_\lambda}, y_{i_\lambda}, \cdots)$, connecting two points $(x', y', \cdots)$ and $(x_{i_\lambda}, y_{i_\lambda}, \cdots)$ with a path Γ such that $\Gamma \subset \Delta_\lambda$, then there is a point $(\bar{x}, \bar{y}, \cdots) \in \Gamma$ such that $\delta_\mu(\bar{x}, \bar{y}, \cdots) = 0$, a contradiction!

By $A \cup B$ denote the set defined by the hypothesis, where A is an open set defined by

$$(\delta_1(x, y, \cdots) < 0) \wedge \cdots \wedge (\delta_s(x, y, \cdots) < 0),$$

that cosists of a number of connected components of Δ and some real zeros of $\delta_0(x, y, \cdots)$, namely $A = Q \cup S$ where $Q = \Delta_1 \cup \cdots \cup \Delta_j$ and S is a set of some real zeros of $\delta_0(x, y, \cdots)$. And B is the whole or partial boundary of A, that consists of some real zeros of $\delta_\mu(x, y, \cdots)$ for $\mu = 1, \ldots, s$.

Now, let us verify whether $\delta_0 < 0$ holds for all the test points in A, one by one. If there is a test point whereat $\delta_0 > 0$, then the proposition is false. Otherwise, $\delta_0 < 0$ holds over Q because every connected component of Q contains a test point and δ_0 keeps the same sign over each component Δ_λ, hence $\delta_0 \le 0$ holds over A by continuity, so it also holds over $A \cup B$, i.e., the proposition is true.

3. Finding Optimal Value

Now let us review Section 1, the Proposition 1 concerns one of the benchmarks from [8], see also [9], h81.mth. We replace the floats with rationals and formulate it as follows:

Problem 1. Find the greatest value of $x_1x_2x_3x_4x_5$, subject to

$$\begin{aligned}
&x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 10,\\
&x_2x_3 = 5\,x_4x_5,\\
&x_1^3 = x_2^3 + 1,\\
&x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ x_4 > 0,\ x_5 > 0,\\
&x_1 \le \frac{23}{10},\ x_2 \le \frac{23}{10},\ x_3 \le \frac{32}{10},\ x_4 \le \frac{32}{10},\ x_5 \le \frac{32}{10}.
\end{aligned}$$

Eliminate $(x_1,\ x_4,\ x_5)$ by solving the 3 equations and convert Problem 1 to the form encoded in $x_2,\ x_3$ as follows.

Problem 1*a*. Find the greatest value of the function

$$f(x_2, x_3) = \frac{1}{5}(x_2^3 + 1)^{\frac{1}{3}} x_2^2\, x_3^2,$$

provided

$$\begin{aligned}
&x_2 > 0,\ x_3 > 0,\\
&x_2^3 + 1 \le \frac{12167}{1000},\\
&x_3 \le \frac{32}{10},\\
&10 - (x_2^3+1)^{\frac{2}{3}} - x_2^2 - x_3^2 - \frac{2}{5}x_2x_3 \ge 0,\\
&\sqrt{250 - 25\,(x_2^3+1)^{\frac{2}{3}} - 25\,x_2^2 - 25\,x_3^2 + 10\,x_2x_3}\\
&\quad + \sqrt{250 - 25\,(x_2^3+1)^{\frac{2}{3}} - 25\,x_2^2 - 25\,x_3^2 - 10\,x_2x_3} \le 32.
\end{aligned}$$

We may apply a naive algorithm to the problem by using BOTTEMA repeatedly to decide a series of inequalities to be true or false. At first it is easy to see that $0 < f_{\max} < 32$. We then use a dichotomous search to find $f_{\max}$ approximately.

Check the inequality	$f(x_2, x_3) \le 16$,	true,
so then check	$f(x_2, x_3) \le 8$,	true,
so then check	$f(x_2, x_3) \le 4$,	true,
so then check	$f(x_2, x_3) \le 2$,	false,
so then check	$f(x_2, x_3) \le 3$,	true,
so then check	$f(x_2, x_3) \le \frac{5}{2}$,	false,
so then check	$f(x_2, x_3) \le \frac{11}{4}$,	false,
$\cdots$	$\cdots$	$\cdots$
so then check	$f(x_2, x_3) \le \frac{95673}{32768}$,	true,
so then check	$f(x_2, x_3) \le \frac{191345}{65536}$,	false,
so then check	$f(x_2, x_3) \le \frac{382691}{131072}$,	true,

so, after proving/disproving 22 inequalities of the same type, we have

$$\frac{191345}{65536} < f_{\max} < \frac{382691}{131072},$$

that is,

$$f_{\min} = 2.919700\cdots$$

with error less than 10^{-5}. This is accurate enough for general purpose. We finished the job on a Pentium 3/450 using time about 200 seconds. It would be concluded from above example that the speed in automated theorem proving is also of importance. On other hand, a parallel process is definitely applicable to such a procedure.

The Proposition 3 concerns another benchmark from [8], see also [9], h73.mth. We replace the floats with rationals and formulate it as follows:

Problem 2. Find the least value of

$$\frac{2455}{100}x_1 + \frac{2675}{100}x_2 + 39\,x_3 + \frac{405}{10}x_4,$$

subject to

$$\begin{aligned}
&x_1 + x_2 + x_3 + x_4 = 1,\\
&0 < x_1 \le 1,\ 0 < x_2 \le 1,\ 0 < x_3 \le 1,\ 0 < x_4 \le 1,\\
&\frac{23}{10}x_1 + \frac{56}{10}x_2 + \frac{111}{10}x_3 + \frac{13}{10}x_4 \ge 5,\\
&12\,x_1 + \frac{119}{10}x_2 + \frac{418}{10}x_3 + \frac{521}{10}x_4 - 21\\
&\quad - \frac{1645}{1000}\sqrt{\frac{28}{100}x_1^2 + \frac{19}{100}x_2^2 + \frac{205}{10}x_3^2 + \frac{62}{100}x_4^2} \ge 0.
\end{aligned}$$

Eliminate x_4 and convert Problem 2 to the form encoded in x_2, x_3 as follows.

Problem 2a. Find the least value of the function

$$g(x_1, x_2, x_3) = -\frac{319}{20}x_1 - \frac{55}{4}x_2 - \frac{3}{2}x_3 + \frac{81}{2},$$

subject to

$$\begin{aligned}
&x_1 > 0,\ x_2 > 0,\ x_3 > 0,\\
&x_1 + x_2 + x_3 \le 1,\\
&10\,x_1 + 43\,x_2 + 98\,x_3 \ge 37,\\
&401\,x_1 + 402\,x_2 + 103\,x_3 + \frac{329}{200}(90\,x_1^2 + 81\,x_2^2 + 2112\,x_3^2\\
&\quad +62 - 124\,(x_1 + x_2 + x_3 - x_1x_2 - x_2x_3 - x_1x_3))^{\frac{1}{2}} \le 311.
\end{aligned}$$

It is easy to see $9 < g_{\min} < 41$. By the same procedure as that for Problem 1a, after proving/disproving 22 inequalities of the same type, we have

$$\frac{3918315}{131072} < g_{\min} < \frac{979579}{32768},$$

that is,

$$g_{\min} = 29.89437\cdots,$$

with error less than 10^{-5}.

4. A Complete Algorithm for Global Optimization

The method described in last section is sufficient for finding the optimal value of a polynomial function under polynomial constraints, but it does not completely solve the problem of global optimization because that procedure cannot give the critical point(s) whereat the optimal value is achieved. In this section we introduce a symbolic-numerical algorithm which completely solves the problems of this kind.

For convenience, any finite set of real numbers which contains all local minima and maxima is called a *finite kernel* which contains more points than that in original sense defined by Wu Wen-tsün [10, 11].

Let us illustrate the algorithm with the following example which is taken from a well-known book [7] on mathematical economics.

Problem 3. Find the least value of the function

$$f(x,y) = \sqrt{x^2+y^4} + 1/x + xy + 1/y^3$$

subject to $x > 0$, $y > 0$.

Step 1. Let T stand for the objective function $f(x,y)$, i.e.

$$\sqrt{x^2+y^4} + 1/x + xy + 1/y^3 - T = 0.$$

Rationalize it, i.e. replace the left hand side with the left polynomial,

$$T^2x^2y^6 - 2\,xy^3(y^3+x^2y^4+x)\,T - x^4y^6 - x^2y^{10} + y^6 + 2\,y^7x^2 + 2\,xy^3 + x^2 + x^4y^8 + 2\,x^3y^4 = 0. \quad (23)$$

Compute the derivative of (23) with respect to x,

$$T^2xy^6 - y^3(3\,x^2y^4+2\,x+y^3)\,T - xy^{10} + 2\,xy^7 + y^3 - 2\,x^3y^6 + 2\,x^3\,y^8 + 3\,x^2y^4 + x = 0. \quad (24)$$

Critical values of T satisfy both (23) and (24), wherefrom we eliminate x,

$$\begin{aligned} & y^{18}\,T^6 - 6\,y^{15}\,T^5 + y^{12}\,(y^{12} - 3\,y^{10} - 10\,y^7 + y^2 + 15)\,T^4 \\ & \quad - 4\,y^9\,(y^{12} - 3\,y^{10} - 10\,y^7 + y^2 + 5)\,T^3 - y^6(-15 + 18\,y^{10} + 14\,y^{12} + 8\,y^9 \\ & \quad + 2\,y^{22} - 3\,y^{20} - 32\,y^{14} + 8\,y^{19} + 60\,y^7 - 6\,y^2 - 2\,y^{17})T^2 + 2y^3(-3 + 20\,y^7 \\ & \quad + 6\,y^{10} + 18\,y^{12} - 32\,y^{14} + 8\,y^{19} - 3\,y^{20} + 2\,y^{22} - 2\,y^{17} + 8\,y^9 - 2\,y^2)T + 1 \\ & \quad - 8\,y^{24} - 10\,y^{22} + 24\,y^{19} + 16\,y^{14} - 19\,y^{12} + 3\,y^{20} - 3\,y^{10} + 2\,y^{17} + y^2 \\ & \quad + 8\,y^{27} - 10\,y^7 - y^{30} + y^{32} - 8\,y^{29} - 32\,y^{21} + 16\,y^{16} - 8\,y^9 + 16\,y^{26} = 0. \end{aligned} \quad (25)$$

Also compute the derivative of (25) with respect to y,

$$\begin{aligned} & 9\,y^{16}\,T^6 - 45\,y^{13}\,T^5 + y^{10}\,(-33\,y^{10} + 90 + 7\,y^2 + 12\,y^{12} - 95\,y^7)\,T^4 \\ & \quad - 2\,y^7\,(21\,y^{12} + 11\,y^2 - 57\,y^{10} + 45 - 160\,y^7)\,T^3 - y^4(126\,y^{12} + 60\,y^9 \\ & \quad - 24\,y^2 + 144\,y^{10} + 390\,y^7 + 28\,y^{22} - 320\,y^{14} - 39\,y^{20} - 45 - 23\,y^{17} \\ & \quad + 100\,y^{19})T^2 + y(200\,y^7 - 9 + 96\,y^9 - 69\,y^{20} + 50\,y^{22} + 270\,y^{12} - 10\,y^2 \\ & \quad + 78\,y^{10} + 176\,y^{19} - 40\,y^{17} - 544\,y^{14})T + 1 + 128\,y^{14} - 35\,y^5 - 116\,y^{27} \\ & \quad + 112\,y^{12} + 30\,y^{18} - 15\,y^8 - 114\,y^{10} - 36\,y^7 + 17\,y^{15} + 16\,y^{30} + 108\,y^{25} \\ & \quad + 208\,y^{24} - 336\,y^{19} - 96\,y^{22} - 15\,y^{28} + 228\,y^{17} - 110\,y^{20} = 0. \end{aligned} \quad (26)$$

Analogously, eliminate y from (25) and (26), we have

$$\begin{aligned}&(1048576\,T^{30} + 515899392\,T^{26} - 1409286144\,T^{25} + 2540961792\,T^{24} - \cdots \\ &\quad + 4964564163953479)\,(72301961339136\,T^{34} - 92137890375936\,T^{32} \\ &\quad - 58912709239296\,T^{31} - 4671310780013760\,T^{30} + 195813679474944\,T^{29} \\ &\quad + \cdots + 540322384839454862827764\,T^2 + 587981569905334239791515 2\,T \\ &\quad - 785335601673232833886399)\,(64\,T^8 - 64\,T^3 + 48\,T^2 - 12\,T + 1) = 0. \qquad (27)\end{aligned}$$

The least value we want must be a positive root of (27), i.e., belongs to the finite kernel[3]

$$K = \{\,0.13455\cdots,\, 0.28526\cdots,\, 0.79816\cdots,\, 1.15159\cdots,\, 4.31535\cdots\}. \qquad (28)$$

The time spent for this step on a Pentium/350 is about 10 seconds.

What we want may not be $0.13455\cdots$, the least member of K, because the equation $f(x,y) = 0.13455\cdots$ may not have a positive solution. So, we have to go on.

Step 2. In general, sort the members of K, the finite kernel received from Step 1,

$$T_1 < T_2 < \cdots < T_s \qquad (s = 5 \ \text{ in this example})$$

and separate them with rational numbers $R_1, \cdots, R_s$ as follows,

$$R_1 < T_1 < R_2 < T_2 < \cdots < R_s < T_s.$$

In this example, we may let

$$R_1 = \frac{1}{8}, \quad R_2 = \frac{1}{4}, \quad R_3 = \frac{1}{2}, \quad R_4 = 1, \quad R_5 = 2.$$

The following fact is then obvious:

Let k be the greatest natural number such that $f > R_k$, then, $f_{\min} = T_k$. If $f > R_1$ does not hold, then the least value of f does not exist.

One thus can convert the least-value-finding problem to verification of finitely many inequalities, $f > R_i$. By means of a dichotomous search, the number of inequalities need to be verified is not greater than $\log_2 s + 1$ which is much less than that in the naive method shown in last section. To this example we need check 3 inequalities only. By applying BOTTEMA, the total time spent on a Pentium/350 is about 4 seconds. Because $f > 2$, we have

$$f_{\min} = 4.315351625\cdots$$

which is the greatest real root of a polynomial equation of degree 34:

$$\begin{aligned}&72301961339136\,T^{34} - 92137890375936\,T^{32} - 58912709239296\,T^{31} \\ &\quad - 4671310780013760\,T^{30} + 195813679474944\,T^{29} + 5685782870701575\,T^{28} \\ &\quad - \cdots \\ &\quad - 770098514370843110413173 0\,T^3 + 540322384839454862827764\,T^2 \\ &\quad + 5879815699053342397915152\,T - 785335601673232833886399 = 0. \qquad (29)\end{aligned}$$

[3]Here the constraints define an open set, otherwise, we should consider critical values on the boundary.

Setp 3. If we want also to find the critical point(s) corresponding to $f_{\min}$, i.e. the positive solution(s) of

$$\sqrt{x^2+y^4}+1/x+xy+1/y^3 = 4.315351625\cdots,$$

the equations (29), (25), (23) form a *triangular system* which is quite easy to be solved.

5. Conclusion

Based on an inequality-proving program, BOTTEMA, We presented here two symbolic-numerical algorithms about global optimization. The former is only for finding the optimal-value, the latter can also give the critical point(s) whereat the optimal value is achieved. The efficiency of both algorithms essentially depend on that of the inequality-proving program.

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