TREAT TECHNOLOGY WITH CARE: SOME TRAPS FOR THE UNWARY

Lynette M Bloom
Edith Cowan University
Perth, Western Australia
Email: l.bloom@ecu.edu.au

Walter R Bloom
Murdoch University
Perth, Western Australia
Email: bloom@murdoch.edu.au

Abstract

The advent of technology in the form of powerful calculators and computer algebra systems brought with it the promise of enormous change in the mathematics curriculum at the secondary and lower tertiary levels. There was the idea that, now that the difficult time-consuming graphing and calculation could be done by the calculator/computer, teachers could fit in so much more 'real mathematics'. In fact, the range of topics covered has not changed significantly. One reason for this is the need to spend time teaching students the judicious use of this powerful new technology. This is needed to combat the very real danger that students will happily accept whatever answers the technology gives them, which in many cases could be totally incorrect. The reasons for these wrong answers are many and varied. The easiest to cope with is when the message ERROR (or its equivalent) appears as a prompt. But what should the student do when a seemingly correct answer is produced? Our experience is that students will just accept the given answer. In this paper we use the computer algebra system *Scientific Notebook v.3.5* to illustrate some of the common pitfalls that arise in both teaching and assessment and discuss measures to avoid and/or redress these.

1 Introduction

The availability and accessibility of (now quite sophisticated) hand-held graphics calculators and user-friendly computer-algebra packages has had a large impact on the teaching of mathematics. At the lower secondary level it is usually the graphics calculator that is used, whereas at the tertiary level, both calculators and computer packages are widespread. Applications in which the computations were previously considered too difficult to be taken as examples in teaching and/or assessment are now commonplace. However, it should be appreciated that some care needs to be taken in introducing students to this technology and some effort needs to be made in teaching them how to use the technology correctly and effectively. Similar sentiments were expressed by Andrews ([2]). Dion ([3]) posed the conjecture that the successful use of calculators requires a higher level of understanding than that required for rote computation or template problem solving. This has certainly been our experience, and it applies to the use of computer software as well as the calculator.

In this paper we discuss a range of examples illustrating some of the problems that arise when technology is used blindly by students carrying out mathematical calculations. Obvious ones are simple data entry and round-off errors, although the latter can be quite subtle. Less easy to detect are the errors that arise from the internal implementation of the particular algebra system used by the calculator or computer package itself.

We give illustrations here with examples from *Scientific Notebook v.3.5*, which is a relatively inexpensive combined word-processing and computer algebra system, operating with both *Maple* and *MuPad* computation engines. Technical details can be found on the *Scientific Notebook* website (http://scinotebook.tcisoft.com), where one can download a free 30-day time-locked version of the package. *Scientific Notebook* is easy to use and students are able to become familiar with this package in a very short time. In our case we found that one workshop was sufficient to enable students to be well on the way to becoming independent users. The user-friendly aspect is particularly important to us since, at both Edith Cowan and Murdoch universities, the majority of our students are studying mathematics as a service subject, with no intention of progressing on to mathematical research.

There will of course be corresponding examples for any of the many available calculating and graphing devices. We distinguish these problems from examples of student misconceptions of the questions asked and misinterpretation of the output provided, some of which are given by Mueller and Forster ([6]) for graphics calculators used under examination conditions.

Anderson *et al* ([1]) drew attention to some specific pitfalls in the use of *Scientific Notebook v.3.0* to solve differential equations. Some of these, none of which was mentioned by Majewski ([5]) or Wilkin ([7]) in their reviews of this package, could arise simply by carelessness on the part of the user while others need to be attended to by the software writers themselves. Examples in the latter category include inconsistency in the interpretation of brackets and providing an incorrect solution to a particular initial value problem.

In this paper we illustrate some of the difficulties encountered due to naive technology use. These examples, with the exception of Section 3.2 below, have arisen in the course of our first and second year university calculus classes.

2 Round-off error

2.1 Calculation of eigenvalues

2.1.1 Exact versus numeric

The following example is quite simple, but many students in a recent class failed to notice the problem. They were asked to find the eigenvalues of the matrix

$$\begin{bmatrix} -0.2 & 0 & 0.2 \\ 0.2 & -0.4 & 0 \\ 0 & 0.4 & -0.2 \end{bmatrix}$$

Using *Scientific Notebook,* COMPUTE, MATRICES, EIGENVALUES students obtained the following:

$$\text{eigenvectors}: \begin{bmatrix} -.666\,67 \\ -.333\,33 \\ -.666\,67 \end{bmatrix} \leftrightarrow 1.\,985\,9 \times 10^{-10},$$

$$\begin{bmatrix} -.867\,69 - 4.196\,8 \times 10^{-2}i \\ -4.196\,8 \times 10^{-2} + .867\,69i \\ .909\,66 - .825\,72i \end{bmatrix} \leftrightarrow -.\,4 + .2i,$$

$$\begin{bmatrix} -.867\,69 + 4.196\,8 \times 10^{-2}i \\ -4.196\,8 \times 10^{-2} - .867\,69i \\ .909\,66 + .825\,72i \end{bmatrix} \leftrightarrow -.\,4 - .\,2i$$

and didn't recognize (for example) that $1.\,985\,9 \times 10^{-10}$ might have been 0. However, if they had written the matrix as

$$\begin{bmatrix} -1/5 & 0 & 1/5 \\ 1/5 & -2/5 & 0 \\ 0 & 2/5 & -1/5 \end{bmatrix}$$

then they would have obtained

$$\text{eigenvectors}: \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \leftrightarrow 0, \begin{bmatrix} i \\ 1 \\ -1-i \end{bmatrix} \leftrightarrow -\frac{2}{5} + \frac{1}{5}i, \begin{bmatrix} -i \\ 1 \\ -1+i \end{bmatrix} \leftrightarrow -\frac{2}{5} - \frac{1}{5}i$$

which is much more illuminating.

2.1.2 Displayed digits versus digits used in computation

In this example we consider the matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 6 & 3 & 1 \\ 5 & 1 & -1 \end{bmatrix}$$

Then COMPUTE, MATRICES, CHARACTERISTIC POLYNOMIAL (together with evaluating the zeros) with the default of 10 digits used in computations and 5 digits in display gives

$$X^3 - 3X^2 - 34X + 18 = 0$$
$$\text{Solution is}: \{X = -4.\,823\,1\}, \{X = .\,510\,34\}, \{X = 7.\,312\,8\}.$$

For the eigenvalue $0.510\,34$ we would naturally expect

$$\det\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 3 & 1 \\ 5 & 1 & -1 \end{bmatrix} - 0.510\,34 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0.$$

But actually computing this gives

$$\det\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 3 & 1 \\ 5 & 1 & -1 \end{bmatrix} - 0.510\,34\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = -1.573 \times 10^{-5}$$

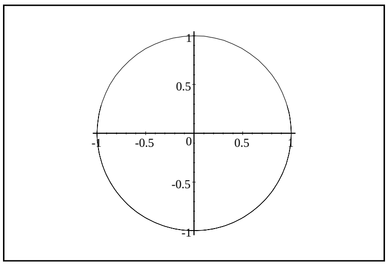
whereas changing the default setting to display 10 digits leads to

$$\det\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 3 & 1 \\ 5 & 1 & -1 \end{bmatrix} - 0.510\,340\,433\,6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 1.0 \times 10^{-8}$$

even though the number of digits for computation is unchanged. In each case the student needs to recognize the answer as being 0.

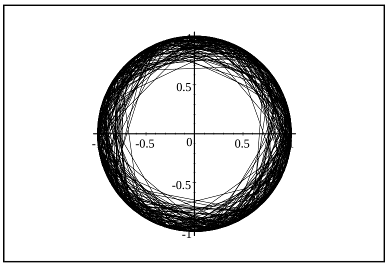
2.2 Graph plotting

The plots of the parametric equations $x = \sin t^n, y = \cos t^n$ where $t \in [0, 2\pi)$ and $n = 1, 2, 3, ...$ should all be identical to that shown in Figure 1 for the case $n = 1$. With *Scientific Notebook* this does not happen, as shown by the plot for the case $n = 4$ in Figure 2 where already a good deal of the interior of the circle is shaded.



$\sin t, \cos t$

Figure 1



$\sin t^4, \cos t^4$

Figure 2

With round-off and sampling error we are in fact obtaining points where the sum of the squares of the coordinates is less than 1, and this worsens as the powers increase. In the above we are using the default settings, and in fact the filling-in disappears to some extent when a larger number of sample points is taken and a larger number of digits used in computations is selected. However, it is not possible to overcome this completely.

2.3 Zeros of functions

Another instance of round-off error which can really mislead students is illustrated by the following example. Consider the solutions of the equation

$$\left(x^2 - 1\right)\ln x = 0.01$$

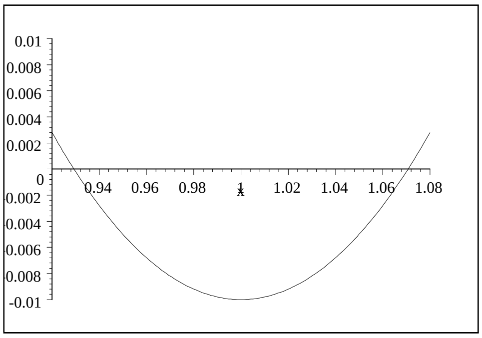
Following the HELP menu for *Scientific Notebook* the student would enter COMPUTE, SOLVE, EXACT and obtain $x = 1.0707$. There is in fact another solution, which can be found by applying COMPUTE, SOLVE, NUMERIC to

$$\begin{bmatrix} (x^2 - 1)\ln x - 0.01 = 0 \\ x \in [0, 1] \end{bmatrix}$$

to obtain $x = 0.9293$ (to 4 decimal places). The latter procedure can also be used to find the first solution, by replacing $[0, 1]$ by $[1, 2]$. (Of course with this approach the solutions need to be isolated from the outset.) Now these are the only two solutions, and they appear to be symmetric about 1. But appearances can be deceptive! Setting the number of digits used in both computation and display to 10 we obtain 1.070 696 935 and 0.929 305 153 respectively, and these are clearly not symmetric about $x = 1$.

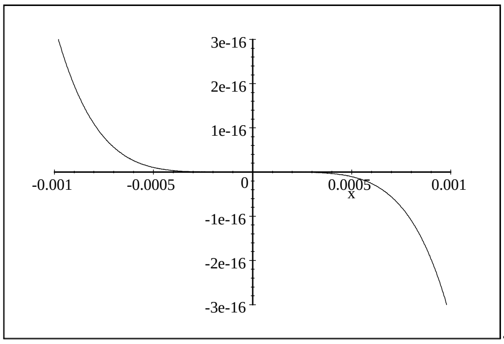
There are two interesting aspects of this calculation. Firstly, the two different zeros are not being returned by the first method, which may lead the student to believe that there is just a single zero. (Even if the student tries the second approach, with $[0, 2]$ replacing $[0, 1]$, still only the smaller zero is returned.) And secondly, which is peculiar to this particular example, there is the need to set a large number of decimal places to dispel the idea that the zeros are symmetric about 1.

Furthermore, by plotting the function $h(x) = (x^2 - 1)\ln x - 0.001$ and changing the scale appropriately, it again appears (see Figure 3) as though h is symmetric about 1. However if we try a different approach and plot $j(x) = h(1 + x) - h(1 - x)$ to obtain the graph in Figure 4



$h(x)$

Figure 3



$j(x)$

Figure 4

then the fact that the zero function is not obtained confirms the lack of symmetry of h about $x = 1$.

3 Programming faults

3.1 Reduced row echelon form of a matrix

Finding the reduced row echelon form of a matrix should be straightforward. For example

$$\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \text{row echelon form} : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and with the variable a replacing the $(1, 2)$-entry,

$$\begin{bmatrix} 1 & a \\ 2 & 2 \end{bmatrix}, \text{row echelon form} : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

But this doesn't always hold, as can be seen by putting $a = 1$,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \text{row echelon form} : \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Most students would be puzzled about what is going on here, but it becomes clearer when the algebra is tracked.

$$\begin{bmatrix} 1 & a \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & a \\ 0 & 2-2a \end{bmatrix} \to \begin{bmatrix} 1 & a - a\frac{2-2a}{2-2a} \\ 0 & \frac{2-2a}{2-2a} \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is only valid for $a \neq 1$. But when simply applying the COMPUTE command all this is hidden, and if the matrix is a little larger and more complicated (a 3×3 matrix will do) then it might not even occur to the user just what has happened.

In general, this would be very difficult for most students to even recognize much less understand. For all but the best students the instructor can only alert the students that with any algebraic manipulation there are invariably singular cases that the computer package is unlikely to incorporate.

3.2 Calculations with polynomials

The Cartier polynomials $P(n, x)$ arise naturally in the study of random walks on polynomial trees. Indexing them by the parameter $\gamma \in \left(0, \frac{1}{2}\right]$ they can be written as

$$\begin{aligned} P(n,x) &= \left(\frac{x(1-2\gamma)+\sqrt{x^2-4\gamma+4\gamma^2}}{2\sqrt{x^2-4\gamma+4\gamma^2}}\right)\left(\frac{x+\sqrt{x^2-4\gamma(1-\gamma)}}{2(1-\gamma)}\right)^n + \\ &+ \left(\frac{-x(1-2\gamma)+\sqrt{x^2-4\gamma+4\gamma^2}}{2\sqrt{x^2-4\gamma+4\gamma^2}}\right)\left(\frac{x-\sqrt{x^2-4\gamma(1-\gamma)}}{2(1-\gamma)}\right)^n \end{aligned}$$

To evaluate $P''(n, 1)$ at $\gamma = \frac{1}{2}$ consider the following three expressions arising in an analysis of these polynomials:

$$R(n) = \lim_{x \to 1} \frac{\partial^2}{\partial x^2} P(n, x),$$

$$S(n) = \frac{1}{(2\gamma-1)^4}\left(\begin{array}{c} 4(-1)^n(-1+\gamma)^{-n+1}\gamma^{n+1}n + 6(-1)^n(-1+\gamma)^{-n+1}\gamma^{n+1} - \\ -2n\gamma - 8(-1)^n(-1+\gamma)^{-n+1}\gamma^{2+n}n + 12n\gamma^2 - n - \\ -8n\gamma^3 - 4n^2\gamma + 4n^2\gamma^2 + n^2 + 6\gamma - 6\gamma^2 \end{array}\right),$$

$$T(n) = \frac{1}{\gamma\left(\frac{1}{\gamma}-2\right)^4}\left(\begin{array}{c} -2\left(-1+\frac{1}{\gamma}\right)^{-n}\left(-1+\frac{1}{\gamma}\right)\left(2\frac{n}{\gamma}+\frac{3}{\gamma}-4n\right)+12\frac{n}{\gamma}-\frac{1}{\gamma^3}n - \\ -8n-\frac{4}{\gamma^2}n^2+\frac{4}{\gamma}n^2+\frac{1}{\gamma^3}n^2+\frac{6}{\gamma^2}-\frac{6}{\gamma}-\frac{2}{\gamma^2}n \end{array}\right).$$

Now using COMPUTE, CHECK EQUALITY we obtain the output

$$R(n), S(n) \text{ is true}$$

and

$$S(n), T(n) \text{ is true}$$

which just says that

$$R(n) = S(n) = T(n).$$

But notwithstanding all of this the computer package gives the obscure

$$\lim_{\gamma \to \frac{1}{2}} S(n) = \text{signum}\left(1 - e^{-in\pi}(-1)^n\right)\infty$$

even though it gives correctly

$$\lim_{\gamma \to \frac{1}{2}} R(n) = -\frac{1}{3}n^2 + \frac{1}{3}n^4 = \frac{1}{3}n^2(n-1)(n+1) = \lim_{\gamma \to \frac{1}{2}} T(n).$$

The problem here is that the program is not coping with these calculations, but of course most students encountering an 'undefined' limit as in the calculation for $\lim_{\gamma \to \frac{1}{2}} S(n)$ would take this answer at face value. On the other hand, if the package had returned 'undecided' then the student could have been alerted to the potential problem.

4 Display of the graph of a function

It is important that students do not simply enter the formula for a function and accept blindly the graphical display produced. Here we consider an example that shows how the student can be seriously misled at various steps in the argument.

4.1 $f(x) = \sin\sqrt{x-1}$

Entering the formula and using PLOT 2D, RECTANGULAR on *Scientific Notebook* produces the graph in Figure 5. This uses the default settings of the *Plot Properties* Dialog Box. In this the default *Plot Components* settings are Domain Interval [-5,5] and sample size 49. The former provides the largest interval on which the program will compute values of the given function and the latter determines the number of values calculated, and hence the detail of the plot. The default *View* setting for the horizontal axis is [-5,5], with automatic scaling provided for the vertical axis.

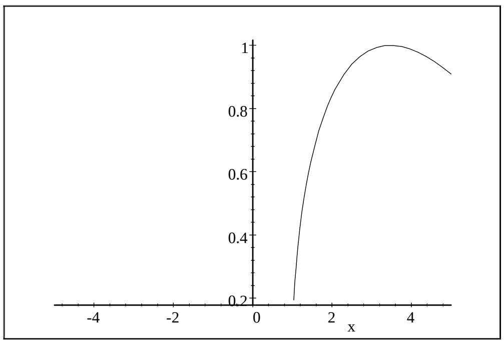


Figure 5

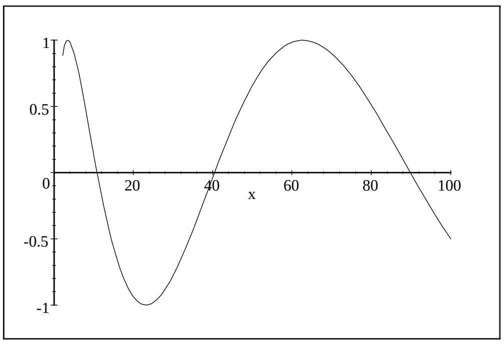


Figure 6

Note that from the graph shown in Figure 5 the function appears not to be defined at x=1, and the periodicity is not evident, nor is it suggested that the function might take any negative values. However, a student who realizes that *sine* is a periodic function with range [-1,1] might change to Domain Interval [0,100] and the horizontal View Interval [0,100] to obtain the graph in Figure 6. The function still appears undefined at x=1. An attempt to view the periodicity more clearly might cause the student to change to Domain Interval [0,500], and horizontal View Interval [0,500], thus obtaining the graph in Figure 7. The shape is now quite different and does appear to give a value of 0 at x=1. However, now there is the problem that the graph seems to start with a section below the horizontal axis, and a naive user would then start to wonder just which was the graph of this function.

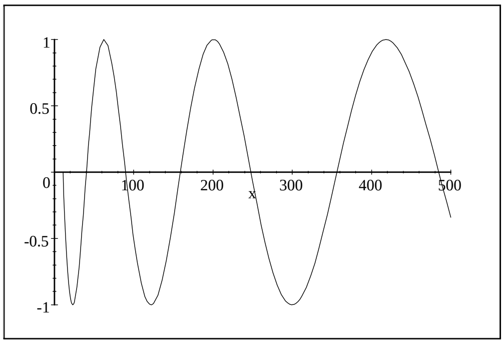


Figure 7

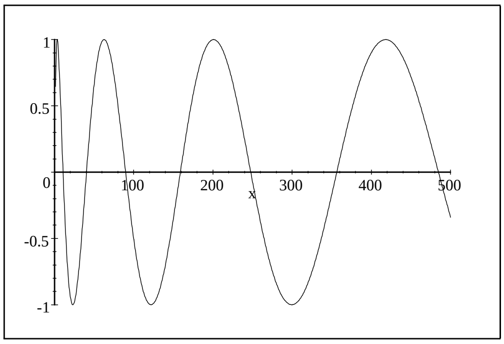


Figure 8

Some mathematical investigation would favour the graph in Figure 6 and it might be thought that the problem around $x = 1$ was caused by a lack of detail in the actual plot. However, a change to Sample Size 999 (the maximum allowed by the package) produces the graph in Figure 8, which is even more confusing. A change to Domain Interval [0,100], horizontal View Interval [0,100] and Sample Size 999 together with some adjustment of the tick marks on the axes gives the graph in Figure 9. There is once again a problem at x=1. This can be overcome by making the single change to Domain Interval [1,100], to obtain the (accurate) graph in Figure 10.

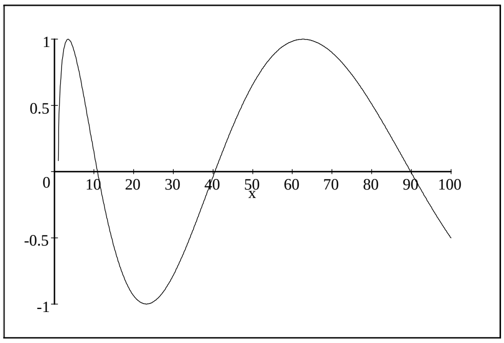


Figure 9

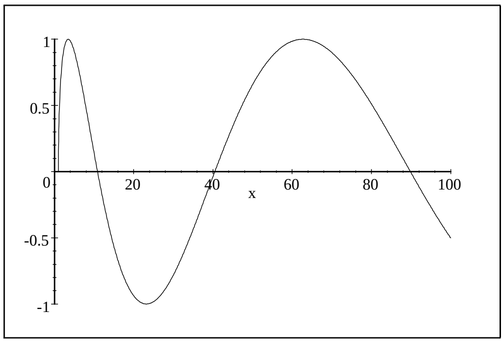


Figure 10

An example such as this can seriously undermine the student's confidence in the use of this technology, so it is essential to have some strategy in place to deal with this when such problems arise. One way students can attempt to combat the problem illustrated in Figure 5, where the effective domain of definition is simply too small to give an accurate idea of the full graph, is to always change from the default domain and view intervals to much larger domain and view intervals and see what effect this has on the graph. It may then be necessary to choose a larger sample size to obtain an accurate plot.

This approach will often be effective, particularly when graphing polynomial functions. For example, consider the function $x^5 + 3x^4 - 392x^3 - 252x^2 + 17392x + 32640$. The default settings give the plot in Figure 11, whereas resetting to a domain interval of [-50, 50] and view intervals [-25, 25] and [-500 000, 500 000] on the horizontal and vertical axes respectively gives the plot in Figure 12.

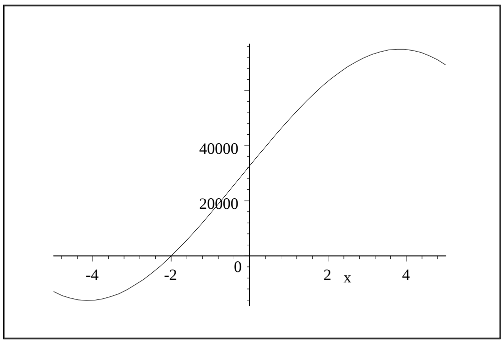


Figure 11

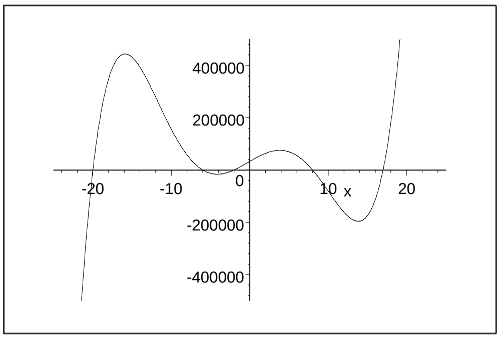


Figure 12

However, this strategy is not a general panacea. For example, the default plot of tan(x) is shown in Figure 13. In this case, a change in the vertical View Interval to [-2, 2] will produce the improvement shown in Figure 14.

In the final analysis there is no substitution for some mathematical sophistication. Goldenberg ([4]) goes even further, with the view that thoughtless use of computer graphing in the classroom may actually obscure what we are trying to teach.

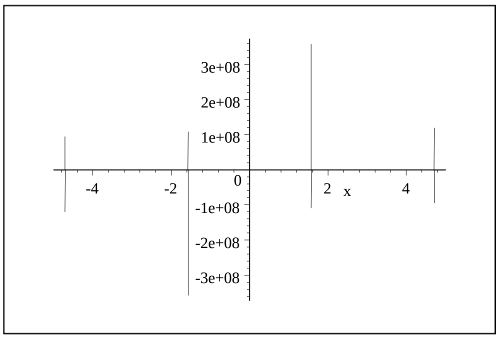


Figure 13

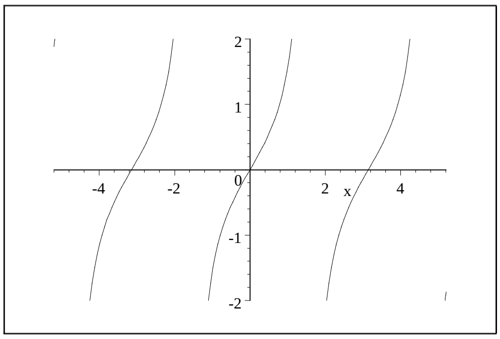


Figure 14

5 Conclusions

We have shown here that, although the technology now available is a very powerful and valuable tool for teaching and learning, teaching time needs to be taken for students to learn how to use the chosen calculator/computer package appropriately, and indeed this technology must be combined with mathematical knowledge for really effective use. There are, as we have shown, many traps for the unwary and the unskilled. However, for the skilled user the advent of modern calculator/computer technology opens up a wide vista of exciting possibilities for mathematics teaching, learning and research. The benefits to be had more than justify the initial effort involved.

There are those who wish to limit student use of technology on the grounds that they will just use it blindly. It is the responsibility of mathematics educators to help students to become intelligent users of the available tools, not to prevent their use. This means we have to teach our students both the mathematics and the use of the technology.

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