A generalization of ‘PolynomialExtendedGCD‘ and perturbed Sylvester Equations

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Abstract

We extend a *Mathematica* command PolynomialExtendedGCD to the case that the coefficients of polynomials contain indeterminates. By virtue of such an extension of the command, one can solve the Sylvester (matrix) equation $AX - XB = C$ explicitly even in the case that the square $m \times m$ matrix A and $n \times n$ matrix B have common eigenvalues, where C is an $m \times n$ matrix, X an (unknown) $m \times n$ matrix.

The asymptotic behavior ($t \to 0$) of the solution of perturbed (non-degenarate) Sylvester Equation $(A+t)X - XB = C$ plays an essential role in our argument.

1 Notation

Throughout the present paper we let k be a field. We denote by k^n the k-vector space of column vectors of size n. If E and F are k-vector spaces, we denote by $Hom_k(E, F)$ the vector space of k-linear maps from E to F. We also denote by $End_k(E)$ the space of k-linear transformations of E, that is, $End_k(E) = Hom_k(E, E)$. Let us denote the k-vector space of $m \times n$ matrices in k by $Mat_{m\times n}(k)$. An $m \times n$ matrix X gives rise to a k-linear map $L_X : k^n \to k^m$ by the rule $k^n \ni v \longmapsto Xv \in k^m$. By this correspondence $Mat_{m\times n}(k) \ni X \longmapsto L_X \in Hom_k(k^n, k^m)$ we identify two k-vector spaces

$Mat_{m\times n}(k)$ and $Hom_k(k^n, k^m)$. We also denote by $Mat_n(k)$ the space of square $n \times n$ matrices in k, that is, $Mat_n(k) = Mat_{n\times n}(k)$.

Let $A \in Mat_m(k)$, $B \in Mat_n(k)$, and $C \in Mat_{m\times n}(k)$, we call the equality

$$AX - XB = C$$

the Sylvester Equation with an unknown matrix $X \in Mat_{m\times n}(k)$.

For $A \in Mat_m(k)$, $B \in Mat_n(k)$, we define a k-linear map $\Phi_{A,B} \in End_k(Mat_{n\times n}(k))$ as follows:

$$\Phi_{A,B} : Mat_{m\times n}(k) \ni X \longmapsto AX - XB \in Mat_{m\times n}(k).$$

Then the Sylvester Equation $AX - XB = C$ has another form

$$\Phi_{A,B}(X) = C.$$

2 Non-degenerate Sylvester Equations

If the two square matrices $A \in Mat_m(k)$ and $B \in Mat_n(k)$ of the coefficients of Sylvester Equation $AX - XB = C$, have disjoint spectrum, that is, $\sigma(A) \cap \sigma(B) = \emptyset$, the linear transformation $\Phi_{A,B}$ is an automorphism of $Mat_{m\times n}(k)$. This is well-known and is also proved by representation theoretic argument in [10]. In such a case, the Sylvester Equation $AX - XB = C$ is called *non-degenerate*. For a non-degenerate Sylvester Equation, we can construct the explicit polynomial formula for the inverse map $\Phi_{A,B}^{-1} : Mat_{m\times n}(k) \ni C \longmapsto$the upper off-diagonal block of a certain polynomial of the block diagonal matrix $T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

For a precise statement we quote a theorem in [10].

Theorem 2.1 *Let A be a square $m\times m$ matrix, B a square $n\times n$ matrix, and C an $m \times n$ matrix. Let $P_A(\lambda)$ and $P_B(\lambda)$ be the characteristic polynomials of A and B, respectively. We assume that $P_A(\lambda)$ and $P_B(\lambda)$ are relatively prime. Let $Q_A(\lambda)$ and $Q_B(\lambda)$ be polynomials satisfying*

$$P_A(\lambda)Q_A(\lambda) + P_B(\lambda)Q_B(\lambda) = 1.$$

Then the unique solution X of the Sylvester Equation $AX-XB = C$ is equal to the upper off-diagonal block of the block triangular matrix $P_B(T)Q_B(T)$, that is,

$$P_B(T)Q_B(T) = \begin{pmatrix} I_m & X \\ 0 & 0 \end{pmatrix},$$

where I_m is the identity matrix of size m and T is a block diagonal matrix

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

3 *Mathematica* programming for the inversion formula of $\Phi_{A,B}$ (non-degenerate case)

Let A be a square $m \times m$ matrix, B a square $n \times n$ matrix, and C an $m \times n$ matrix. We assume the characteristic polynomials $P_A(\lambda)$ and $P_B(\lambda)$ of A and B, respectively, are relatively prime. A computing system *Mathematica* contains a package PolynomialExtendedGCD which gives a pair of polynomials $Q_A(\lambda)$ and $Q_B(\lambda)$ such that $P_A(\lambda)Q_A(\lambda) + P_B(\lambda)Q_B(\lambda) = 1$ as a list of length 2 for an input $\{P_A(\lambda), P_B(\lambda)\}$. By virtue of the package PolynomialExtendedGCD, we can translate Theorem 2.1 into *Mathematica* language.

The final output of the following programming is equal to $\Phi^{-1}_{A,B}(C)$.

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------------------programming-----------------

  <<Algebra` PolynomialExtendedGCD`
T=Join[Transpose[Join[Transpose[A],Transpose[Const]]],
 Transpose[
 Join[Transpose[Table[Table[0,{Length[A]}],{Length[B]}]],
Transpose[B]]]]

Coef=CoefficientList[
 Det[\[Lambda] MatrixPower[B,0] -B]
   PolynomialExtendedGCD[Det[\[Lambda] MatrixPower[A,0] -A],
      Det[\[Lambda] MatrixPower[B,0] -B]][[2]][[2]],\[Lambda]];

TT:=Coef.Table[MatrixPower[T,n],{n,0,Length[Coef]-1}]//Simplify
```

```
Transpose[
Table[Transpose[Table[TT[[i]],{i,Length[A]}]][[Length[A]+j]],
{j,Length[B]}]]

--------------------END----------------------
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According to the above script of *Mathematica* whose essential parts are polynomial operations and calculations, we define a new command $Sylvester[\{A, B, C\}]$ which gives a unique solution X of a non-degenerate Sylvester Equation $AX - XB = C$. We may reconstruct PolynomialExtendedGCD by Euclidean mutual division algorithm using built-in commands PolynomialQuotient and PolynomialRemainder recursively, in which we can specify the indeterminate of polynomials. Such a modification or generalization of PolynomialExtendedGCD enable us to treat a field extension of coefficients of Sylvester Equations.

4 Perturbation of Sylvester Equations

In this section we consider a degenerate Sylvester Equation $AX - XB = C$, that is, $\sigma(A) \cap \sigma(B) \neq \emptyset$. The main idea of our treatment is as follows. We introduce an indeterminate t and perturb linearly the degenerate Sylvester Equation $AX - XB = C$ such as $(A + t)X - XB = C$. For almost all (that is, at most finitely many exceptions) scalar value of t in k the Sylvester Equation $(A + t)X - XB = C$ is non-degenerate, so we have the unique solution $\Phi^{-1}_{A+t,B}(C) = Sylvester[\{A + t, B, C\}]$ by matrix operations and polynomial calculations. Then we observe the asymptotic behavior of $Sylvester[\{A+t, B, C\}] \ \ (t \to 0)$ or the principal part of Laurent expansion of $Sylvester[\{A + t, B, C\}]$ with respect to the center 0

$$Sylvester[\{A + t, B, C\}]//Series[\#, \{t, 0, 0\}]\&.$$

In general degenerate Sylvester Equations may be inconsistent or indeterminate. Inconsistency and indeterminacy of the equations are closely related to the singularity of $Sylvester[\{A + t, B, C\}]$ at $t = 0$.

The following theorem is a corollary of the general theory of linear equations, which describes a beautiful connection between solvability of the equation and removability of singularity in the case of at most simple pole singularity. The relation between solvability of the equation and removability of perturbation singularity is much more complicated in the higher singularity case.

Theorem 4.1 *Let A be a square $m \times m$ matrix, B a square $n \times n$ matrix. Let $X(t, C)$ be a unique solution of the perturbed Sylvester Equation $(A + t)X - XB = C$, for an arbitrary $m \times n$ matrix C and a scalar $t \in k$ (at most finite number of exception of values). We assume that there exist a non-zero linear functional Ω on the space of $m \times n$ matrices $Mat_{m\times n}(k)$ and a non-zero matrix $Y \in Mat_{m\times n}(k)$ such that the difference*

$$Z(t, C) = X(t, C) - \frac{\Omega(C)}{t} Y$$

is a rational function of t whose denominator does not vanish at $t = 0$. Then we have the following;

1. *The Sylvester Equation $AX - XB = C$ has a solution if and only if $\Omega(C) = 0$.*

2. *For each matrix C satisfying $\Omega(C) = 0$, the solution space of the Sylvester Equation $AX - XB = C$ is equal to the line $\{\tau Y + Z(0, C) \mid \tau \in k\}$.*

5 A numerical example of perturbation and asymptotics

In this section we give a simple and nontrivial numerical example of degenerate Sylvester Equation.

Let $A_t = \begin{pmatrix} 1+t & 2 \\ 2 & 4+t \end{pmatrix}$ with an indeterminate t, $B = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$, and $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

We consider the following perturbed Sylvester Equation

$$A_t X - XB = C,$$

where X is an unknown 2×2 matrix and t is the indeterminate.

Then the characteristic polynomials $P_{A_t}(\lambda)$ of A_t and $P_B(\lambda)$ of B are the following;

$$P_{A_t}(\lambda) = 5t + t^2 - 5\lambda - 2t\lambda + \lambda^2$$

$$P_B(\lambda) = -11\lambda + \lambda^2$$

Generalized PolynomialExtendedGCD gives us two polynomials $Q_A(\lambda)$ and $Q_B(\lambda)$ satisfying $P_{A_t}(\lambda)Q_A(\lambda) + P_B(\lambda)Q_B(\lambda) = 1$.

$$Q_A(\lambda) = \frac{66 - 17t + t^2 - 6\lambda + 2t\lambda}{t(330 - 19t - 12t^2 + t^3)}$$

$$Q_B(\lambda) = \frac{3t^2 + 6(-5 + \lambda) - t(7 + 2\lambda)}{t(330 - 19t - 12t^2 + t^3)}$$

The solution $X_t(C)$ of $A_tX - XB = C$ has the following asymptotic expansion t tends to 0;

$$X_t(C) = \frac{\Omega}{t}Y + Z(t, C),$$

where $\Omega(C) = \frac{1}{55}(8p - 6q - 4r + 3s)$, $Y = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$, and

$$Z(t, C) = \begin{pmatrix} \frac{p-482q+302r-364s}{6050} + O(t) & \frac{-482p-601q-364r-2s}{9075} + O(t) \\ \frac{151p-182q+227r-514s}{3025} + O(t) & -\frac{2(182p+q+514r+302s)}{9075} + O(t) \end{pmatrix}.$$

Applying Theorem 4.1 to the above, we have the solution of the degenerate Sylvester Equation

$$A_0X - XB = C.$$

6 Concluding remark

In the present paper we consider the polynomial solution of a non-degenerate Sylvester Equation $AX - XB = C$ and a perturbation approach to a degenerate Sylvester Equation. The solution of a perturbed Sylvester Equation is explicitly described by virtue of a generalization of a *Mathematica* package PolynomialExtendedGCD which permits indeterminates in coefficients of polynomials.

We can read solvability of the degenerate Sylvester Equation in the asymptotic behavior of such a computing system solution of the associated perturbed equation.

In the case of simple pole singularity the relation between solvability and removability of singularity is revealed. If the asymptotic expansion contains a higher singularity, the solvability condition and the solution space of the degenerate equation may be determined the principal part of singularity.

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