Graphics Calculator Use in the Public Examination of Calculus: Experience of the First Year

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Abstract

Graphics calculators were allowed in all tertiary entrance mathematics examinations in Western Australia for the first time in 1998. In this paper we present an analysis of graphics calculator use in the Calculus examination. In particular, we discuss misconceptions attributable to the technology and misuses of it and we give an example of how the multirepresentation capabilities on it were used effectively for checking. These aspects of calculator utilisation will be illustrated via four of the questions on the examination paper.

1 Introduction

The use of graphics calculators in the public examination of mathematics subjects is gaining acceptance in various states of Australia. In Western Australia graphics calculators were expected to be used by all students for the first time in the Tertiary Entrance Examinations (TEE) of 1998. In this paper we consider effective and problematic aspects of their use and associated misconceptions as evidenced in the examination scripts of the first Calculus TEE with graphics calculators.

2 Background

The incorporation of graphics calculators into the West Australian secondary school mathematics curricula was formalized in 1995 through the announcement by the Secondary Education Authority that the calculators would be required in all tertiary entrance examinations (TEE) from 1998 onwards. Key considerations leading to this decision included the adoption of graphics calculators by teachers in some schools for the teaching mathematics from the early 1990s on, equity issues regarding access to the technology and the advent of scientific calculators with preprogrammed advanced functions [1].The first list of graphics calculators that would be allowed appeared in 1996. The Western Australian government assisted in the introduction by making available funds to schools for purchasing class sets of these calculators. The policy decision was followed by a period of extensive professional development initiatives to facilitate teachers' adoption of the new technology and a gradual incorporation of the calculators into school examinations.

In Western Australia, Calculus is a Year 12 secondary-school subject that is studied in the second year of a two-year tertiary entrance programme. it covers the following topics: calculus of trigonometric functions, functions and limits, theory and techniques of the calculus, applications, vector calculus and complex numbers. Most students who were candidates in the 1998 Calculus TEE would have either owned a graphics calculator, or had long-term access to one through school borrowing schemes for the two years leading up to the examination. Graphics calculators without symbolic processing and the Hewlett Packard HP38G with limited symbolic processing were approved [2]. In contrast, calculators with non-symbolic capabilities only were allowed for the Victorian Certificate of Education for 1998 [3] and the policy for the similar standard US Advanced Placement (AP) Calculus examination [4] was to state the minimum level of capabilities assumed in setting questions, but to allow full symbolic capabilities.

A non-prescriptive approach was taken for the three hour 1998 Calculus TEE with regard to the working required for calculator-assisted answers. No explicit instructions were given to use a graphics calculator in accordance with the syllabus goal to "select and use appropriate technology". However, the examination paper contained prompts such as "show analytically" to ensure the use of traditional methods for two part questions and one question could not be solved without using the technology. There was no requirement to clear memories of the calculators before the examination and so stored programs and text could be used. Two approved graphics calculators and any number of scientific calculators were allowed and, because the text storage capacities of the various brands of graphics calculators differ, students were permitted four A4 pages (two sheets) of notes.

3 Data Collection

Before the examination we selected six of the nineteen questions in the examination paper for detailed analysis. The topics covered in the questions included complex numbers, graphing, determination of limits, integration and soluitons of equations. They satisfied the criteria 'Graphics calculators are expected to be used . . . Graphics calculators are expected to be used by some students but not by others' [5]. For these questions examination markers were asked by the Curriculum Council to record the part marks awarded for answers and to circle the methods students used from a list of choices. Systematic sampling of the scripts of the first two candidates listed on each normal marks recording sheet resulted in a sample of 404 (21%) out of the total 1882 scripts.

Qualitative data were obtained from four sources. Firstly, 172 (9%) of the 1882 examination scripts were perused to ascertain the details of students' working. These were all papers in six bundles of scripts assigned randomly to us for marking. Students were asked, and most complied with the request, to write the brand of their calculator on their script. All commonly used brands of calculators allowed for the examination were represented in this set of scripts. Secondly, three female and three male examination candidates of differing abilities were interviewed in the week following the examination with a view to ascertaining students' calculator use not apparent in written answers. Third, two teachers of Calculus in 1998 were interviewed after their having worked the examination paper. Fourth, two experienced examination markers were asked about their perceptions of the form and adequacy of students' solutions.

4 The Questions

We focus our discussion on three questions of the six questions selected for analysis and one question that fell into the category "Graphics calculators are not expected to be used. [5]" . For each of the first three questions we outline the possible solution strategies and give an overview over the adoption of these methods by the students in the sample. Note that even though the sample size was 404, the total number of students for whom a method was recorded is usually less than 404. This is attributable to either students not

Table 1: Solution strategy adopted for Question 1

	Algebraic	**Direct**
No students choosing the method	227	138
No students with recorded mark	205	123
No students with full marks	105	66
Mean mark (total: 4)	2.7	2.8

having followed the specified approach or markers having been unable to determine which approach was used.

Question 1 Let $f(x) = \frac{1}{1+e^{-x}}$ for $-\infty < x < \infty$.

(a) Determine the range of f.

(b) Show analytically that f is increasing.

(c) Sketch the graph of the inverse of f, clearly indicating all intercepts and asymptotes.

The first two parts of this question were intended to set the scene for obtaining the graph of the inverse in the last part and will not be discussed in what follows. There are three methods for obtaining the graph of the inverse. In the first method, the Algebraic method (see Table 1), an algebraic expression for the inverse function is obtained first and then graphed. The second method, which we call the Direct method (see Table 1), is to graph f and then transcribe the reflection of the graph of f in the line $y = x$. As a third approach the definition of the graph of the inverse function as the set of ordered pairs $\{(f(x), x) \mid -\infty < x < \infty\}$ is used to obtain the graph of the inverse function. Of these possible approaches, only the first two can be easily distinguished in the scripts as the third is but a variation of the second. The first, more traditional method was the one adopted most frequently by the students in our sample (see Table 1)

Students most likely used their graphics calculators for graphing in both methods. Perusal of scripts showed that some students who chose the traditional method of calculating the inverse function failed to recognized the asymptotes at $x = 1$ and $x = 0$. This misconception is an artifact of the screen resolution and results from an inappropriate choice of scale (see Figure 1).

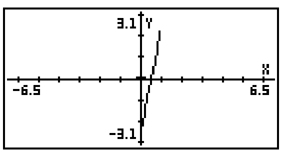


Figure 1: Graph of f^{-1}

Another problem exhibited in students' written answers that is attributable to the calculator display (see Figure 1) was not showing the curvature of the graph, or showing it incorrectly. This was usually caused by choosing too small a vertical viewing interval and can easily be remedied by zooming out vertically. However, correctly deducing the asymptotic behavior of the inverse function requires a thorough understanding of the relationship between the graph of the function and the graph of the inverse. Here, repeated zooming in near the asymptotes can allow insight into the asymptotic behavior. The mean mark of 2.7 out of a possible 4 (see Table 1) is explained by these errors and by errors in performing the necessary algebra. However, 105 (51%) out of 205 students in the sample (see Table 1) who chose to calculate the inverse function scored full marks, an indication of competent calculator usage. Errors with the direct method included traditional problems of students misplacing the asymptotes and inverting the curvature on parts of the reflected curve. Alternatives adopted by the 39 (10%) out of 404 students (see Table 1) not recorded as using the methods described above include graphing the reciprocal of f rather than its inverse.

Question 2 Determine the following limits showing your reasoning.

(a) $\lim_{x \to \infty} \frac{e^x+4}{2e^x}$

(b) $\lim_{x \to -1} \frac{x^3+x^2+5}{x^3+3}$

(c) $\lim_{t \to 0} \frac{\tan^2(3t)}{t}$

For all three parts of this question, graphical, tabular or symbolic methods would have led to acceptable solutions. (see [9]) In all three cases, less than 10% of all students in the sample opted for a graphical approach. More than 50% opted for a symbolic approach in (a) and (c) while the percentages of students using a tabular or symbolic approach for part (b) were almost equal (see [8]). Anecdotal evidence suggests that the words "showing your reasoning" prompted students to use a symbolic approach rather than the other two methods. The mean marks achieved for all limits for graphical and tabular approaches, and the proportions of students having chosen one of these approaches and receiving full marks, were lower than for traditional symbolic approaches for the limits. We will limit our discussion to Question 2(a). Table 2 summarizes the results for question 2(a).

For those students who used graphics calculators the most common problems lay in the justification of their answer. In the case when a graph was provided

Table 2: Solution strategy adopted for Question 2(a)

	Graphical	**Tabular**	**Symbolic**
No students choosing the method	26	67	290
No students with recorded mark	24	60	265
No students with full marks	9	10	166
Mean mark (total: 2)	1.2	0.9	1.4

in support of the answer the horizontal asymptote at $y = .5$ was frequently omitted and in the case of justification by means of a table, the number of values shown was frequently too low to establish any trend. Some students gave an answer of one for the limit in (a). This error is caused by the limitations to storing very large or very small numbers in graphics calculators resulting in the graphical and calculation effects illustrated in Figures 2 and 3 for the case of the Hewlett Packard HP38G.

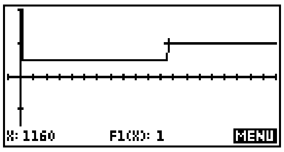
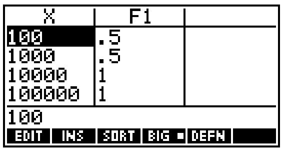


Figure 2: Graph of $\frac{e^x+4}{2e^x}$



X	F1
100	.5
1000	.5
10000	1
100000	1

Figure 3: Table of values

Question 3 A function is defined by $f(x) = 1 - \frac{x}{(x-1)^2}$. Sketch the graph of f, indicating all asymptotes and the co-ordinates of all turning points.

The majority of students in the sample appeared to use their graphics calculators by providing the graph without working. However, the mean mark of 3.7 and that only 66 (23%) out of 289 students providing the graph only scored full marks, indicates that students encountered difficulties in interpreting the calculator screen display.

Errors were the inclusion of a turning point at $x = 1$, possibly found by running the cursor along to the bottom of the graph on the calculator screen; failure to identify the turning point at $x = -1$; and for the left branch of the graph to drop below the horizontal asymptote. In fact, close to half the students whose scripts were sighted failed to include the horizontal asymptote.

Table 3: Solution strategy adopted for Question 3

	Graphical	**Symbolic**
No students choosing the method	321	63
No students with recorded mark	289	59
No students with full marks	66	18
Mean mark (total: 5)	3.7	3.6

However, comparing with a mean mark of 3.6 for students showing working, students who chose to provide the graph only did not score significantly differently than those who used symbolic working.

Question 4 Consider the function

$$f(x) = -3\,|x| + \sqrt{1 + 6\,|x| - 9x^2}$$

for $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

(a) Rewrite f as a piece-wise defined function.

(b) Justify that f is continuous at $x = 0$.

(c) Write the derivative of f as a piecewise function and justify whether or not f is differentiable at $x = 0$.

For some questions in the Calculus TEE the use of a graphics calculator had not been anticipated. Questions of this group were those primarily concerned with ascertaining students ' understanding of properties of functions. The first two parts of question 1 fall into this category as does the question 4.

While the question could not be answered by supplying a graph, graphing the function on the calculator could potentially assist students in determining limits, function values and derivative values for (b) and (c). The interviews conducted with students after the examination showed that the graphics calculator was used as a means to check answers. The approach of plotting the graphs of the functions appearing in the piecewise definition (see Figure 4) together with the graph of the original function is a viable method for checking the answer for part (a). However, the graphical display can be confusing if the domains of the pieces are not restricted. Domains can be restricted by setting the horizontal range of the viewing window appropriately and multiplying the function as shown in Figure 4.

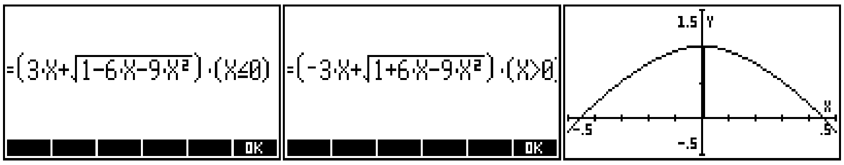
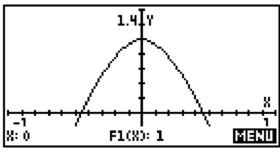


Figure 4: Graph of f

However, interviews with students showed that the graph of the function caused some students confusion as they could not reconcile the shape of the function with the fact that its definition involved the absolute value function. An expectation was that the graph should have only positive function values and cusps. One strategy indicated in students' scripts was to cope with the unexpected nature of the graph by approximating it with curves with which they were familiar, for example, $g(x) = 1 - 5x^2$. The coefficients of this parabola can be obtained by solving the system of linear equations

$$\begin{aligned} a - 0.5b + 0.25c &= f(-0.5) \\ a &= f(0) \\ a + 0.5b + 0.25c &= f(0.5) \end{aligned}$$

and subsequent rounding. Students who tried to use this approach had to "step sideways" and employ methods learnt in another subject. The close resemblance of the graphs of f and g (see Figures 5 and 6) is evident. In a problem-solving scenario this approach of replacing the given function by an approximation with more familiar functions may have served a starting point for understanding the function.



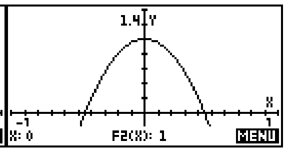


Figure 5: Graph of f Figure 6: Graph of g

Interviews [10] ascertained that students who did not use their calculators as the primary solution tool in some cases used them as a device for checking answers. This checking involved reading limiting function values and $f(0)$ from the graph to establish continuity, and reading limiting derivative values from the graph to justify differentiability. Such graphical checking of symbolic solutions is a non-trivial skill, as is the co-ordination of any different forms of representation.

5 Discussion

The four questions discussed in this paper highlight the main difficulties students seemed to encounter in using their graphics calculators in the examination. The most commonly encountered problems with the use of the technology relate to the transcription of graphs from the calculator screen to the examination script. They include the failure to recognize and consequent omission of asymptotes, the inclusion of non-existent turning points, and choice of inappropriate scale, the latter often leading to the depiction of the graph without showing its curvature adequately. Similar observations were reported in [6] and [7]. Another area of concern is the lack of knowledge about the nature of machine capacity with regard to storing and manipulating large numbers which led to a fallacious conclusion about its value in the case of one limit.

Overall, there appeared to be an under-utilization of graphics calculators in the examination. The calculators were used for those questions directly related to the graphing of functions, and the numeric calculation of integrals, but there was a surprisingly low uptake of calculator based approaches to determining limits and solving algebraic equations. Even though syllabus statements relating to the treatment of limits clearly state that this should be informal, students opted for symbolic methods to determine limits not making use of the graphical or tabulation capabilities of their calculators to help establish limits.

6 Conclusions

The results reported above suggest areas of difficulty experienced by students that warrant being the subject of instruction. These are mainly concerned with intepretation of the screen display of graphs. However, the results indicate that choosing to use graphics calculators was not associated with higher (or lower) marks than those obtained via more traditional methods. The apparent reluctance to use the technology in areas where there would be an advantage to do so, for example in evaluating limits, indicates that there is uncertainty as to when to use the technology. This outcome may cease to be a problem as familiarity with the calculators increases.

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