The Teaching of Conformal mapping

Zhang, Jinhao
Department of Mathematics
Fudan University
Shanghai 200433 China
email: zhangjhk@online.sh.cn

Abstract:

Conformal mapping is a very important concept in complex analysis. It's also a difficult subject for students in their first learning, in particular, when they construct a conformal mapping by element functions step by step. In this paper we show that how to teach "Conformal Mapping" using animation by Matlab.

0. Let $w = f(z)$ be a holomorphic function defined on a domain $G \subset \mathbb{C}$, which means $f(z)$ is a differentiable and satisfies Cauchy-Riemann equation

$$\frac{\partial f(z)}{\partial \bar{z}} = 0$$

everywhere in G. If one wants to emphasize the geometric aspects of f, then f is usually called a holomorphic mapping. A mapping preserving angles of intersection at points of intersection of two curves is called a conformal mapping. It is well know that a holomorphic mapping is conformal if its derivative function is nowhere vanishing. Among conformal mappings, the one-to-one conformal mappings, which usually called by univalent mapping or biholomorphic mapping, are more interesting for us. For example, the fractional linear mapping

$$w = \frac{az+b}{cz+d} \quad (ad - bc \neq 0), \qquad z \in \mathbb{C} \cup \{\infty\},$$

exponential mapping

$$w = e^z, \qquad y_0 < \Im z < y_0 + 2\pi,$$

univalent branch of logarithm mapping

$$w = \log z \qquad \alpha < \arg z < \alpha + 2\pi,$$

and power mapping

$$w = z^p \qquad \alpha < \arg z < \alpha + \frac{2\pi}{p}$$

are all biholomorphic in their defining domain.

There are two questions: the first is to determine the image of a given domain under a biholomorphic mapping; the second is to find a biholomorphic mapping which maps a given domain to unit disk or vice versa.

In the past, we only explained the answers for these two questions with few examples in the classroom. Now we can demonstrate the process with the help of software developed by Matlab.

1. Given a domain G and a biholomorphic mapping f, to determine the image $f(G)$ is the first step of studding conformal mapping. Moreover, to understand the correspondence between the boundaries of G and $f(G)$ is also important. For example:

$$G = \{z : |z| < 1, \Im z > 0\}, \qquad f(z) = \frac{z-1}{z+1}.$$

The domain G is upper half of unit disk. Two curves of upper half of unit circle and diameter intersect at points -1 and 1 with angle of $\pi/2$. Under the holomorphic mapping $f(z)$, -1, 1 correspond to ∞ and 0 respectively, and the two curves correspond to two half line intersecting at 0. Since $f(x) < 0$ for every point x at the diameter, one of the half lines is negative real axis, then another one is upper half image axis by the preservability of angle. Therefore, the image $f(G)$ is the second quadrant. Now it will be shown by following procedure:

```
t=100;
set(gcf,'position',[0 120 800 420])
subplot(1,2,1);
a=plot(0,0,'g.');
line([0,0],[-1.2,1.2])
line([-1.2,1.2],[0,0])
axis square
set(a,'EraseMode','none','Markersize',6)
subplot(1,2,2);
b=plot(0,0,'r.');
line([0,0],[-1,2])
line([-2,1],[0,0])
axis square
set(b,'EraseMode','none','Markersize',6)
text(-1.2,-0.5,'Re w¡0, Im w¿0')

for i=1:t
    x=0:0.002:(i/t)*pi;
    y=max(2*(2*x-pi)/pi,0);
    set(a,'XData',cos(pi+2*x),'YData',max(sin(pi+2*x),0))
    set(b,'XData',min(2*(2*x-pi)/pi,0),'YData',y)
    drawnow;
end
```

From the above animation, we can see clearly not only the image $f(G)$, but also the correspondence between the boundaries. It is also easily to modify the parameters of above procedure if the domain and f are changed.

Riemann mapping theorem tell us that given a simply domain $G \subset \mathbb{C}$, $G \neq \mathbb{C}$ there are biholomorphic mappings which map G to unit disk or upper half of plane. In general, there are no good way to find the biholomorphic mapping explicitly. In the classroom we show students how to construct a biholomorphic mapping only for very special simply domains, which boundary are consisting of segments, rays, lines or arcs. For example,

$$G = \{z : |z| < 1, 0 < \arg z < \pi/4\}.$$

At first we take mapping

$$z_1 = z^4,$$

which transforms G to upper half of unit disk. Then

$$z_2 = \frac{1+z_1}{1-z_1}$$

transforms the upper half of unit disk to the first quadrant. And mapping

$$w = z_2^2$$

transform the first quadrant to upper half of plane. After taking composition of $z_1.z_2, w$, we find a biholomorphic mapping which maps G to upper half of plane:

$$w = \left(\frac{1+z^4}{1-z^4}\right)^2.$$

We have written a procedure by Matlab to show the above process more clearly. One just input domain G and mapping functions. The mapping animation will complete automatically step by step. Although the length of the procedure is too long to include it here, but the principle is very simple, which is just putting three procedures similar to the one in the first paragraph of this section together such that these mappings can be shown on one window step by step.

2. The examples shown in section 1 are very simple. In this section we will discuss the posibility of finding a biholomorphic mapping with aid of computer. We call a simply domain the E-domain, if its boundary consists of no-negative real axis, segments with one end point at origin, and rays which oppositely directed extension pass through the origin. After inspecting these examples used in classroom, we find that either of them are E-domains, or will be E-domains by a fractional linear, power or exponential transforms. For example,

$$G = \{z : |z| < 1\} \setminus [a, 1], \quad -1 < a < 1.$$

The unit disk is a part of the boundary of G. And the fractional linear mapping

$$f(z) = i\frac{1+z}{1-z}$$

transforms G to upper half plane deleting a ray with end point $i\dfrac{a+1}{1-a}$ and perpendicular to real axis. Now the boundary of $f(G)$ consists of a ray and a line. If we look the real axis consists of no-negative real axis $[0, +\infty)$ and ray $(-\infty, 0]$, then $f(G)$ is a E-domain. Here is another example,

$$G = \{z : 0 < \Im z < 2\pi\} \setminus \{z : \Im z = \pi, \Re z \in [0, +\infty)\}.$$

This is a string deleting a ray which pararells to real axis. The exponential mapping

$$f(z) = e^z$$

transforms the domain G to a plane deleting no-negative real axis and a ray $(-\infty, -1]$. It is clear $f(G)$ is also an E-domain.

In the following we are going to give a systematic approach of how to find a biholomorphic mapping which maps upper half plane to a given E-domain. Since every part of boundary for a E-domain is "strait", we can look E-domain as a degenerated polygon. It is well known that every biholomorphic mapping from upper half plane (or disk) to n-polygon can be represented by Schwarz-Christoffel formula (see [3]):

$$f(z) = C_0 \int_{z_0}^{z} \prod_{k=0}^{n} (\zeta - a_k)^{\beta_k} \, \mathrm{d}\zeta + C_1,$$

where $\beta_1, \beta_2, \cdots, \beta_n$ are exterior angles of the polygon, $\beta_1 + \cdots + \beta_n = 2\pi$, and $a_1, a_2, \cdots, a_n$ are points on real axis corresponding to the vertices of polygon, C_0, C_1 are constants. If the polygon is bounded, then the integral can't be integrated by element functions even for triangles or rectangles. But for degenerated polygon the things is much better. So the Schwarz-Christoffel formula is main tool for us to find a biholomorphic mapping.

Let G be an E-domain. Denote by $l_1, \cdots, l_{p-1}$, $l'_1, \cdots, l'_{q-1}$ the segments and rays in the boundary of G respectively. Let $l_j = [0, B_j]$, $j = 1, \cdots, p-1$, and D_k be the initial point of ray l'_k, $k = 1, \cdots, q-1$. Since the no-negative real axis is also a part of boundary, the E-domain G as a degenerated polygon has p inner angles at origin and q inner angles at ∞. Let $\phi_j \pi$ be the angle between l_{j-1}, l_j, $j = 1, \cdots, p$, where $l_0 = l_p$ denote the no-negative real axis. Let $\psi_k \pi$ be the angle between l'_{k-1}, l'_k at ∞ (which is negative of the angle at 0), $k = 1, \cdots, q$, where $l'_0 = l'_q = l_0$. From Schwarz-Christoffel formula we obtain that the biholomorphic mapping from upper half plane to G can be represented as (see [1])

$$f(z) = C \frac{\prod_{j=1}^{p} (z - a_j)^{\phi_j}}{\prod_{k=1}^{q} (z - c_k)^{\psi_k}}, \tag{2.1}$$

where $a_1, \cdots, a_p$ are points on real axis corresponding to p vertices of G at 0, and $c_1, \cdots, c_q$ are points on real axis corresponding to q vertices of G at ∞. Let $b_1, \cdots, b_{p-1}$, $d_1, \cdots, d_{q-1}$ be the points on real axis corresponding to $B_1, \cdots, B_{p-1}$, $D_1, \cdots, D_{q-1}$ respectively. Then one can show that they satisfy following equation (see [1]):

$$\sum_{j=1}^{p} \frac{\phi_j}{z - a_j} - \sum_{k=1}^{q} \frac{\psi_k}{z - c_k} = 0. \tag{2.2}$$

In general, to determine the $p + q + 1$ parameters a_j, c_k and C in the representation of f is very hard. It will be reduced to solve a system of algebraic equations if ϕ_j, ψ_k are different with integral multiple, which is most of the cases occuring in teaching.

Denote by

$$\theta = \min(\phi_1, \cdots, \phi_p, \psi_1, \cdots, \psi_q).$$

Set

$$\phi_j = n_j\theta,\ \psi_k = m_k\theta, \quad j = 1, \cdots, p,\ k = 1, \cdots, q.$$

Suppose that ϕ_j, ψ_k are different with integral multiple. Then n_j, m_k are integers, and (2.1) can be written as

$$f(z) = \left[C^{\frac{1}{\theta}} \frac{\prod_{j=1}^{p}(z - a_j)^{n_j}}{\prod_{k=1}^{q}(z - c_k)^{m_k}} \right]^{\theta}. \tag{2.3}$$

From (2.3) and

$$f(b_\mu) = B_\mu,\ f(d_\nu) = D_\nu, \quad \mu = 1, \cdots, p-1,\ \nu = 1, \cdots, q-1,$$

we have

$$B_\mu^{\frac{1}{\theta}} \prod_{k=1}^{q}(b_\mu - c_k)^{m_k} = C^{\frac{1}{\theta}} \prod_{j=1}^{p}(b_\mu - a_j)^{n_j}, \quad \mu = 1, \cdots, p-1,$$
$$D_\nu^{\frac{1}{\theta}} \prod_{k=1}^{q}(d_\nu - c_k)^{m_k} = C^{\frac{1}{\theta}} \prod_{j=1}^{p}(d_\nu - a_j)^{n_j}, \quad \nu = 1, \cdots, q-1.$$

In addition, from (2.2) we have

$$\sum_{j=1}^{p} \frac{n_j}{b_\mu - a_j} - \sum_{k=1}^{q} \frac{m_k}{b_\mu - c_k} = 0, \quad \mu = 1, \cdots, p-1,$$
$$\sum_{j=1}^{p} \frac{n_j}{d_\nu - a_j} - \sum_{k=1}^{q} \frac{m_k}{d_\nu - c_k} = 0, \quad \nu = 1, \cdots, q-1.$$

Now we can use Wu's theory to determine $2p+2q-1$ parameters C, a_j, b_μ, c_k, d_ν from above $2p+2q-4$ algebraic equations (see [2]). Then the biholomorphic mapping (2.1) follows.

So the first step of determing a biholomorphic mapping from upper half plane to a given domain is to reduce it to be a E-domain.

Reference

[1] Li, Zhaohui: *A discussion on representation of Schwarz-Christoffel formula by composition of element mappings*. Thesis, 2000.

[2] Shi, He: *Introduction to Mechanical Mathematics* (in Chinese). Hunan Education Press, 1998.

[3] Zhang, Jinhao & Qiu, Weiyuan: *Function theory of complex variable* (in Chinese). Higher Education Press, 2000.