Forward Displacement Analysis of a Special Stewart-Gough Platform

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Abstract

In this report we study the forward displacement analysis of a special Stewart-Gough platform. The polynomial system describing the problem is solved by matrix eigenproblem approach using exact and numeric computation. The problem on the perturbation of the coefficients of the system is studied. A special procedure for computing the Gröbner basis of such problem is presented to save time. Numerical examples are given.

1 Introduction

The number of solutions of the multivariate polynomial system describing the forward displacement analysis of Stewart-Gough platform is at most 40 [1,2]. But how to compute these solutions and their numerical stability are still interesting problems.

In this report we present an approach for forward displacement analysis of a special Stewart-Gough platform related to a hexapod-based machine VAMT1Y newly designed in China [5].

The resulted polynomial system describing the problem is composed of 6 quadratic polynomials in 6 variables. We propose to solve this system by matrix eigenproblem approach[3], i.e. to form a 40×40 matrix from the Gröbner basis of the system and to read the solutions from its eigenvectors.

A special procedure for computing the Gröbner basis of the system in total degree ordering is designed for given geometric size of the platforms but leaving the lengths of legs as parameters. The Gröbner basis is composed of 36 polynomials. 6 of them are linear combinations of the polynomials of the original system, and others are obtained from 30 specific precisely selected S-polynomials. Once it is computed, the Gröbner basis for any given set of lengths of legs can be obtained within 20 sec. even on micro computer PII.

In general, the number of solutions of the multivariate polynomial system is sensitive to the perturbation of coefficients of the system. For the system concerned it may be changed from 40 to 64 if the coefficients of terms with high degree are perturbed and the perturbed system may not still describe the problem studied. But perturbations often appear in many cases, it causes troubles to the calculation. In section 4 a simple way is proposed to compensate for the perturbation and the effects are illustrated by examples in section 6.

The computation of Gröbner basis and matrix is carried exactly by Maple, and its eigenvectors by MATLAB using floating arithmetic double precision.

The main advantage of the proposed approach is that the computational errors arise only in the last step, i.e. eigenvector computation, and uniformly in some sense with respect to all 6 variables. This would be superior than computing the values of variables from single variable polynomials one by one successively for each solution.

Three testing examples are given for illustration and comparison.

The structure of the report is as follows: In section 2 we establish polynomial system. In section 3 we briefly describe the main steps for solving the polynomial system. In section 4 we deal with the perturbation. In section 5 A simple procedure for computing Gröbner basis is presented. Three numerical examples and some conclusions are given in section 6 and section 7.

2 The polynomial system

2.1 Coordinate

The 6 joints B_j of the base platform lie on two parallel planes with distance z_1. B_1, B_3 and B_5 lie on a plane and form an equilateral triangle, similarly for B_2, B_4 and B_6 on the other plane. The line connecting the centers of these triangles is perpendicular to planes. The sides B_1B_3 and B_2B_4 are parallel etc. We may represent B_j in rectangular coordinates $OXYZ$ as follows (Fig 1):

$B_1\ (x_1,\ \ 0,\ \ z_1),\qquad B_2\ (x_2,\ \ 0,\ \ 0)$

$B_3\ (-\frac{1}{2}x_1,\ \frac{\sqrt{3}}{2}x_1,\ \ z_1),\ \ B_4\ (-\frac{1}{2}x_2,\ \ \frac{\sqrt{3}}{2}x_2,\ \ 0)$
$B_5\ (-\frac{1}{2}x_1,\ \ -\frac{\sqrt{3}}{2}x_1,\ \ z_1),\ \ B_6\ (-\frac{1}{2}x_2,\ \ -\frac{\sqrt{3}}{2}x_2,\ \ 0)$
where $x_1 > 0,\ x_2 > 0,\ z_1 > 0$.
The 6 joint points M_j on the moving platform are coplanar and lie on a circle centered at M. $M_1M_3M_5$ and $M_2M_4M_6$ are two equilateral triangles. Let MPQ be a plane rectangular coordinate on this plane and represent M_j by $M_1\ (p_1,\ \ q_1),\qquad M_2\ (p_1,\ \ -q_1)$
$M_3\ (-\frac{1}{2}p_1 - \frac{\sqrt{3}}{2}q_1,\ \ -\frac{1}{2}q_1 + \frac{\sqrt{3}}{2}p_1),\ M_4\ (-\frac{1}{2}p_1 + \frac{\sqrt{3}}{2}q_1),\ \ \frac{1}{2}q_1 + \frac{\sqrt{3}}{2}p_1)$
$M_5\ (-\frac{1}{2}p_1 + \frac{\sqrt{3}}{2}q_1,\ \ -\frac{1}{2}q_1 - \frac{\sqrt{3}}{2}p_1),\ \ M_6\ (-\frac{1}{2}p_1 - \frac{\sqrt{3}}{2}q_1,\ \ \frac{1}{2}q_1 - \frac{\sqrt{3}}{2}p_1)$
to describe the geometry, where $p_1 > 0,\ q_1 < 0$ (Fig 2):

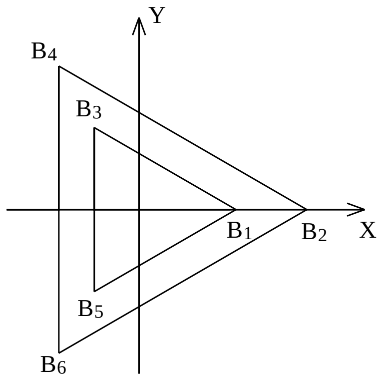

Fig. 1

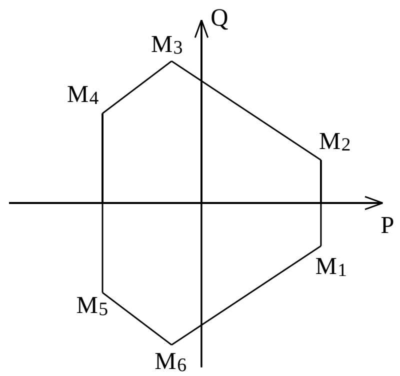

Fig. 2

Let $u_1,\ u_2,\ u_3$ be the direction cosines of $\overrightarrow{MP}$ in $OXYZ$, $v_1,\ v_2,\ v_3$ for $\overrightarrow{MQ}$ and the coordinates of M be (x, y, z). Then the coordinates of $M_j (j = 1, \cdots, 6)$ in $OXYZ$ are

$$(p_j\,u_1 + q_j\,v_1 + x,\ p_j\,u_2 + q_j\,v_2 + y,\ p_j\,u_3 + q_j\,v_3 + z).$$

2.2 The polynomial system

Let the lengths between B_j and M_j be l_j, we have 6 distance equations $f_1, \cdots, f_6$.
Since $\overrightarrow{MP}, \overrightarrow{MQ}$ are two orthogonal unit vectors, we have:
$f_7 := u_1^2 + u_2^2 + u_3^2 - 1$

$f_8 := v_1^2 + v_2^2 + v_3^2 - 1$
$f_9 := u_1\, v_1 + u_2\, v_2 + u_3\, v_3$
These 9 quadratic polynomials above with 9 unknowns $u_1, u_2, u_3, v_1, v_2, v_3, x, y, z$ form the polynomial system describing problem.

3 The system solving

The main steps for solving the system are the following.

3.1 The system simplification

• We introduce 3 new variables u, v, w in order to simplify computation as in [4]:
$f_{10} := u - u_1\, x - u_2\, y - u_3\, z$
$f_{11} := v - v_1\, x - v_2\, y - v_3\, z$
$f_{12} := w - x^2 - y^2 - z^2$
Now we have 12 quadratic polynomials.
• $f_1, \cdots, f_6$ can be simplified by $f_7, \cdots, f_{12}$, the polynomials resulted are still named $f_1, \cdots, f_6$, but they are linear in $u_1, u_2, u_3, v_1, v_2, v_3, u, v, w, x, y, z$.
• Solve $u_1, u_2, u_3, v_1, v_2, v_3$ in terms of u, v, w, x, y, z from $f_1, \cdots, f_6$
• Substitute $u_1, u_2, u_3, v_1, v_2, v_3$ in $f_7, \cdots, f_{12}$ by their linear expressions just obtained, then we get 6 quadratic polynomials $g_1, \cdots, g_6$.
The problem becomes to solve the quadratic polynomial system $g_1, \cdots, g_6$ with unknowns u, v, w, x, y, z.

3.2 Gröbner basis computation

The Gröbner basis of $g_1, \cdots, g_6$ may be computed by Maple function *gbasis* in total degree, inverse lexicographic ordering and $u \succ v \succ w \succ x \succ y \succ z$, but we prefer to do this by the proposed procedure in §5 to save time.

3.3 Matrix construction

Once we have the Gröbner basis, we may construct the matrix needed, with respect to any variable, e.g. z as in [3].

3.4 Eigenvectors computation

Compute eigenvectors of constructed matrix by numeric software MATLAB and read the solutions u, v, w, x, y, z from them.

3.5 The coordinates of joint points of moving platform

Now it is easy to compute the coordinates of M_j for real solutions reading from eigenvectors.

4 On the perturbation

When the values of parameters x_1, x_2, p_1, q_1, z_1 are rational, the coefficients of the high degree terms in $g_1, \cdots, g_6$ computed above are rational, but the coefficients of lower degree terms contain $\sqrt{3}$. If we compute Gröbner basis of $g_1, \cdots, g_6$ by Maple function *gbasis*, the coefficients of $g_1, \cdots, g_6$ must be rational, so we must first replace $\sqrt{3}$ by its approximate rational value. Let $\tilde{g}_1, \cdots, \tilde{g}_6$ be the polynomials resulted. They may be not a system describing problem even for another set of lengths of legs.

But leaving the values of x_1, x_2, p_1, q_1, z_1 unchanged, we may compute another set of $\hat{l}_j$'s such that $\sqrt{3}$ does not appear in the corresponding polynomials $\hat{g}_j$ and $\hat{g}_4 \equiv \tilde{g}_4, \;\; \hat{g}_5 \equiv \tilde{g}_5$. It just needs to solve a linear system to determine $\hat{l}_j$'s. In addition, it is easy to see that $\alpha_j := \hat{g}_j - \tilde{g}_j$ $(j = 1, 2, 3)$ are constants. And when $\tilde{g}_i \to g_i$, then $\alpha_i \to 0, \hat{l}_j \to l_j$.

Since $\hat{g}_1, \cdots, \hat{g}_6$ form a polynomial system describing the problem, its solutions can be considered as good approximation of those solutions required, and modified by any numerical method if necessary.

5 The process for computing Gröbner basis

5.1 Preparation

For given values of parameters x_1, x_2, z_1, p_1, q_1 of platforms, and leaving l_j as parameters, let a_1, a_2, a_3 denote the coefficients of x, y, z in g_4 and a_4, a_5, a_6 denote the coefficients of x, y, z in g_5 respectively. Since a_j's are linear combination of l_j^2, we may express l_j^2 linearly in terms of a_j's. Substituting l_j's by their expressions in a_j's in $g_1, \cdots, g_6$, we get $f_1, \cdots, f_6$. The Gröbner basis of $f_1, \cdots, f_6$ is composed of 36 polynomials $gb_1, \cdots, gb_{36}$.

5.2 $gb_1, \cdots, gb_6$

Let $gb_1 := f_6$. gb_2 and gb_3 are linear combination of f_4, f_5 in order that the leading power products of them are $v\,x$ and $u\,x$ respectively. gb_4, gb_5 and gb_6 are linear combination of f_1, f_2 and f_3 in order that the leading power products of them are v^2, $u\,v$, and u^2 respectively. Reduce gb_i by gb_j for $j < i$ and $i \le 6$.

5.3 $gb_7, gb_{10}, \cdots, gb_{14}, gb_{16}, gb_{17}$

Compute 8 S-polynomials of gb_i and gb_j, where (i,j) are (1,2), (1,3), (2,3), (2,4), (3,5), (3,6), (5,6), (4,5). Then compute the linear combination of them in order that the leading power products are $w\,y^2, w\,x\,y, w^2\,y, v\,w\,y, u\,w\,y, w^2\,x, w^2\,v, w^2\,u$ and name them $gb_7, gb_{10}, \cdots, gb_{14}, gb_{16}, gb_{17}$ respectively. Reduce gb_i by gb_j for $j < i$ and $i \le 17$.

5.4 $gb_8, gb_9, gb_{15}, gb_{29}, \cdots, gb_{35}$

Compute 10 S-polynomials of gb_i and gb_j, where (i,j) are (3,10), (3,13), (3,14), (4,12), (5,12), (5,13), (5,16), (5,17), (6,13), (6,17). Then compute the linear combination of them in order that the leading power products are $v\,y^2, u\,y^2, w^3, w^2\,z^2, v\,w\,z^2, u\,w\,z^2, y^3\,z, x\,y^2\,z, y^4, x\,y^3$ and name them $gb_8, gb_9, gb_{15}, gb_{29}, \cdots, gb_{35}$ respectively. Reduce gb_i by gb_j for $j < i$ and $i \leq 35$.

5.5 $gb_{23}, \cdots, gb_{28}$

Compute 6 S-polynomials of gb_i and gb_j, where (i,j) are (2,8), (3,9), (7,8), (7,9), (5,8), (4,8). Then compute the linear combination of them in order that the leading power products are $y^2\,z^2, x\,y\,z^2, w\,y\,z^2, v\,y\,z^2, u\,y\,z^2, w\,x\,z^2$ and name them $gb_{23}, \cdots, gb_{28}$ respectively. Reduce gb_i by gb_j for $j < i$ and $i \leq 35$.

5.6 $gb_{18}, \cdots, gb_{22}$

Compute 5 S-polynomials of gb_i and gb_j, where (i,j) are (7,23), (7,25), (8,23), (8,26),(9,27). Then compute the linear combination of them in order that the leading power products are $y\,z^3,\ x\,z^3,\ w\,z^3,\ v\,z^3, u\,z^3$ and name them $gb_{18}, \cdots, gb_{22}$ respectively. Reduce gb_i by gb_j for $j < i$ and $i \leq 35$.

5.7 gb_{36}

Compute S-polynomial of gb_1 and gb_{19}. Then it is reduced by gb_i's in order that the leading power product is z^5 and name it gb_{36}.

5.8 Check these gb_i being Gröbner basis of $f_1, \cdots, f_6$

We remark that 30 S-polynomials computed yields 30 Gröbner basis polynomials, and there is no S-polynomial which would be reduce to zero in the process of reduction. Furthermore, the leading coefficients of all polynomials appearing in the process of computation are constants, independent on a_j's i.e. l_j's.

6 Examples

We give some results of 3 examples in the following. The values of parameters of the platforms are the same for all examples:

$$x_1 = \frac{357}{200},\quad x_2 = \frac{189}{125},\quad z_1 = \frac{267}{1000},\quad p_1 = \frac{37}{200},\quad q_1 = \frac{-19}{250}$$

but the sets of lengths l_j of legs are slightly different. All matrices in examples are constructed with respect to z. And every matrix has 2 real eigenvectors and 38 complex ones.

Ex. 1 :

The lengths of l_j's are :

$$l_1 = l_3 = l_5 = \frac{3}{1000}\sqrt{618785}\ (\approx 2.35989)$$

$$l_2 = l_4 = l_6 = \frac{3}{1000}\sqrt{640745}\ (\approx 2.40140)$$

All coefficients of g_j's are rational.

Ex. 2 :

Modify l_j's in Ex.1 to:

$$l_1 = \frac{47}{20},\quad l_3 = \frac{237}{100},\quad l_5 = \frac{119}{50},\quad l_2 = \frac{241}{100},\quad l_4 = \frac{239}{100},\quad l_6 = \frac{12}{5}$$

$\sqrt{3}$ appears in g_j, and it is approximated by $\frac{173}{100}$

Ex. 3 :

The l_j's are determined by method in §4 from those of Ex. 2.

$$l_1 = \frac{47}{20} = 2.35,\quad l_2 = \frac{241}{100} = 2.41(\mathit{unchanged})$$

$$l_3 = \sqrt{\frac{112813}{20000} - \frac{3287}{240000}\sqrt{3}}\ (\approx 2.37001)$$

$$l_5 = \sqrt{\frac{112813}{20000} + \frac{3287}{240000}\sqrt{3}}\ (\approx 2.37999)$$

$$l_4 = \sqrt{\frac{114721}{20000} - \frac{82867}{6000000}\sqrt{3}}\ (\approx 2.39001)$$

$$l_6 = \sqrt{\frac{114721}{20000} + \frac{82867}{6000000}\sqrt{3}}\ (\approx 2.39999)$$

$$\alpha_1 \approx -.8492e-4,\ \alpha_2 \approx -.3794e-4,\ \alpha_3 \approx .5137e-4$$

The result of 3 examples are given in table 1 and 2, where L_j is the distance between the computed M_j and given B_j, and $d_j = L_j - l_j$.

The results obtained from one real eigenvector:(table 1)

	Ex. 1	Ex. 2	Ex. 3
u	-.00000002113295	.1384593466955643	.1384317581249299
v	.00000001600577	.3735338626155642	.3734994203428016
w	4.00000002698952	4.004893647362623	4.0049132100452
x	-.00000000509285	.01925939437717648	.0192545199547591
y	.00000000232906	.04646279876284222	.04645454613271308
z	2.00000000753983	2.000590915314096	2.00059601471922
u_1	1.00000005813645	.9947550226737925	.9947739151343103
u_2	-.00000001330652	-.08150225172587074	-.081489232238558
u_3	-.00000001579784	.0613073130515653	.061513274485924
v_1	.0000000194937	.07048695643770426	.070523030743525
v_2	1.00000002487649	.9840330397009512	.9840746127318993
v_3	.00000001611049	.1633183358570836	.1631647786029377
$\sum u_i^2$	1.000000116	.9999387588047272	.9999995201404542
$\sum v_i^2$	1.000000048	.9999623130780718	.9999990862652355
$\sum u_i\, v_i$	.6000000565e-8	-.7104621543464e-4	-.21344446755e-6
L_1	2.359886649587556	2.349956568499387	2.349999973377820
L_2	2.401396469683635	2.409974340900660	2.409999972311437
L_3	2.359886652769989	2.370028004847917	2.370005921429820
L_4	2.401396472084417	2.390043300490847	2.390005935375818
L_5	2.359886651519058	2.380013799038510	2.379994090386574
L_6	2.401396468523582	2.399981458092913	2.399994078136531
d_1	-.232727e-9	-.43431500613e-4	-.26622180e-7
d_2	.959655e-9	-.25659099340e-4	-.27688563e-7
d_3	.2949706e-8	.28004847917e-4	-.11224078e-7
d_4	.3360437e-8	.43300490847e-4	.2826375e-8
d_5	.1698775e-8	.13799038510e-4	-.1871919e-8
d_6	-.200398e-9	-.18541907087e-4	-.14018378e-7

The results obtained from other real eigenvector:(table 2)

	Ex. 1	Ex. 2	Ex. 3
u	-.00000000437184	.1974148137228609	.1974300502673288
v	.00000002473035	.4056654265456342	.405719849361412
w	2.87274877695932	2.865564782863905	2.86554980183621
x	-.00000000142812	.02756465660071426	.02756585119552948
y	.00000000520151	.05313558950324184	.05314723172182714
z	1.69491851893156	1.691739265577902	1.691734381904142
u_1	-.7375428871	-.7696201738654509	-.7696586780356295
u_2	.675300363	.6289047770341943	.628997706081115
u_3	-.600652254e-8	.1092611609629994	.1094834832251653
v_1	-.675300337	-.5851643114599547	-.585090963680784
v_2	-.7375428985	-.7635050109826924	-.7635120198435977
v_3	.493347215e-8	.493347215e-8	.2733447919800437
$\sum u_i^2$	1.000000091	.999774431892099	.9999992280299726
$\sum v_i^2$	1.000000072	1.00012972048407	.9999994155291722
$\sum u_i v_i$	-.2690000003e-7	.5930435890991e-4	-.23146240359e-6
L_1	2.359886649587556	2.349961452701369	2.349999991982228
L_2	2.401396469683635	2.409977878703793	2.409999980188563
L_3	2.359886652769988	2.370024705418375	2.370005898050535
L_4	2.401396472084417	2.390035297403388	2.39000589711429
L_5	2.359886651519058	2.380012352598532	2.37999406804247
L_6	2.401396468523582	2.399985965234307	2.399994081300655
d_1	-.232727e-9	-.38547298631e-4	-.8017772e-8
d_2	.959655e-9	-.22121296207e-4	-.19811437e-7
d_3	.2949705e-8	.24705418375e-4	-.34603363e-7
d_4	.3360437e-8	.35297403388e-4	-.35435153e-7
d_5	.1698775e-8	.12352598532e-4	-.24216023e-7
d_6	-.200398e-9	-.14034765693e-4	-.10854254e-7

7 Conclusions

The numerical results of examples suggest that the matrix eigenproblem approach for polynomial system solving cooperated with suitable Gröbner basis computation works well for forward displacement analysis of Stewart-Gough platform.

For example 1, there is an exact solution:

$$x = 0, \quad y = 0, \quad z = 2, \quad u = 0, \quad v = 0, \quad w = 4$$

$$u_1 = 1, \quad u_2 = 0, \quad u_3 = 0, \quad v_1 = 0, \quad v_2 = 1, \quad v_3 = 0$$

The computed solution in column 1 of table 1 is very close to it.

In example 1 and 3 the d_j's are small in comparison with those in example 2. One of the reasons is that the perturbed system is no longer a system describing the problem associated to a set of leg lengths. It looks a good strategy to use $\hat{g}_j$'s instead of $\tilde{g}_j$'s.

References

[1] M. L. Husty, An Algorithm For Solving the Direct Kinematics of General Stewart-Gough Platforms, Mech.Mach.Theory, Vol.31, No.4, pp365-380,1996

[2] C.W.Wampler, Forward Displacement Analysis of General Six-in -parallel SPS(Stewart) Platform Manipulators Uing Soma Coordinates, Mech.Mach.Theory, vol 31, No.3, pp331-337,1996

[3] Hans J. STETTER, Multivariate Polynomial Equations as Matrix Eigenproblems, WSSIAA 2(1993) pp.355-371

[4] W.D.Wu and Y.Z.Huang,The direct kinematic solution of the planar Stewart platform with coplanar ground points, Journal of Computational Mathematics, Vol.14, No.3, 263-272,1996

[5] T.Huang, J.Wang, C.Gosselin and D.J.Whitehouse, Design of Hexapod-based Machine Tools with Specified Orientation Capability and Well-conditioned Dexterity, in Proc. of the 10th World Congress for Theory of Machines and Mechanisms, Oulu, Finland, June 20-24, 1999.