HP48G and Calculus Teaching

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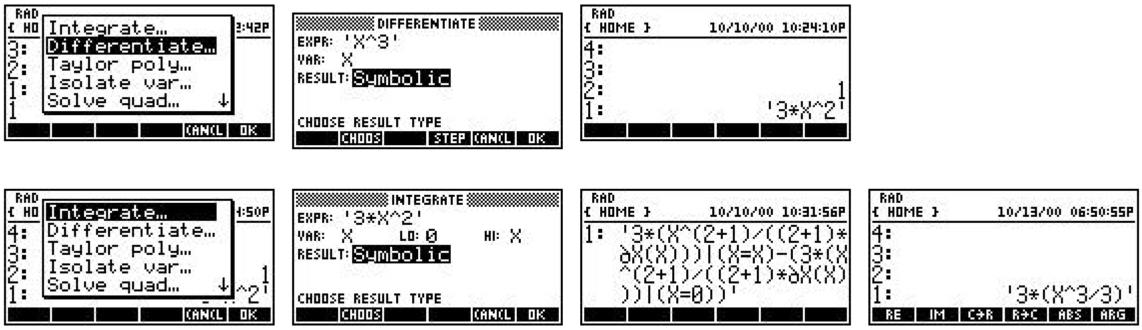
Abstract
Our topic is the use of the calculator in assisting calculus students to understand the Fundamental Theorem of Calculus, which states that if F is the indefinite integral of f then f is the derivative of F. Although this theorem is important, many students exhibit little understanding of it. We present here some activities with calculator (Specially, the HP48G) that seems to help students grasp this theorem. Our activities take advantage of the modern calculator's capacity to quickly draw the graph of a given function. For a basic elementary function, f, the calculator can find the derivative f' and draw the graph, and can find $\int_a^x f(t)dt$ and draw the graph.

Introduction
In order to understand calculus, we have to understand the Fundamental Theorem of Calculus that is an essential part of calculus. By Fundamental Theorem of Calculus we mean here the statement that f(x) is the derivative of $\int_a^x f(t)dt$ (there are various equivalent formulations). This theorem, which relates the concepts of derivative and definite integral, is one of the most important propositions encountered in any calculus course. Yet few students seem able to state the theorem precisely or give an intuitive explanation of it. Perhaps part of the trouble is that in most courses students promptly become pre-occupied with practicing the various techniques for producing anti-derivatives of specific functions and do not have much time to think about the meaning and implications of the Fundamental Theorem. We want students to see that calculus is not divided into two distinct, separate compartments, but that actually the Fundamental Theorem bridges the differential and the integral calculus.

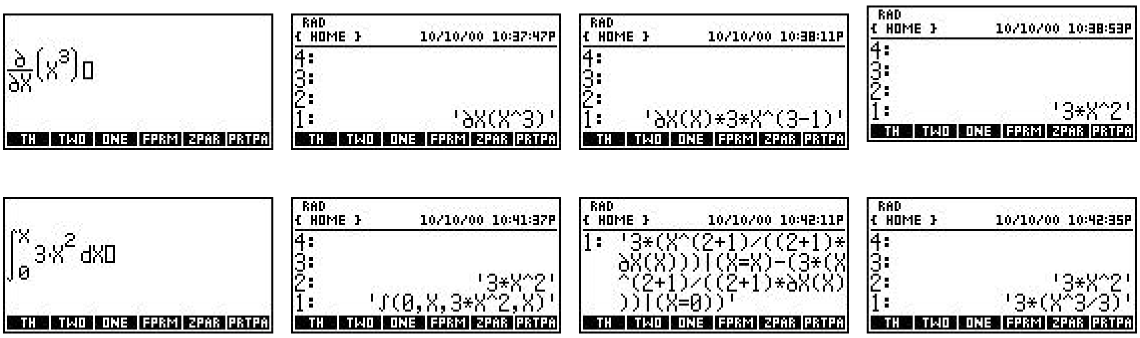
We have tried some calculator-aided activities to illustrate and reinforce the connection embedded in the Fundamental Theorem. Our hope is that by using a calculator's built-in ability for graphing derivatives and indefinite integrals of a given function students will gain an understanding of how the Fundamental Theorem works. We describe these activities and discuss what we have learned from them.

The Teaching and Learning Tool
The calculator we used as the teaching and learning tool in our class is HP48G. One can instruct this calculator to get the derivative of a given function via the instruction $Y_2=\frac{d}{dx}(Y_1)$.
It will also get a specific anti-derivative function, say the anti-derivative vanishing at the origin, via the instruction $Y_2=\int_0^x Y_1 dx$. These capabilities would be built into any of the newer calculators. Most of our activities are based on the use of these two calculator operations.

The HP48G Calculator has several ways to produce the required results. In our class we introduce two ways. One is through the instructions of Symbolic Function Keys, such as in the following sequence of pictures.



The other is using the Equation Writer Function Key. The Equation Writer produces the step by step procedures just as we would write on paper.



The Inverse Relationship—Power Functions

In Activity 1 (see Table 1 below) students do the differentiation and indefinite integration operations on power functions. After doing some experimentation and comparison of using Symbolic Function Keys and Equation Writer Function Key, they should be able to draw the conclusion that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ and $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + c\right) = x^n$ are inverse operations of each other. From such a table as Table 1 they should infer the inverse relationship in the following diagram:

f	Indefinite Integral		Differentiation	f

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

f	Differentiation		Indefinite Integral	f

$$\int \left[\frac{d}{dx} f(x) \right] dx = f(x) + c$$

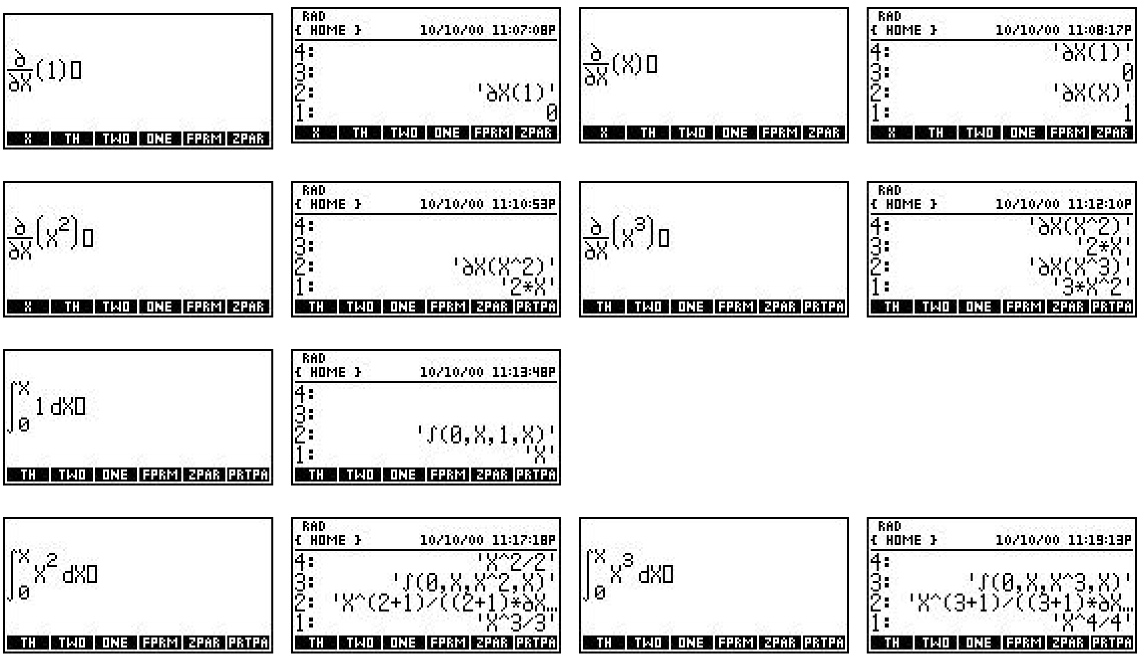
Of course, the second diagram is not always correct except for simple power functions. We return to this point in a later calculator activity.

Activity 1: Is integral the inverse operation of derivative for power functions?

Table1: The power functions

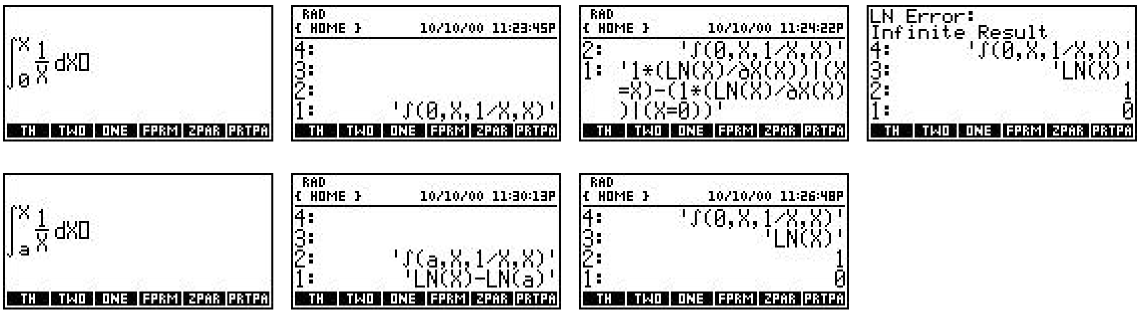
Function	Derivative		Undo	Integral
$y = c$	$\frac{dy}{dx}$ =y`=0			
$y = x + c$	$\frac{dy}{dx}$ =y`=1		$\int 1dx = x + c$	$\int dx = x + c_1$
$y = x^2 + c$	$\frac{dy}{dx}$ =y`=2x		$\int 2xdx = x^2 + c$	$\int xdx = \frac{x^2}{2} + c_1$
$y = x^3 + c$	$\frac{dy}{dx}$ =y`=3x^2		$\int 3x^2dxy = x^3 + c$	$\int x^2dx = \frac{x^3}{3} + c_1$
$y = x^4 + c$	$\frac{dy}{dx}$ =y`=4x^3		$\int 4x^3dxy = x^4 + c$	$\int x^3dx = \frac{x^4}{4} + c_1$
…			…	
$y = x^n + c$	$\frac{dy}{dx}$ =y`= nx^{n-1}		$\int nx^{n-1}dx = x^n + c$	$\int x^ndx = \frac{x^{n+1}}{n+1} + c_1$

We provide students the above table with certain selected functions and leave the rest of the table blank. We let them copy down all the results the HP48G can produce. After the students fill out the blanks by copying the results from the calculator, we hope they can see the pattern and come up with the proper conclusion for the nth power function.



The Constraint

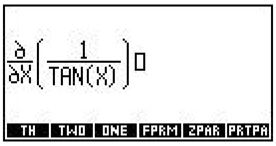
In Activity 2, we extend the considerations of Activity 1 to the other elementary functions. By applying our calculator operations to one column and comparing results, students can generate strong calculator evidence for the validity of Table 2, which illustrates the Fundamental Theorem for the elementary functions. In some cases, the calculator will exhibit an error message. For instance, in the case of $\int \frac{1}{x}dx$, the calculator will exhibit an error message, such as "LN Error" and "Indefinite Result" if we try to start the indefinite integration at 0, as shown below. However, if we start the indefinite integration at other numbers like 1 or 2, or even a generic number a, the calculator gives us the correct answer $\ln(x) - \ln(a)$ (as also shown below). This gives us a chance to talk about the necessity for the more general indefinite integral $\int_a^x f(t)dt$ and the reason why the denominator can not take the value zero for the function $f(t) = \frac{1}{t}$, (or, in general why as must make sure f has no discontinuities between a and x, a major hypothesis of the Fundamental Theorem).



Pursuing the integration of $\frac{1}{x}$ further, we can make the point that blind reliance on a calculator can be misleading. Our calculator actually gave us an answer for the integral $\int_{-1}^{1} \frac{1}{x} dx$, namely a recognizable decimal approximation of $-pi$. This is, of course, $\ln(1) - \ln(-1)$, consistent with the general formula $\ln(x) - \ln(a)$, but can not be regarded as a reasonable answer for the integration of a real-valued function.

The Inverse Relationship—Other Functions

In the following activity, we explore functions other than power functions. The calculator can produce the right answers. Students can compare the two columns for differentiation and integration, and try to come up with the same conclusion as for the power function. However, beside the constraint we mentioned above, there are other things the students need to think about. For example, they will ask 1) what am I supposed to do if the calculator does not have the required keys? and 2) what am I supposed to do when the result from the calculator is different from my textbook? We use two examples to illustrate students questions. The first is that because the calculator does not have cotangent key, the students have to use one over tangent as a substitution.



The second is when the result of reading off the derivative of cotangent from the calculator is different from the textbook statement. The textbook gives the derivative f cotangent as $-\csc^2 x$, where as the calculator gives the answer $-\left(\dfrac{1+\tan^2 x}{\tan^2 x}\right)$. We use this opportunity to talk about the necessity of learning trigonometry and the trigonometric identities. The students prove that these two expressions are equal as follows:

$$\frac{d}{dx}[\cot x] = -\csc^2 x = -\left(\frac{1}{\sin^2 x}\right) = -\left(\frac{\sin^2 x+\cos^2 x}{\sin^2 x}\right) = -\left(\frac{\sin^2 x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}\right) = -\left(1+\cot^2 x\right)$$

$$= -\left(1+\frac{1}{\tan^2 x}\right) = -\left(\frac{\tan^2 x}{\tan^2 x}+\frac{1}{\tan^2 x}\right) = -\left(\frac{1+\tan^2 x}{\tan^2 x}\right)$$

After students show the identity of these two expressions, they will understand that the home -work exercises they did in high school have finally paid off.

Activity 2: Looking for the relationships; is integration the inverse operation of differentiation?

Table 2:The Elementary Functions and their Operations

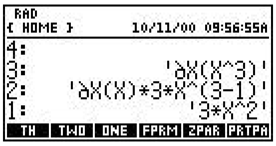
Rules	Differentiation	Integration
Constant Rule	$\frac{d}{dx}[ax] = a$	$\int a dx = ax + c$
Power Rule	$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$
Exponential Rule	$\frac{d}{dx}[e^x] = e^x$	$\int e^x dx = e^x + c$
Logarithm Rule	$\frac{d}{dx}[\ln x] = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln\lvert x\rvert + c$
Sine Rule	$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x = \sin x + c$
Cosine Rule	$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x dx = -\cos x + c$
Tangent Rule	$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x dx = \tan x + c$
Cotangent Rule	$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x dx = -\cot x + c$
Secant Rule	$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + c$
Cosecant Rule	$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x dx = -\csc x + c$
Arcsine Rule	$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$

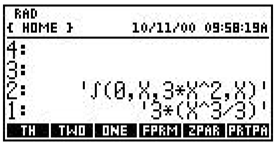
Continuation: The Elementary Functions and their Operations

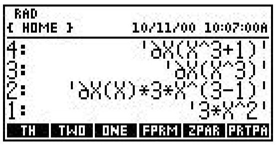
Arccosine Rule	$\frac{d}{dx}\left[\arccos x\right] = -\frac{1}{\sqrt{1-x^2}}$	$\int \frac{-dx}{\sqrt{1-x^2}} = \arcsin x + c$
Arctangent Rule	$\frac{d}{dx}\left[\arctan x\right] = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \arctan x + c$
Arccotangent Rule	$\frac{d}{dx}\left[arc\cot x\right] = -\frac{1}{1+x^2}$	$\int \frac{-dx}{1+x^2} = arc\cot x + c$
Arcsecant Rule	$\frac{d}{dx}\left[arc\sec x\right] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{dx}{x\sqrt{x^2-1}} = arc\sec x + c$
Arccosecant Rule	$\frac{d}{dx}\left[arc\csc x\right] = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{-dx}{x\sqrt{x^2-1}} = arc\sec x + c$

The Additive Constant

The additive constant appearing in integration confuses many students. Calculator-aided activities can help students understand the role of the additive constant. As one such, we discuss our Activity 3 which is finding the constants of the integration to fill out the table 3. Let us have our calculator take Y_1=f(x) via the operations $Y_2 = f'(x) = \frac{d}{dx} f(x)$ and $Y_3 = f(x) = \int_0^x f'(t)dt$. When Y_1 is taken to be x^3 we find that Y_3 agrees with Y_1. When Y_1 is taken to be x^3+1 we find that Y_3 no longer agrees with Y_1, but is the same shape as Y_1 and can be made to coincide with Y_1 by adding an appropriate constant. Thus the final part of this activity is to determine what constant must be added to Y_3 in order to get back to Y_1.







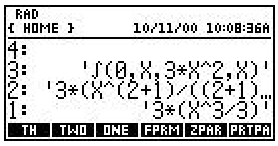
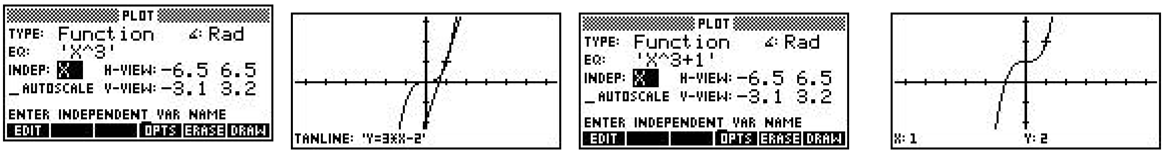


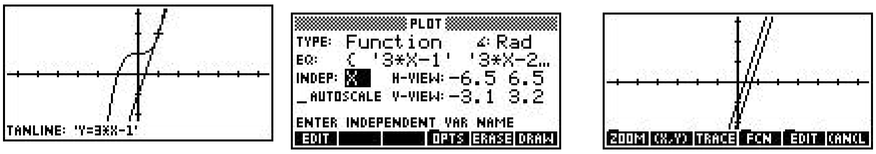
Table 3 The Additive Constant

Function	Derivative	anti-derivative	Constant c
$f(x) = x^3$	f`(x)= $3x^2$	$\int 3x^2 dx = 3\frac{x^3}{3} = x^3 + c$	c=?
$f(x) = x^3 + 3$	f`(x)= $3x^2$	$\int 3x^2 dx = 3\frac{x^3}{3} = x^3 + c$	c=?
$f(x) = x^3 - 3$	f`(x)= $3x^2$	$\int 3x^2 dx = 3\frac{x^3}{3} = x^3 + c$	c=?
$f(x) = x^3 + a$	f`(x)= $3x^2$	$\int 3x^2 dx = 3\frac{x^3}{3} = x^3 + c$	c=?

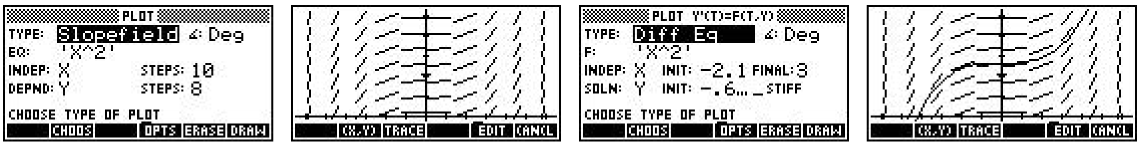
The Same Slope of Different Tangent Lines

We can also illustrate by having the calculator draw the tangent line to Y_1 at x=1. According as we take $Y_1(X) = X^3$ or $Y_1(X) = X^3 + 1$ this tangent line changes, but not its slope. Thus the position of the tangent line at x=1 depends on the additive constant c, but the slope of this tangent line does not depend on c.





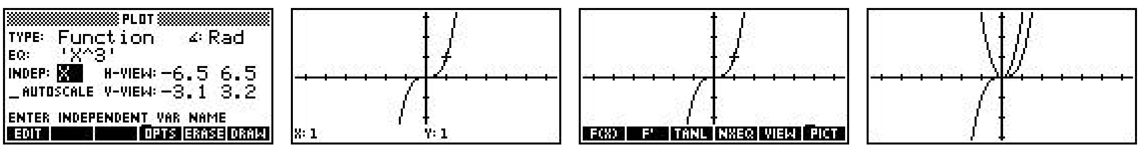
If the students have strong feeling for the additive constants, they will be well prepared for the slope-field idea. They can tell by the shape of the slope-field that the differential equation embedded here has the cubic function as solution.

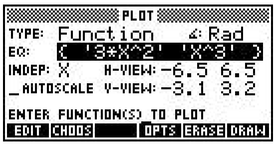


The Graphs of the Fundamental Theorem of Calculus

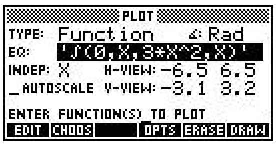
Activity 4 also illustrates the Fundamental Theorem by applying our two calculator operations successively. Except rather than finding the results of differentiation or integration, we find their graphs. For instance, feed in the function $Y_1(x) = 3x^2$, and ask the calculator to graph $Y_2(x) = \int_0^x 3t^2 dt$, then ask the calculator to graph $Y_3(x)=Y_2'(x)$. The result looks identical to $Y_1(x) = 3x^2$. The calculator has to work hard enough to produce Y_3 to make the end result somewhat dramatic. (Actually, our calculator can graph Y_1 and Y_2' on the same screen). Interestingly, the calculator did not give us a formula for Y_2 (namely x^3) so that it appears in this exercise to be a function defined by its graph. It would be interesting to do the same activity, but with $Y_1(x) = e^{x^2}$ (for which actually there is no elementary anti-derivative Y_2). Unfortunately the HP48G does not have the capacity to graph the anti-derivative of $Y_1(x) = e^{x^2}$.

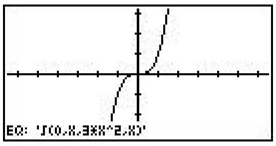
The following pictures are given the function $Y_1(x) = x^3$ and finding the graph of its derivative $Y_2(x) = 3x^2$. The PLOT function will show the equations for both functions.

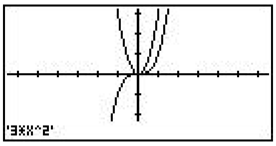


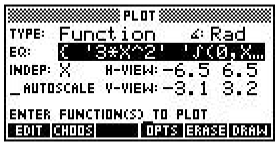


The following pictures are obtained by giving the integration of function $Y_2(x) = 3x^2$ and asking the calculator to draw the derivative of the integration of $Y_2(x) = 3x^2$.









Discussion

The Fundamental Theorem of Calculus, as we have remarked, unifies calculus by showing that differentiation and integration are closely related operations on functions in that they are essentially inverses of each other. The word "essentially" here hides the role of the additive arbitrary constant, a role which is confusing to some students. For one thing, there is some semantic confusion in the notion of an *arbitrary* constant. Also, students at this stage of development are used to problems which have a unique answer. Students tend to forget to carry along the additive constant or, more importantly, to understand its importance.

A modern graphing calculator will graph $\int_a^x f(t)dt$ when any of the basic elementary function f is the input (and for any choice of a as long as f is continuous on [a, x]) and also graphing calculators have the capacity to graph f' for any given elementary f. Doing the first operation to a given f (let's call this operation Ia, for "integration") and then doing the second operation (let's call it D) to the result will bring us back to f, suggesting that Ia and D are inverse operations (one undoes the other).

However, this of course does not always work when the operations are done in the other order. For example, if we ask the calculator to differentiate e^x, and then integrate the result with the most natural choice of a (namely a=0), we don't get back to e^x, we get e^x-1. (That is, $I_0(D(e^x))=e^x -1$). Students can be encouraged to try to understand what is happening here, and frequently ask questions of their own record. What will happen if we use various other choices of a? We never get back to e^x for any choice of a, but we never get graphs which look much different from e^x. In fact, it will be clear by graphical evidence that we are always getting vertical translates of e^x. Thus although our operations are not quite inverse the basic graph is preserved.

Similar exercises can be based on applying D and Ia starting with the function cos x. (Now, of course, IaD brings us back to cos x for a certain choice of a). Of course the concept we want to get across is that the indefinite integral, or anti-derivative, is determined only to within an additive constant, and that it is important to understand this additive constant. Technology can be used to help students at this point.

Modern graphing calculators have the pedagogically useful feature of being able to quickly draw slope fields for equations

$$\frac{dy}{dx} = g(x, y)$$

(g a given function). It is not hard to show students how to follow a trajectory from a given initial point, and it becomes clear that the slope field determines a family of curves. When we specialize by making g a function of x alone, our trajectories become a family of parallel curves. Thus we associate the indefinite integral with a direction field of the form $\frac{dy}{dx} = f(x)$. When we have the calculator draw the direction field for, for example, $\frac{dy}{dx} = x^2$, it is not hard to trace a few trajectories and see that they are all appear to be vertical translates of the graph of $y = \frac{x^3}{3}$.

As a further exercise, one might, for example, ask students to graphically find a function which has the same derivative as x^3 and contains the point (1,2). One could also discuss a few examples of slope fields where the function g is not independent of y, such as the equation

$$\frac{dy}{dx} = cy.$$

Conclusion

With the help of the graphing calculator, the students seemed to enjoy the class more. They arrived earlier to the classroom in order to get their hands on the graphing calculator. They talked to each other about mathematics. They showed other students their works or their discoveries. They were excited about seeing the graph of a given function in a short while and also being able to identify the characteristics of the graph. Although they did not know quite well how to compute the derivative or the indefinite integral of a given function, they knew that these two concepts are inverse operations of each other. Intuitively they could talk about what we mean by the Fundamental Theorem of Calculus. However, students still wondered what the various constants were doing in the function. What were their roles in the learning of Fundamental Theorem of Calculus? With the help of calculator and the exhibition of the slope-fields, the students were finally realizing the importance of additive constants.

Our experience indicates that these calculator activities described in this paper help students to understand the Fundamental Theorem of Calculus, the concept of families of functions, integration and the role of the constants of integration. Further, as we discuss near the end of the paper, use of calculators gives students a first look at slope fields and differential equations, providing some germination time for concepts they will be working with later. According to the philosophy of the "spiral curriculum" it is a good idea to take a first look at important concepts early in the student's development.

References

Lee, B. (1999). Functions and HP48G Graphics Calculator. Workshop material presented at Asian Technology Conference in Mathematics.