Outer Product Factorization in Clifford Algebra

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Abstract

Clifford algebra plays an important role in mathematics and physics, and has various applications in geometric reasoning, computer vision and robotics. When applying Clifford algebra to geometric problems, an important technique is parametric representation of geometric entities, such as planes and spheres in Euclidean and spherical spaces, which occur in the form of homogeneous multivectors. Computing a parametric representation of a geometric entity is equivalent to factoring a homogeneous multivector into an outer product of vectors. Such a factorization is called outer product factorization.

When no signature constraint is imposed on the vectors whose outer product equals the homogeneous multivector, a classical result can be found in the book of Hodge and Pedoe (1953), where a sufficient and necessary condition for the factorability is given and called quadratic Plücker relations (p-relations). However, the p-relations are generally algebraically dependent and contain redundancy. In this paper we construct a Ritt-Wu basis of the p-relations, which serves as a much simplified criterion on the factorability. When there are signature constraints on the vectors, we propose an algorithm that can judge whether the constraints are satisfiable, and if so, produce a required factorization.

1 Introduction

Clifford algebra is an important tool in modern mathematics and physics. Because of its invariant representation for geometric computation, Clifford algebra is gaining wider and wider recognition in fields like geometric reasoning, computer vision and robotics. This short paper contributes to solving an often encountered problem in applying Clifford algebra, the problem of outer product factorization.

Let $\mathcal{K}$ be a field whose characteristic $\neq 2$, and let $\mathcal{V}^n$ be a $\mathcal{K}$-vector space of dimension n. The Grassmann algebra $\Lambda(\mathcal{V}^n)$ generated by $\mathcal{V}^n$ is a graded $\mathcal{K}$-vector space, whose grades range from 0 to n. The multiplication is denoted by "$\wedge$", called the outer product. An element in the Grassmann algebra is called a multivector, and an r-graded element is called an r-vector. Any r-vector is called a homogeneous multivector.

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of $\mathcal{V}^n$. It generates a basis $\{e_{i_1} \wedge \cdots \wedge e_{i_r} \mid 1 \leq i_1 < \ldots < i_r \leq n\}$ for the space $\Lambda^r(\mathcal{V}^n)$ of r-vectors. Let A_r be an r-vector, then

$$A_r = \sum_{1 \leq i_1 < \ldots < i_r \leq n} a_{i_1 \ldots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r}. \tag{1}$$

The list $(a_{i_1 \ldots i_r} \mid 1 \leq i_1 < \ldots < i_r \leq n)$ is called the Plücker coordinates of A_r. In this paper, we allow any order among the suffixes by requiring that $a_{i_1 \ldots i_r}$ be anti-symmetric with respect to its suffixes.

The outer product of r vectors is called an r-extensor. For an r-extensor $A_r = a_1 \wedge \cdots \wedge a_r$, where the a's are vectors, a vector $x \in \mathcal{V}^n$ is in the subspace $\mathbf{A_r}$ spanned by the a's if and only if $x \wedge A_r = 0$. Therefore we can use A_r to represent the space $\mathbf{A_r}$. Since an extensor represents a subspace, an important question is how to judge whether or not a homogeneous multivector is an extensor, and how to factor an extensor. The factorization will be unique up to a special linear transformation in the subspace.

In the book of Hodge and Pedoe (1953), there are two theorems that answer the above question:

Theorem 1.1. An i-vector A_r is an extensor if and only if its Plücker coordinates $a = (a_{l_1 \ldots l_r} \mid 1 \leq l_1, \ldots, l_r \leq n)$ satisfy the following quadratic Plücker relations (p-relations): for any $1 \leq i_1 < \ldots < i_{r-1} \leq n$ and any $1 \leq j_1 < \ldots < j_{r+1} \leq n$, the following equality holds:

$$F_{i_1 \ldots i_{r-1}, j_1 \ldots j_{r+1}}(a): \sum_{\lambda=1}^{r+1} (-1)^\lambda a_{i_1 \ldots i_{r-1} j_\lambda} a_{j_1 \ldots j_{\lambda-1} j_{\lambda+1} \ldots j_{r+1}} = 0. \tag{2}$$

Theorem 1.2. Let A_r be an r-extensor with Plücker coordinates $(a_{i_1 \ldots i_r} \mid 1 \leq i_1, \ldots, i_r \leq n)$, where $a_{l_1 \ldots l_r} \neq 0$. Let $b_j = (b_{l_j 1}, \ldots, b_{l_j n})^T$, where $b_{l_j k} = a_{l_1 \ldots l_{j-1} k l_{j+1} \ldots l_r}$. Then

$$A_r = (a_{l_1 \ldots l_r})^{1-r} b_1 \wedge \cdots \wedge b_r. \tag{3}$$

In the book of Iversen (1992), where the Grassmann algebra is replaced by a nondegenerate Clifford algebra, Theorem 1.1 is reformulated elegantly as follows, where the dot denotes the inner product in the Clifford algebra:

Theorem 1.3. An r-vector A_r is an extensor if and only if for any $(r-1)$-vector X_{r-1},

$$A_r \wedge (A_r \cdot X_{r-1}) = 0. \tag{4}$$

The set of p-relations (2) forms a Gröbner basis for a convenient monomial order. Furthermore, it has a structure of Hodge algebra (see DeConcini, Eisenbud and Procesi, 1982). However, the p-relations are generally algebraically dependent, and contain redundancy when used as a criterion on the factorability of a homogeneous multivector.

In this paper, we construct a Ritt-Wu basis (Wu, 1978), i. e., an irreducible characteristic set, of (2). The number of relations in the basis is much smaller than that in (2). The basis not only simplifies the criterion on the factorability, but also yields an efficient way to reduce a polynomial of Plücker coordinates by the Plücker relations.

When $\mathcal{V}^n$ is equipped with an inner product, the Clifford algebra $\mathcal{C}(\mathcal{V}^n)$ generated by $\mathcal{V}^n$ (*cf.* Crumeyrolle, 1990) is linearly isomorphic to $\Lambda(\mathcal{V}^n)$. When $\mathcal{K} = \mathcal{R}$, a nonzero vector x is said to be positive, or negative, or null, if $x \cdot x > 0$, or < 0, or $= 0$, respectively. The sign of $x \cdot x$ is called the signature of x. Let $A_r = a_1 \wedge \cdots \wedge a_r$, where the a's are mutually orthogonal vectors and p of which are positive, q of which are negative. The triplet $(p, q, r-p-q)$ is called the signature of A_r.

The problem of outer product factorization in Clifford algebra is, given an r-extensor, factor it into the outer product of p positive vectors, q negative ones and $r-p-q$ null ones, if the factorization is possible. An algorithm is proposed in this paper to solve the problem.

2 Factorization in Grassmann Algebra

Let A_r be a nonzero r-vector with Plücker coordinates $(a_{l_1 \ldots l_r} \mid 1 \leq l_1, \ldots, l_r \leq n)$. Assume that $A_r = a_1 \wedge \cdots \wedge a_r$, where $a_i = (a_{i1} \ldots a_{in})^T$ with respect to the basis $\{e_1, \ldots, e_n\}$ of $\mathcal{V}^n$. Then

$$a_{i_1 \ldots i_r} = \begin{vmatrix} a_{1i_1} & \cdots & a_{1i_r} \\ \vdots & \ddots & \vdots \\ a_{ri_1} & \cdots & a_{ri_r} \end{vmatrix}. \tag{5}$$

Since A_r is nonzero, at least one of its Plücker coordinates is nonzero. We assume that $a_{1\ldots r} \neq 0$, otherwise we simply change the suffixes of the basis $\{e_1, \ldots, e_n\}$ to achieve this. Denote $a_0 = a_{1\ldots r}$, $a_{\overset{\vee}{ij}} = a_{1\ldots(i-1)j(i+1)\ldots r}$. First we derive the classical p-relations.

The r-extensor A_r can be represented by the row vectors of the $r \times n$ matrix

$$A = \begin{pmatrix} a_1^T \\ \vdots \\ a_r^T \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rn} \end{pmatrix}. \tag{6}$$

Since $a_0 \neq 0$, the matrix

$$A_0 = \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix} \tag{7}$$

is invertible. Multiplying A from the left by A_0^{-1}, and using

$$A_0^{-1} \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ri} \end{pmatrix} = \frac{1}{a_0} \begin{pmatrix} a_{\overset{\vee}{1i}} \\ \vdots \\ a_{\overset{\vee}{ri}} \end{pmatrix}, \tag{8}$$

we get

$$A_0^{-1}A = \begin{pmatrix} b_1^T \\ \vdots \\ b_r^T \end{pmatrix} = \begin{pmatrix} 1 & & & \frac{a_{\overset{\vee}{1(r+1)}}}{a_0} & \frac{a_{\overset{\vee}{1(r+2)}}}{a_0} & \cdots & \frac{a_{\overset{\vee}{1n}}}{a_0} \\ & \ddots & & \vdots & \vdots & \ddots & \vdots \\ & & 1 & \frac{a_{\overset{\vee}{r(r+1)}}}{a_0} & \frac{a_{\overset{\vee}{r(r+2)}}}{a_0} & \cdots & \frac{a_{\overset{\vee}{rn}}}{a_0} \end{pmatrix}, \tag{9}$$

where the b's are vectors in $\mathcal{V}^n$.

Geometrically, the multiplication of A from the left by A_0^{-1} induces an invertible linear transformation in the space represented by A_r. The row vectors of matrix $A_0^{-1}A$ represents a factorization of A_r divided by the determinant $\det(A_0^{-1}) = a_0^{-1}$ of the transformation, i. e.,

$$A_r = a_0 b_1 \wedge \cdots \wedge b_r. \tag{10}$$

This formula provides a factorization of A_r into the outer product of vectors represented by its Plücker coordinates.

Now we consider the constraints that the Plücker coordinates of A_r satisfy. Apply (5) to A_r in its new form (10), we get

$$a_{i_1\ldots i_r} = a_0 \begin{vmatrix} b_{1i_1} & \cdots & b_{1i_r} \\ \vdots & \ddots & \vdots \\ b_{ri_1} & \cdots & b_{ri_r} \end{vmatrix}, \tag{11}$$

where $b_i = (b_{i1} \ldots b_{in})^T$, and $b_{ij} = a_{\underset{ij}{\vee}}/a_0$. The equality becomes trivial when $(i_1, \ldots, i_r)$ equals $(1, \ldots, r)$ or differs from it by one element. The number of nontrivial relations is

$$C_n^r - 1 - C_r^{r-1} C_{n-r}^1 = C_n^r - r(n-r) - 1. \tag{12}$$

We can further simplify (11) to quadratic one. Expanding the determinant on the right-hand side of (11) with respect to the last row, we get

$$\begin{array}{rcl} a_{i_1\ldots i_r} & = & a_0 \sum_{j=1}^r (-1)^{j+r} b_{ri_j} \begin{vmatrix} b_{1i_1} & \cdots & b_{1i_{j-1}} & b_{1i_{j+1}} & \cdots & b_{1i_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{(r-1)i_1} & \cdots & b_{(r-1)i_{j-1}} & b_{(r-1)i_{j+1}} & \cdots & b_{(r-1)i_r} \end{vmatrix} \\ & = & a_0^{-1} \sum_{j=1}^r a_{\underset{ri_j}{\vee}} a_{i_1\ldots i_{j-1} r i_{j+1} \ldots i_r}, \end{array}$$

where the second step follows from (11). Writing $a_{i_1\ldots i_{j-1} r i_{j+1} \ldots i_r}$ as $a_{i\underset{j}{\vee}r}$, we get

$$a_0 a_{i_1\ldots i_r} = \sum_{j=1}^r a_{\underset{ri_j}{\vee}} a_{i\underset{j}{\vee}r}. \tag{13}$$

Conversely, let A_r be an r-vectors whose Plücker coordinates satisfy (13). Let $b_i = (b_{i1} \ldots b_{in})^T$, where $b_{ij} = a_{\underset{ij}{\vee}}/a_0$. The above procedure shows that under the assumption $a_0 \neq 0$, (11) is equivalent to (13). Therefore (10) holds and A_r is an extensor.

Theorem 2.1. let A_r be an r-vectors in $\mathcal{V}^n$ with Plücker coordinates $(a_{l_1\ldots l_r} \mid 1 \leq l_1, \ldots, l_r \leq n)$. If $a_{1\ldots r} \neq 0$, A_r is an extensor if and only if (13) holds for any $1 \leq i_1 < \ldots < i_r \leq n$, where at least two i's are greater than r.

When expanding the determinant on the right-hand side of (11) with respect to different rows and columns, we get different quadratic relations. The set of all these relations is just the p-relations. The number of p-relations is

$$C_n^{r-2} C_{n-r+2}^4 + \sum_{i=3}^r C_n^{r-i} C_{n-r+i}^{i-1} C_{n-r+1}^{i+1}. \tag{14}$$

Now we define an order among the Plücker coordinates of A_r. Let $\#(a_{l_1\ldots l_r})$ be the number of elements in $l_1, \ldots, l_r$ that are greater than r, then

1. if $\#(a_{i_1 \dots i_r}) < \#(a_{j_1 \dots j_r})$, set $a_{i_1 \dots i_r} \prec a_{j_1 \dots j_r}$;
2. if $\#(a_{i_1 \dots i_r}) = \#(a_{j_1 \dots j_r})$, but $i_1, \dots i_r \prec j_1, \dots, j_r$ in lexical order, set $a_{i_1 \dots i_r} \prec a_{j_1 \dots j_r}$.

Theorem 2.2. Under the above order, the set

$$\{a_0 a_{i_1 \dots i_r} = \sum_{j=1}^{r} a_{\overset{\vee}{r} i_j} a_{i \overset{\vee}{j} r} \mid \#(a_{i_1 \dots i_r}) > 1, \quad 1 \leq i_1 < \dots < i_r \leq n\} \tag{15}$$

is a Ritt-Wu basis of the p-relations.

Proof. Any relation in (2) is reduced by (15) to a polynomial one in the variables $\{a_0, a_{\overset{\vee}{i} j} \mid 1 \leq i \leq r, r < j \leq n\}$. By Theorem 2.1, there is no constraint among these variables. So any relation in (2) is reduced to $0 = 0$ by (15). The characteristic set (15) is linear with respect to every of its leading variables and has only one initial a_0, and so is irreducible.

Example 1. When $n = 5$, $r = 2$, there are 5 Plücker relations:

$$\begin{array}{ll} F_{1,234}: & a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0, \\ F_{1,235}: & a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23} = 0, \\ F_{1,245}: & a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24} = 0, \\ F_{1,345}: & a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34} = 0, \\ F_{2,345}: & a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34} = 0. \end{array} \tag{16}$$

When $a_{12} \neq 0$, a Ritt-Wu basis is $F_{1,234}, F_{1,235}, F_{1,245}$. For $n > 4$ and $r = 2$, (2) contains C_n^4 relations, while the basis contains C_{n-2}^2 relations.

3 Factorization in Clifford Algebra

Let $\mathcal{K} = \mathcal{R}$, A_r be an extensor with signature (p', q', o'). Consider the problem of factoring A_r into the outer product of r vectors which are p positive, q negative, and $o = r - p - q$ null ones. We have the following criterion on the availability of the signature constraints decided by the triplet (p, q, o):

Theorem 3.1. (1) If $p' = q' = 0$ (or $p' = o' = 0$, or $q' = o' = 0$), then $p = q = 0$ (or $p = o = 0$, or $q = o = 0$). (2) If $p' = 0$ but $q' \neq 0$ (or $q' = 0$ but $p' \neq 0$), then $p = 0$ (or $q = 0$), and o varies from 0 to o'. (3) In other cases, p, q, o can take any values from 0 to r.

Proof. (1) is obvious. Let $e_1, \dots, e_r$ be an orthogonal basis of the subspace represented by A_r. For (2), when $p' = 0$ but $q' \neq 0$, let the first q' vectors of the basis be negative. Then e_1+e_i is negative for any $q'<i\leq r$. For (3), when $p',q'\neq 0$, let the first p' vectors of the basis be unit positive and the subsequent q' vectors be unit negative. Then $e_i+\lambda e_j$ is positive, null or negative for $1\leq i\leq p'$, $p'<j\leq p'+q'$, if and only if $|\lambda|<1$, $=1$ or >1 respectively. This and (2) guarantee that p,q,o can take any values from 0 to r.

Let $\mathbf{A_r}$ be the space represented by A_r. We can first apply (10) to get $A_r=b_1\wedge\cdots\wedge b_r$, where the b's are vectors. The inner product matrix of the b's is $M=(b_i\cdot b_j)_{r\times r}$. A special linear transformation T in $\mathbf{A_r}$ changes M to TMT^T. From linear algebra, we know that there exists a T that changes M to a diagonal matrix whose first p' diagonal elements are positive and the subsequent q' diagonal elements are negative. To obtain the required triplet (p,q,o), we only need to find a special linear transformation X in $\mathbf{A_r}$ that changes TMT^T to a matrix which has p positive, q negative and o zero, diagonal elements. In matrix form, let $B^T=(b_1\ \ldots\ b_r)$, then a factorization of A_r satisfying the signature constraints can be realized by the row vectors of the matrix XTB.

The following algorithm realizes the above idea by choosing X according to the triplets (p,q,o) and (p',q',o'). When $o=0$, an input A_r of rational coefficients leads to an output whose vectors are of rational components. The algorithm is implemented with Maple 5 Release 3.

Input: $A_r=b_1\wedge\cdots\wedge b_r$, where the b's are vectors in numeric form.

Step 1. Construct the inner product matrix $M=(b_i\cdot b_j)_{r\times r}$.

Step 2. Diagonalize M with the following transformation:

$$M\mapsto TMT^T,\ \text{where}\ \det(T)=1.$$

T can be chosen to be the composition of some elementary row transformations that replace a row a of a matrix with $a+\lambda b$, where λ is a scalar and b is another row of the matrix.

Let $D=\operatorname{diag}(d_1,\ldots,d_r)=TMT^T$.

Step 3. Rearrange the diagonal elements in D such that $d_1,\ldots,d_{p'}>0$, $d_{p'+1},\ldots,d_{p'+q'}<0$, and $d_{p'+q'+1},\ldots,d_r=0$. This can be realized by a linear transformation T' of determinant ±1, which is the composition of some elementary row transformations that swap two rows of a matrix. Denote $o'=r-p'-q'$.

Step 4. The triplet (p',q',o') is the signature of A_r. Using Theorem 3.1, if the constraint (p,q,r) is not satisfiable, output this message and exit.

Step 5. There are the following possibilities:

Case 1. If $p' = q' = 0$, output $A_r = b_1 \wedge \cdots \wedge b_r$ and exit.

Case 2. If $p' \neq 0$, $q' = 0$ (or $q' \neq 0$, $p' = 0$), then $p \geq p'$, $q = 0$ (or $q \geq q'$, $p = 0$). Define

$$X = \begin{pmatrix} I_{p'} & & \\ J_{(p-p')\times p'} & I_{p-p'} & \\ & & I_{o'} \end{pmatrix} \quad (\text{or } \begin{pmatrix} I_{q'} & & \\ J_{(q-q')\times q'} & I_{q-q'} & \\ & & I_{o'} \end{pmatrix}),$$

where I_k represents the $k \times k$ unit matrix, $J_{(p-p')\times p'}$ (or $J_{(q-q')\times q'}$) is a matrix whose first column is $(1 \ldots 1)^T$ and other columns are zeros.

Case 3. If $p', q' \neq 0$, $p' \geq p$, $q' \geq q$, then $o' \leq o$. Define

$$X = \begin{pmatrix} I_p & & & & \\ & I_{p'-p} & J_{(p'-p)\times q} & & \\ & & I_q & & \\ J'_{(q'-q)\times p} & & & I_{q'-q} & \\ & & & & I_{o'} \end{pmatrix},$$

where the J's are matrices whose nonzero elements are the first columns; the first column of $J_{(p'-p)\times q}$ is

$$(\sqrt{-d_{p+1}/d_{p'+1}}, \sqrt{-d_{p+2}/d_{p'+1}}, \ldots, \sqrt{-d_{p'}/d_{p'+1}})^T;$$

the first column of $J'_{(q'-q)\times p}$ is

$$(\sqrt{-d_{p'+q+1}/d_1}, \sqrt{-d_{p'+q+2}/d_1}, \ldots, \sqrt{-d_{p'+q'}/d_1})^T.$$

Case 4. If $p', q' \neq 0$, $p' \leq p$, $q' \leq q$, then $o' \geq o$. Define

$$X = \begin{pmatrix} I_{p'} & & & & \\ & I_{q'} & & & \\ & & I_o & & \\ J_{(p-p')\times p'} & & & I_{p-p'} & \\ & J'_{(q-q')\times q'} & & & I_{q-q'} \end{pmatrix},$$

where the J's are matrices whose first columns are $(1 \ldots 1)^T$ and other columns are zeros.

Case 5. If $p', q' \neq 0$, $p' \geq p$, $q' \leq q$ (or $p' \leq p$, $q' \geq q$), and $o' \leq o$, define

$$X = \begin{pmatrix} I_p & & & & \\ & I_{q-q'} & & J_{(q-q')\times q'} & \\ & & I_{o-o'} & J'_{(o-o')\times q'} & \\ & & & I_{q'} & \\ & & & & I_{o'} \end{pmatrix} \quad (\text{or } \begin{pmatrix} I_{p'} & & & & \\ & I_q & & & \\ J_{(p-p')\times p'} & & I_{p-p'} & & \\ J'_{(o-o')\times p'} & & & I_{o-o'} & \\ & & & & I_{o'} \end{pmatrix}),$$

where the J's are matrices whose nonzero elements are the first columns; the first column of $J_{(q-q')\times q'}$ is

$$([\sqrt{-d_{p+1}/d_{p'+1}}]+1,[\sqrt{-d_{p+2}/d_{p'+1}}]+1,\ldots,[\sqrt{-d_{p+q-q'}/d_{p'+1}}]+1)^T;$$

the first column of $J_{(p-p')\times p'}$ is

$$([\sqrt{-d_{p'+q+1}/d_1}]+1,[\sqrt{-d_{p'+q+2}/d_1}]+1,\ldots,[\sqrt{-d_{p+q}/d_1}]+1)^T;$$

the first column of $J'_{(o-o')\times q'}$ is

$$(\sqrt{-d_{p+q-q'+1}/d_{p'+1}},\sqrt{-d_{p+q-q'+2}/d_{p'+1}},\ldots,\sqrt{-d_{p'}/d_{p'+1}})^T;$$

the first column of $J'_{(o-o')\times p'}$ is

$$(\sqrt{-d_{p+q+1}/d_1},\sqrt{-d_{p+q+2}/d_1},\ldots,\sqrt{-d_{p'+q'}/d_1})^T.$$

Case 6: If $p',q'\neq 0$, $p'\geq p$, $q'\leq q$ (or $p'\leq p$, $q'\geq q$), and $o'\geq o$, define

$$X=\begin{pmatrix} I_p & & & & \\ & I_{p'-p} & J_{(p'-p)\times q'} & & \\ & & I_{q'} & & \\ & & & I_o & \\ & & J'_{(o'-o)\times q'} & & I_{o'-o}\end{pmatrix}\quad (\text{or}\ \begin{pmatrix} I_{p'} & & & & \\ & I_q & & & \\ J_{(q'-q)\times p'} & & I_{q'-q} & & \\ & & & I_o & \\ J'_{(o'-o)\times p'} & & & & I_{o'-o}\end{pmatrix}),$$

where the J's are matrices whose nonzero elements are the first columns; the first column of $J_{(p'-p)\times q'}$ is

$$([\sqrt{-d_{p+1}/d_{p'+1}}]+1,[\sqrt{-d_{p+2}/d_{p'+1}}]+1,\ldots,[\sqrt{-d_{p'}/d_{p'+1}}]+1)^T;$$

the first column of $J_{(q'-q)\times p'}$ is

$$([\sqrt{-d_{p'+q+1}/d_1}]+1,[\sqrt{-d_{p'+q+2}/d_1}]+1,\ldots,[\sqrt{-d_{p'+q'}/d_1}]+1)^T;$$

the first columns of $J'_{(o-o')\times q'}$ and $J'_{(o-o')\times p'}$ are $(1\ldots 1)^T$.

Output: $A_r=\det(T')a_1\wedge\ldots\wedge a_r$, where a_i^T is the i-th row of the matrix $XT'TB$, and $B^T=(b_1\ldots b_r)$.

Example 2. Let $\{e_1,e_2,e_3\}$ be an orthogonal basis of $\mathcal{R}^{2,1}$, $e_1\cdot e_1=e_2\cdot e_2=-e_3\cdot e_3=1$. Let $A_2=e_1\wedge e_2+3e_1\wedge e_3+e_2\wedge e_3$.

By the factorization formula (10), we get

$$A_2 = (e_1 - e_3) \wedge (e_2 + 3e_3);$$

when $(p, q, o) = (3, 0, 0)$, using the above algorithm, we get

$$A_2 = (e_1 + \frac{3}{8}e_2 + \frac{1}{8}e_3) \wedge (-3e_1 - \frac{1}{8}e_2 + \frac{21}{8}e_3);$$

when $(p, q, o) = (0, 3, 0)$, we get

$$A_2 = (e_1 - \frac{5}{8}e_2 - \frac{23}{8}e_3) \wedge (e_2 + 3e_3);$$

when $(p, q, o) = (0, 0, 3)$, we get

$$A_2 = (\frac{1}{2}e_1 - \frac{1}{2}e_3) \wedge (\frac{8}{3}e_1 + 2e_2 + \frac{10}{3}e_3).$$

In geometric applications, an extensor in a Minkowskii space plays an important role. For example, in the hyperboloid model of hyperbolic geometry, a Minkowskii r-extensor represents an $(r-1)$-plane; factoring it into an outer product of negative vectors corresponds to representing the plane with its points. In the homogeneous model of Euclidean geometry, a Minkowskii r-extensor represents an $(r-2)$-sphere (or plane); factoring it into an outer product of null vectors corresponds to representing the sphere (or plane) with its points. An r-extensor can also represent an $(r-1)$-bunch of spheres and hyperplanes; factoring it into an outer product of positive vectors corresponds to representing the bunch with its spheres and hyperplanes.

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