# Evaluating $\zeta(2n)$ using *Mathematica*

Shigeki Matsumoto*

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**Abstract**

In the present paper we give one of the simplest and the most elementary methods of evaluating the values $\zeta(s)$ of Riemann zeta function for all the positive even integers $s$ by using a uniform estimation of a trigonometirc polynomial $\sum_{k=1}^{n}(-1)^{k-1}\frac{\sin kx}{k}$ on a closed interval $[0, \frac{\pi}{2}]$. All materials in our argument belong to the basics of Calculus and our presentation is self-contained. We prove a linear recurrence equation for $\zeta(s)$ for even integers $s > 1$ and evaluate all of them with *Mathematica.*

## 1 Uniform estimation of a trigonometric polynomial $\sum_{k=1}^{n}(-1)^{k-1}\frac{\sin kx}{k}$

The following is a well known equality in the theory of Fourier analysis.

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin kx}{k} = \frac{x}{2} \qquad (-\pi < x < \pi). \tag{1}$$

It is also known that the convergence of the left hand of the above (1) is not uniform on an open interval $-\pi < x < \pi$ but uniform on any compact subset $K$ of the open interval.

**Theorem 1.1** *For an arbitrary compact subset $K$ of the open interval $(-\pi, \pi)$ there exists a positive constant $C(K)$ such that the following inequalities are valid for $\forall n > 0$ and $\forall x \in K$*

$$-\frac{C(K)}{n} < \sum_{k=1}^{n}(-1)^{k-1}\frac{\sin kx}{k} - \frac{x}{2} < \frac{C(K)}{n}. \tag{2}$$

*Faculty of Science, Konan University

Our proof of the above theorem is quite elementary and straightforward. In fact, integrating the both sides of the following easily verified equation

$$\sum_{k=1}^{n}(-1)^{k-1}\cos kx = \frac{1}{2} + (-1)^{n-1}\frac{\cos\left(n+\frac{1}{2}\right)x}{2\cos\frac{x}{2}}, \tag{3}$$

we have

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{\sin kx}{k} = \frac{x}{2} + (-1)^{n-1}\int_0^x \frac{\cos\left(n+\frac{1}{2}\right)t}{2\cos\frac{t}{2}}dt. \tag{4}$$

By virtue of the formula of integration by parts, the right hand side of (4) is equal to

$$\frac{x}{2} + \frac{(-1)^{n-1}}{2n+1}\left(\frac{\sin\left(n+\frac{1}{2}\right)x}{\cos\frac{x}{2}} - \int_0^x \frac{\sin\frac{t}{2}\sin\left(n+\frac{1}{2}\right)t}{2\cos^2\frac{t}{2}}dt\right). \tag{5}$$

The conclusion of the theorem easily comes out from the above (5).

## 2 Another proof of $\zeta(2) = \frac{\pi^2}{6}$ L.Euler missed

Applying Theorem 1.1 to the case $K = [0, \frac{\pi}{2}]$, we have the following estimate

$$-\frac{C}{n} < \sum_{k=1}^{n}(-1)^{k-1}\frac{\sin kx}{k} - \frac{x}{2} < \frac{C}{n}, \tag{6}$$

where $C$ is a positive constant which does not depend neither $n$ nor $x \in [0, \frac{\pi}{2}]$.

Integrating each side of the above estimate (6) with respect to $x$ from 0 to $\frac{\pi}{2}$ and taking the limit as $n$ goes to the infinifty, we have

$$\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^{\frac{\pi}{2}}\frac{\sin kx}{k}dx = \int_0^{\frac{\pi}{2}}\frac{x}{2}dx = \frac{\pi^2}{16}. \tag{7}$$

We can easily verify the left hand side of (7) is a rational multiple of $\zeta(2)$.

That is, we arrive at the goal $\zeta(2) = \frac{\pi^2}{6}$ walking with short steps starting from (7).

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\left(1-\cos\frac{k\pi}{2}\right)}{k^2} = \frac{\pi^2}{16}. \tag{8}$$

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} + \sum_{k=1}^{\infty}\frac{-2}{(4k-2)^2} = \frac{\pi^2}{16}. \tag{9}$$

$$\left(1-\frac{2}{2^2}\right)\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{16}. \tag{10}$$

$$\left(1-\frac{2}{2^2}\right)\left(\sum_{k=1}^{\infty}\frac{1}{k^2}-\sum_{k=1}^{\infty}\frac{1}{(2k)^2}\right)=\frac{\pi^2}{16}. \tag{11}$$

$$\left(1-\frac{2}{2^2}\right)\left(1-\frac{1}{2^2}\right)\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{16}. \tag{12}$$

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\frac{\pi^2}{16}}{\left(1-\frac{2}{2^2}\right)\left(1-\frac{1}{2^2}\right)}=\frac{\pi^2}{6}. \tag{13}$$

## 3 The higher order primitive function and the remainder of Taylor's theorem

Let $M$ be a positive integer. Multiplying each side of the estimate (6) in the previous section by $\left(\frac{\pi}{2}-x\right)^{2M}$ and integrating it with respect to $x$ from 0 to $\frac{\pi}{2}$ respectively, we have the following

$$\sum_{k=1}^{\infty}\int_0^{\frac{\pi}{2}}(-1)^{k-1}\frac{\sin kx}{k}\left(\frac{\pi}{2}-x\right)^{2M}dx=\int_0^{\frac{\pi}{2}}\frac{x}{2}\left(\frac{\pi}{2}-x\right)^{2M}dx. \tag{14}$$

The value of the right hand side of the above is calculated by hand or by *Mathematica* as follows;

$$Integrate\left[\frac{x}{2}\left(\frac{\pi}{2}-x\right)^{2M},\left\{x,0,\frac{\pi}{2}\right\},Assumptions\to\{M>0\}\right] \tag{15}$$

$$\frac{2^{-2M-4}\pi^{2M+2}}{(M+1)(2M+1)}$$

That is, we have

$$\int_0^{\frac{\pi}{2}}\frac{x}{2}\left(\frac{\pi}{2}-x\right)^{2M}dx=\frac{2^{-2M-4}\pi^{2M+2}}{(M+1)(2M+1)}. \tag{16}$$

On the other hand the definite integral

$$\int_0^{\frac{\pi}{2}}(-1)^{k-1}\frac{\sin kx}{k}\left(\frac{\pi}{2}-x\right)^{2M}dx, \tag{17}$$

on the left hand side of (14) is equal to the following;

$$(-1)^M(2M)!\left(\frac{(-1)^{k-1}\left(1-\cos\frac{k\pi}{2}\right)}{k^{2M+2}}-\sum_{s=1}^{M}\frac{(-1)^{s-1}}{(2s)!}\left(\frac{\pi}{2}\right)^{2s}\frac{(-1)^{k-1}}{k^{2M-2s+2}}\right). \quad (18)$$

This is a special case of Taylor's theorem.

**Theorem 3.1** *Let $n$ be a positive integer and $f$ be an (n+1)-th continuously differentiable function defined on an open interval $I$. Then we have*

$$f(x)-\sum_{\nu=0}^{n}\frac{f^{(\nu)}(a)}{\nu!}(x-a)^{\nu}=\int_a^x\frac{f^{(n+1)}(t)}{n!}(x-t)^n dt, \quad (19)$$

*for $a\in I, x\in I$.*

That is, we can evaluate the definite integral (17) applying Taylor's theorem to the case, $f(x)=\cos kx, n=2M, a=0, x=\frac{\pi}{2}$.

# 4 A recurrence formula for $\zeta(2m)(m=1,2,3,...)$ and *Mathematica* computation

Summing up the value (18) of the definite integral (17) in the previous section with respect to $k$ from 1 to $\infty$, we have an equality

$$(-1)^M(2M)!\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\left(1-\cos\left(\frac{k\pi}{2}\right)\right)}{k^{2M+2}}-\sum_{s=1}^{M}\frac{(-1)^{s-1}}{(2s)!}\left(\frac{\pi}{2}\right)^{2s}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{2M-2s+2}}\right)=\frac{2^{-2M-4}\pi^{2M+2}}{(M+1)(2M+1)}, \quad (20)$$

so special values of Riemann zeta function $\zeta(2m)(m=1,2,3,...)$ satisfy the following recurrence formula.

$$\zeta(2M+2)=\frac{\sum_{s=1}^{M}\frac{(-1)^{s-1}}{(2s)!}\left(\frac{\pi}{2}\right)^{2s}\left(1-\frac{2}{2^{2M-2s+2}}\right)\zeta(2M-2s+2)+\frac{(-1)^M}{2(2M+2)!}\left(\frac{\pi}{2}\right)^{2M+2}}{\left(1-\frac{1}{2^{2M+1}}\right)\left(1-\frac{1}{2^{2M+2}}\right)}, \quad (21)$$

after reversing the summation direction, which can be rewritten as

$$\zeta(m)=(-1)^{\frac{m}{2}-1}\left(\frac{\pi}{2}\right)^m\frac{\left(\sum_{s=1}^{\frac{m}{2}-1}\frac{(-1)^s}{(m-2s)!}\left(\frac{\pi}{2}\right)^{-2s}\left(1-\frac{1}{2^{2s-1}}\right)\zeta(2s)\right)+\frac{1}{2m!}}{\left(1-\frac{1}{2^{m-1}}\right)\left(1-\frac{1}{2^m}\right)}. \quad (22)$$

This linear recurrence for $\zeta(2n)$ is closely related to that of Bernoulli numbers $B_n$ in the literature of Mathematics.

In *Mathematica* programming, we can define a sequence in a recursive fashion. Therefore, if we define a sequence $\{Z[2m]\}(m = 1, 2, 3, ...)$ by an initial condition and a recurrence equation such as

$$Z[2] = \frac{\pi^2}{6}$$

and

$$Z[m\_] := Z[m] = \frac{\sum_{s=1}^{\frac{m-2}{2}} \frac{(-1)^{s-1}}{(2s)!} \left(\frac{\pi}{2}\right)^{2s} \left(1 - \frac{1}{2^{m-2s-1}}\right) Z[m-2s] + \frac{(-1)^{\frac{m-2}{2}}}{2m!} \left(\frac{\pi}{2}\right)^m}{\left(1 - \frac{1}{2^{m-1}}\right)\left(1 - \frac{1}{2^m}\right)},$$

*Mathematica* generates all the values of $Z[2m] = \zeta(2m)(m = 1, 2, 3, ...)$ in principle.

First few terms are as follows;

$$Table[\{Z[k], Zeta[k]\}, \{k, 2, 30, 2\}]$$

| | |
|---|---|
| $\frac{\pi^2}{6}$ | $\frac{\pi^2}{6}$ |
| $\frac{\pi^4}{90}$ | $\frac{\pi^4}{90}$ |
| $\frac{\pi^6}{945}$ | $\frac{\pi^6}{945}$ |
| $\frac{\pi^8}{9450}$ | $\frac{\pi^8}{9450}$ |
| $\frac{\pi^{10}}{93555}$ | $\frac{\pi^{10}}{93555}$ |
| $\frac{691\pi^{12}}{638512875}$ | $\frac{691\pi^{12}}{638512875}$ |
| $\frac{2\pi^{14}}{18243225}$ | $\frac{2\pi^{14}}{18243225}$ |
| $\frac{3617\pi^{16}}{325641566250}$ | $\frac{3617\pi^{16}}{325641566250}$ |
| $\frac{43867\pi^{18}}{38979295480125}$ | $\frac{43867\pi^{18}}{38979295480125}$ |
| $\frac{174611\pi^{20}}{1531329465290625}$ | $\frac{174611\pi^{20}}{1531329465290625}$ |
| $\frac{155366\pi^{22}}{13447856940643125}$ | $\frac{155366\pi^{22}}{13447856940643125}$ |
| $\frac{236364091\pi^{24}}{201919571963756521875}$ | $\frac{236364091\pi^{24}}{201919571963756521875}$ |
| $\frac{1315862\pi^{26}}{11094481976030578125}$ | $\frac{1315862\pi^{26}}{11094481976030578125}$ |
| $\frac{6785560294\pi^{28}}{564653660170076273671875}$ | $\frac{6785560294\pi^{28}}{564653660170076273671875}$ |
| $\frac{6892673020804\pi^{30}}{5660878804669082674070015625}$ | $\frac{6892673020804\pi^{30}}{5660878804669082674070015625}$ |

# References


[1] I. Yamaguchi, Number Theory, Sangyo Tosyo, 1994.