A Recursive Triangular Factorization Procedure for Inverse Vandermonde Matrices

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Abstract

This paper is concerned with the factorization of inverse Vandermonde matrices as a product of two triangular matrices. Recursive algorithms for determining the entries of the triangular matrices in the factorization are presented.

1 Introduction

Given n pairwise distinct numbers $\lambda_1, \cdots, \lambda_n$ we define the $n \times n$ Vandermonde matrix $V = V(\lambda_1, \lambda_2, \cdots, \lambda_n)$ by

$$V(\lambda_1, \lambda_2, \cdots, \lambda_n) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}.$$

It is well known (see Graybill [3]) that the determinant of V is given by

$$\det V = \prod_{1 \le j < i \le n} (\lambda_i - \lambda_j).$$

The λ_i's being all distinct, it follows that V is invertible. Inverting this matrix appears naturally in many applications.

Since all the principal submatrices of V are also nonsingular Vandermonde matrices, V has an LU factorization requiring no pivots.

Triangular factors of Vandermonde matrices have been obtained by Björk and Pereyra (see [2] and the references therein). In this paper we present a factorization formula expressing the inverse V^{-1} as a product of two triangular matrices whose entries are easily computed by means of recursive algorithms. These algorithm are both suitable for symbolic as well as numerical computations.

2 Main Results

Consider the Newton polynomials :

$$\begin{cases} \varphi_1(s) &= 1, \\ \varphi_{j+1}(s) &= (s-\lambda_j)\varphi_j(s), \quad j = 1, 2, \ldots, n-1. \end{cases}$$

Let L be the $n \times n$ matrix whose rows are associated with the coefficients of the Newton polynomials $\varphi_j(s)$. Notationally,

$$\begin{bmatrix} \varphi_1(s) \\ \varphi_2(s) \\ \vdots \\ \varphi_n(s) \end{bmatrix} = L \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix}.$$

It is clear from the construction of the polynomials, $\varphi_j(s)$, that the rows of L can be recursively computed, and that L is lower triangular with 1's on the main diagonal (the leading coefficient in $\varphi_i(s)$ is 1).

Then the inverse V^{-1} factors into two $n \times n$ matrices as

$$V^{-1} = HL.$$

The $n \times n$ matrix H appearing in this factorization is characterized by the following theorem (see Hou [1]).

Theorem 1 *The matrix* $H = [h_n, h_{n-1}, \cdots h_1]$ *is upper triangular, and its column vectors* h_k *may be computed recursively as follows:*
Let

$$\triangle(s) = \begin{bmatrix} \lambda_1 - s, & \lambda_2 - s, & \cdots & \lambda_n - s \end{bmatrix}^T.$$

Then

$$h_{k+1} = \triangle(\lambda_{n-k}) * h_k, \qquad k = 0, 1, \ldots, n-1,$$

with the initial vector $h_1 = \begin{bmatrix} c_1, & c_2, & \cdots & c_n \end{bmatrix}^T$ *as determined by the partial fraction expansion*

$$\frac{1}{(s-\lambda_1)\cdots(s-\lambda_n)} = \frac{c_1}{s-\lambda_1} + \frac{c_2}{s-\lambda_2} + \cdots + \frac{c_n}{s-\lambda_n}.$$

Remark. The symbol $*$ denotes the (Hadamard) vector multiplication as defined by

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} * \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 \\ u_2 v_2 \\ \vdots \\ u_n v_n \end{bmatrix}$$

(See Van Loan [4]).

3 Illustrative Example

The following is a 4×4 Vandermonde matrix $V(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ for which $\lambda_1 = -1, \lambda_2 = 3, \lambda_3 = 2, \lambda_4 = -2$:

$$V = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 3 & 2 & -2 \\ 1 & 9 & 4 & 4 \\ -1 & 27 & 8 & -8 \end{bmatrix}$$

whose inverse is

$$V^{-1} = \begin{bmatrix} \frac{12}{12} & \frac{-4}{12} & \frac{-3}{12} & \frac{1}{12} \\ \frac{-4}{20} & \frac{-4}{20} & \frac{1}{20} & \frac{1}{20} \\ \frac{6}{12} & \frac{7}{12} & \frac{0}{12} & \frac{-1}{12} \\ \frac{-6}{20} & \frac{-1}{20} & \frac{4}{20} & \frac{1}{20} \end{bmatrix}.$$

In this case the Newton polynomials $\varphi_j(s)$ are easily recursively computed as

$$\begin{aligned} \varphi_1(s) &= 1, \\ \varphi_2(s) &= (s+1)\varphi_1(s) = 1 + s, \\ \varphi_3(s) &= (s-3)\varphi_2(s) = -3 - 2s + s^2, \\ \varphi_4(s) &= (s-2)\varphi_3(s) = 6 + s - 4s^2 + s^3, \end{aligned}$$

so that

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & -2 & 1 & 0 \\ 6 & 1 & -4 & 1 \end{bmatrix}.$$

To find the H matrix in the factorization $V^{-1} = HL$, we proceed by first defining the vector

$$\triangle(s) = \begin{bmatrix} -1-s, & 3-s, & 2-s, & -2-s \end{bmatrix}^T,$$

and expanding

$$\frac{1}{(s+1)(s-3)(s-2)(s+2)} = \frac{\frac{1}{12}}{s+1} + \frac{\frac{1}{20}}{s-3} + \frac{\frac{-1}{12}}{s-2} + \frac{\frac{-1}{20}}{s+2}$$

to get the initial vector

$$h_1 = \begin{bmatrix} \frac{1}{12} & \frac{1}{20} & \frac{-1}{12} & \frac{-1}{20} \end{bmatrix}^T.$$

Thus according to the recursive algorithm given in Theorem 1, we have

$$\begin{aligned} h_2 &= \triangle(-2) * h_1 \\ &= \begin{bmatrix} 1 & 5 & 4 & 0 \end{bmatrix}^T * \begin{bmatrix} \frac{1}{12} & \frac{1}{20} & \frac{-1}{12} & \frac{-1}{20} \end{bmatrix}^T \\ &= \begin{bmatrix} \frac{1}{12} & \frac{1}{4} & \frac{-1}{3} & 0 \end{bmatrix}^T \end{aligned}$$

$$\begin{aligned} h_3 &= \triangle(2) * h_2 \\ &= \begin{bmatrix} -3 & 1 & 0 & -4 \end{bmatrix}^T * \begin{bmatrix} \frac{1}{12} & \frac{1}{4} & \frac{-1}{3} & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} \frac{-1}{4} & \frac{1}{4} & 0 & 0 \end{bmatrix}^T \end{aligned}$$

$$\begin{aligned} h_4 &= \triangle(3) * h_3 \\ &= \begin{bmatrix} -4 & 0 & -1 & -5 \end{bmatrix}^T * \begin{bmatrix} \frac{-1}{4} & \frac{1}{4} & 0 & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T. \end{aligned}$$

Hence

$$H = \begin{bmatrix} 1 & \frac{-1}{4} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{20} \\ 0 & 0 & \frac{-1}{3} & \frac{-1}{12} \\ 0 & 0 & 0 & \frac{-1}{20} \end{bmatrix} = \begin{bmatrix} \frac{12}{12} & \frac{-3}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{5}{20} & \frac{5}{20} & \frac{1}{20} \\ 0 & 0 & \frac{-4}{12} & \frac{-1}{12} \\ 0 & 0 & 0 & \frac{-1}{20} \end{bmatrix}.$$

It is easy to check that $V^{-1} = HL$, that is

$$\begin{bmatrix} \frac{12}{12} & \frac{-4}{12} & \frac{-3}{12} & \frac{1}{12} \\ \frac{-4}{20} & \frac{-4}{20} & \frac{1}{20} & \frac{1}{20} \\ \frac{6}{12} & \frac{7}{12} & \frac{0}{12} & \frac{-1}{12} \\ \frac{-6}{20} & \frac{-1}{20} & \frac{4}{20} & \frac{1}{20} \end{bmatrix} = \begin{bmatrix} \frac{12}{12} & \frac{-3}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{5}{20} & \frac{5}{20} & \frac{1}{20} \\ 0 & 0 & \frac{-4}{12} & \frac{-1}{12} \\ 0 & 0 & 0 & \frac{-1}{20} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & -2 & 1 & 0 \\ 6 & 1 & -4 & 1 \end{bmatrix}.$$

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References

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