Building Dynamic Mathematical Models with Geometry Expert*

III. A Geometry Deductive Database

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Abstract

Geometry Expert (GEX) is a software system for dynamic diagram drawing and automated geometry theorem proving and discovering. For a given diagram, we can use GEX to generate a database which contains all the properties of this diagram that can be deduced from a fixed set of geometric rules or axioms, and for each geometric property in the database GEX can generate an elegant proof for it. Based on this software, we introduce the concept of *dynamic logic model* which can do reasoning itself. Logic models can be used for intelligent educational tasks, such as automated generation of test problems, automated evaluation of students' answers, intelligent tutoring, etc.

1 Introduction

Using tools to teach abstract mathematical concepts and to assist students to do reasoning was used to be quite popular [5, 10, 16]. Currently, this approach was revived due to the invention of dynamic geometry softwares [4, 5, 8, 9, 10]. Visual models built with dynamic geometry softwares have many advantages over models built with real materials. But they still can only assist people to do reasoning. In this paper, we will introduce the concept of *dynamic*

* This work was partially supported by an Outstanding Youth Grant (No. 69725002) from Chinese NSF.

logic model which can do reasoning itself. Logic models can be used for more intelligent educational tasks, such as automated generation of test problems, automated evaluation of students' answers, intelligent tutoring, etc.

The dynamic logic model is based on software *Geometry Expert (GEX)* [7] for dynamic diagram drawing and automated geometry theorem proving and discovering. In Parts I and II of this series papers [8, 9], we discussed how to use Geometry Expert to build visual models for various mathematical concepts. In this paper, we will show how to use GEX to build *dynamic logic models*. Precisely speaking, for a given geometric configuration, GEX can generate a database which contains all the properties of this configuration that can be deduced from a fixed set of geometric rules or axioms, and for each geometric property in the database GEX can generate an elegant proof for it. Our experiments show that GEX can discover most of the well-known results and often some unexpected ones for hundreds of geometry diagrams. This software is based on the current research work on automated reasoning [3, 2, 15].

With this feature, GEX can be used as a Dynamic Geometry Dictionary. With it, teachers can easily make exercises and test problems; students can enhance their ability of solving problems by fully exploring the properties of a given diagram. The advanced part of the system can be used by geometers to solve challenge problems or conjectures.

In [11], a similar software system for solid geometry is developed, which also uses the techniques of deductive database and has close connections with the geometry textbooks.

2 Fixpoint and Deductive Database

Let D_0 be the given geometric properties in a geometry configuration and R a set of geometric rules. We may use the *breadth-first forward chaining method* to find new properties of this diagram. Basically speaking, the breadth-first forward chaining method works as follows

$$\boxed{D_0} \overset{R}{\subset} \quad \boxed{D_1} \overset{R}{\subset} \quad \ldots \overset{R}{\subset} \quad \boxed{D_k} \quad \text{(Fixpoint)}$$

where D_{i+1} is the union of D_i and the set of new properties obtained by applying rules in R to properties in D_i. If at certain step $D_k = D_{k+1}$, i.e.,

$$R(D_k) = D_k,$$

then we say that a *fixpoint* (of reasoning) for D_0 and R is reached.

The naive form of breadth-first forward chaining is notorious for its inefficiency. But, in the case of geometry, by using techniques from the theory of deductive database [6] and by introducing new search techniques, we manage to build a very effective prover based on this simple idea [3].

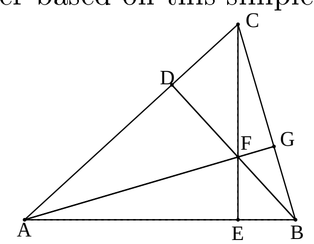

Figure 1

Example 2.1 (The Orthocenter Theorem) *Show that the three altitudes of a triangle are concurrent.*

Let D and E be the feet of the perpendicular lines drawn from points B and C to AC and AB respectively, F the intersection of BD and CE, G the intersection of BC and AF. Then the hypotheses are:

$$D_0 = \left\{ \begin{array}{l} collinear(D,A,C), perpendicular(B,D,A,C), \\ collinear(E,A,B), perpendicular(C,E,A,B), \\ collinear(F,B,D), collinear(F,C,E), \\ collinear(G,B,C), collinear(G,A,F). \end{array} \right\}$$

Reaching the fixpoint for D_0 with GEX costs 0.75 second on a SUN SPARC-20. The fixpoint contains 151 geometry properties:

collinear point sets: 6
perpendicular pairs: 3
co-cyclic points sets: 6
equal angle pairs: 24
similar triangles sets: 7
equal ratio pairs: 105.

We will explain the contents of the fixpoint in later sections.

3 Geometric Rules

In this section, we will give a brief introduction to the geometric rules or axioms used in GEX. A geometric rule used in GEX has the following form

$$Q(x) :\Leftrightarrow P_1(x), \cdots, P_k(x) \quad \text{meaning}$$
$$\forall x[(P_1(x) \cap \cdots \cap P_k(x)) \Rightarrow Q(x)]$$

where the x are the points occurring in the geometry predicates $P_1, \cdots, P_k$, and Q. The following are several rules used in GEX.

- $EF \parallel BC$:- midp(E, A, B), midp(F, A, C), where midp stands for midpoint.
- midp(F, A, C) :- midp(E, A, B), $EF \parallel BC$,coll(F, A, C), where coll stands for collinearity.
- simtri(A, B, C, P, Q, R) :- $\angle ABC = \angle PQR$, $\angle ACB = \angle PRQ$, $\neg$ coll(A, B, C), where simtri stands for similar-triangles.
- contri(A, B, C, P, Q, R) :- simtri(A, B, C, P, Q, R), $AB = PQ$, where congtri stands for congruent-triangles.

One of the central geometric concept is the full-angle. Intuitively, a *full-angle* $\angle[u, v]$ is the angle from line u to line v. Note that u and v are not rays as in the definition for the ordinary angles. Two full-angles $\angle[l, m]$ and $\angle[u, v]$ are equal if there exists a rotation K such that $K(l) \parallel u$ and $K(m) \parallel v$. If A, B and C, D are distinct points on l and m respectively, then $\angle[l, m]$ is also denoted by $\angle[AB, CD]$, $\angle[BA, CD]$, $\angle[AB, DC]$, and $\angle[BA, DC]$.

The introduction of full-angles greatly simplifies the predicate of the angle congruence. For instance, we have the following rule about parallel lines and angles.

R1. $AB \parallel CD$ if and only if $\angle[AB, PQ] = \angle[CD, PQ]$ (Figure 2).

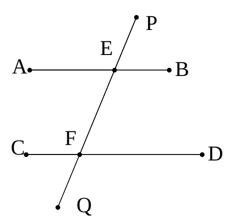

Figure 2

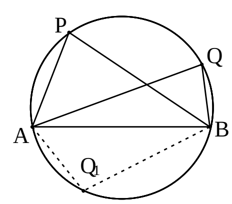

Figure 3

If using ordinary angles, we need to specify the relations among eight angles and we need to use order relations (inequalities) to distinguish the cases. For instance, we have: "if points B, D are on the same side of line PQ and points P, C are on the different sides of line AB (the order relations), then $AB \parallel CD \Leftrightarrow \angle PEB = \angle PFD$." This rule is very difficult to use and may lead to branchings during the deduction. The following two rules also show why full-angle is crucial to our approach.

R2. $\angle[PA, PB] = \angle[QA, QB]$:- cyclic(A, B, P, Q) (Figure 3).

R3. cyclic(A, B, P, Q) :- $\angle[PA, PB] = \angle[QA, QB]$, $\neg$ coll(P, Q, A, B) (Figure 3).

In rule R2, if using the ordinary angle, we need two conditions (Figure 3): $\angle APB = \angle AQB$ or $\angle APB + \angle AQ_1B = 180°$ and to distinguish these two cases, we need to know "points P and Q are on the same or different sides of line AB." Using full-angles, the two cases can be treated uniformly.

The program uses about seventy rules (see [3]).

4 Automated Theorem Discovering

Forward chaining is a natural way of discovering "new" properties for a given geometric configuration. Any thing obtained in the forward chaining may be looked as a "new" result. Our experiments show that GEX can discover most of the well-known results and often some unexpected ones.

Example 4.1 *(Continue from Example 2.1) Take the simple configuration (Figure 1) related to the orthocenter theorem as an example. GEX discovered the often mentioned properties about this configuration:*

1. *The three altitude are concurrent* ($AG \perp BC$)*.*
2. $\angle EGA = \angle AGD$.

The fixpoint also contains six groups of co-cyclic points:

$$A, D, E, F; B, C, D, E; C, D, F, G; A, B, D, G; A, C, E, G; B, E, F, G$$

and seven sets of similar triangles

$$\triangle DBA \sim \triangle DCF \sim \triangle EBF \sim \triangle ECA;$$
$$\triangle DCB \sim \triangle DFA \sim \triangle GFB \sim \triangle GCA;$$
$$\triangle EFA \sim \triangle EBC \sim \triangle GBA \sim \triangle GFC;$$
$$\triangle FBC \sim \triangle FED \sim \triangle GBD \sim \triangle GEC;$$
$$\triangle ACB \sim \triangle AED \sim \triangle GCD \sim \triangle GEB;$$
$$\triangle CED \sim \triangle CAF \sim \triangle GAD \sim \triangle GEF;$$
$$\triangle FBA \sim \triangle EBD \sim \triangle FGD \sim \triangle EGA.$$

Another amazing fact is that this simple configuration contains 105 nontrivial ratios!

Example 4.2 (The Nine-Point-Circle Theorem) *For a triangle ABC, let H be its orthocenter. Then the three midpoints of its three sides, the three feet on its three sides, and the three midpoints of AH, BH and CH are on the same circle.*

The fixpoint is reached for 454.45 seconds and contains 6046 facts. The fact that the nine points $D, E, F, G, H, L, M, N, O$ are on the same circle is in the database. The database contains the following facts.

midpoints: 6
collinear lines: 6
parallel lines: 6
perpendicular lines: 36
cyclic: 7
equal angles: 5699
similar triangles 13
congruent triangles 31
congruent segments 7
equal ratios: 235.

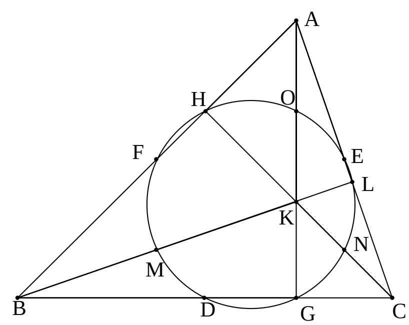

Figure 4

5 Automated Geometry Theorem Proving

For each geometric property in the database, GEX can produce a proof of traditional style. The following is the proof for the Orthocenter Theorem (Example 2.1) produced by GEX. Notice that the proof is in an "analysis style", i.e., it starts from the conclusion and goes all the way to the hypotheses of the statement.

1. $AG \perp BC$,
   because $AC \perp BD$(hypothesis), (2)$\angle[AC, BD] = \angle[BC, AF]$.
2. $\angle[AC, BD] = \angle[BC, AF]$,
   because (3)$\angle[AC, BC] = \angle[BD, AF]$. (This is a rule in GEX).
3. $\angle[CA, CB] = \angle[BD, AF]$,
   because (4)$\angle[CA, CB] = \angle[DE, AB]$, (5)$\angle[BD, AF] = \angle[DE, AB]$.
4. $\angle[CA, CB] = \angle[DE, AB]$,
   because (6)co-cyclic$[B, D, C, E]$. (Rule R2)
5. $\angle[BD, AF] = \angle[DE, AB]$,
   because (7)co-cyclic$[A, D, E, F]$. (Rule R2)
6. co-cyclic[B,D,C,E],
   because $DC \perp DB$(hypothesis), $EC \perp EB$(hypothesis). (Rule R3)
7. co-cyclic$[A, D, E, F]$,
   because $DF \perp DA$(hypothesis), $EF \perp EA$(hypothesis). (Rule R3)

The first step of the proof can be understood as follows. $AG \perp BC$ is true because $AC \perp BD$ which is a hypothesis and $\angle[AC, BD] = \angle[BC, AF]$ which will be proved in the second step. The other steps can be understood similarly.

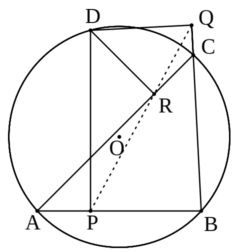

Figure 5

Example 5.1 (Simson's Theorem) *Let D be a point on the circumcircle of triangle ABC. From D three perpendiculars are drawn to the three sides BC, AC, and AB of triangle ABC. Let Q, R, and P be the three feet respectively. Show that P, Q and R are collinear.*

The Machine Proof

1. coll$[PQR]$, because (2)para$[PQ, PR]$. (para means parallel)
2. para$[PQ, PR]$, because (3)$\angle[DP, DP] = \angle[PQ, PR]$.
3. $\angle[DP, DP] = \angle[PQ, PR]$, because (4)$\angle[DP, PQ] = \angle[DP, PR]$.
4. $\angle[DP, PQ] = \angle[DP, PR]$, because

   (5) $\angle[DP, PQ] = \angle[AD, AC]$, (6) $\angle[DP, PR] = \angle[AD, AC]$.
5. $\angle[DP, PQ] = \angle[AD, AC]$, because

   (7) $\angle[DP, PQ] = \angle[BD, BC]$, (8) $\angle[AD, AC] = \angle[BD, BC]$.
6. $\angle[DP, PR] = \angle[AD, AC]$, because

   (9) $\angle[DP, PR] = \angle[BD, BC]$, (8)$\angle[AD, AC] = \angle[BD, BC]$.
7. $\angle[DP, PQ] = \angle[BD, BC]$, because

   (hyp)coll$[CBQ]$, (10)$\angle[PD, PQ] = \angle[BD, BQ]$.
8. $\angle[AD, AC] = \angle[BD, BC]$, because (hyp)circle$[ABCD]$.
9. $\angle[DP, PR] = \angle[BD, BC]$,

   because (hyp)coll$\angle[RPQ]$, (hyp)coll[CBQ], (10)$\angle[PD, PQ] = \angle[BD, BQ]$.
10. $\angle[PD, PQ] = \angle[BD, BQ]$, because (11)circle$[BPDQ]$.
11. circle$[BPDQ]$, because (hyp)perp$[PD, PB]$, (hyp)perp$[QD, QB]$. (perp means perpendicular)

6 Constructing Auxiliary Points

We know that constructing new points or lines is one of the most basic methods of solving geometry problems. One advantage of GEX is that it can automatically add auxiliary points to prove a geometry statement if needed.

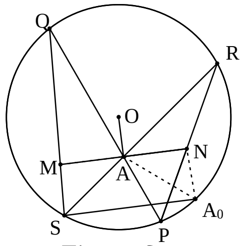

Figure 6

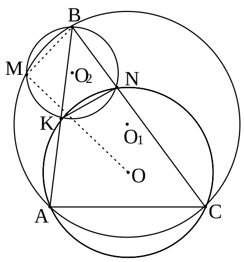

Figure 7

Example 6.1 (The Butterfly Theorem) *P, Q, R, and S are on the same circle with center O. A is the intersection of PQ and SR. The line passing through A and perpendicular to OA meets PR and QS in N and M respectively. Show that A is the midpoint of NM. (Figure 6)*

The conclusion is not in the first fixpoint. GEX then automatically adds an auxiliary point A_0 which is the intersection of the line passing through S and parallel to AN and the circle O. With the point A_0, GEX reachs a fixpoint which contains the conclusion.

Example 6.2 (International Mathematics Olymiad, 1985) *A, C, K, and N are four points on a circle. $B = AK \cap CN$. M is the intersection of the circumcircle of triangles BKN and BAC. Show that $BM \perp MO$ (Figure 7).*
An auxiliary point: $A_0 = NK \cap OO_2$ is added. GEX generates a database contains 480 properties including the conclusion and the following interesting results.

OO_1BO_2 *is a parallelogram.*

$A, N, O, M; C, O, K, M; O, O_2, O_1, M$ *are co-cyclic point sets.*

$$\triangle MCN \sim \triangle MAK \sim \triangle OO_2O_1 \sim \triangle MO_1O_2 \sim \triangle BO_1O_2 \sim \triangle CKB$$
$$\sim \triangle ANB \sim \triangle ONO_2 \sim \triangle OKO_2 \sim \triangle AOO_1 \sim \triangle COO_1.$$

7 Other Reasoning Methods in GEX

Geometry Expert (GEX) is a powerful computer program for geometric reasoning. It implements some of the most effective methods for geometric reasoning introduced in past twenty years. Within its domain, it invites comparison with the best of human geometry provers. Here is a short introduction to the methods used in GEX.

Wu's method is the most powerful method in terms of proving difficult geometry theorems [15, 1, 14]. Wu's method is a coordinate-based method. It first transfers geometry conditions into polynomial equations in the coordinates of the involving points, then deals with the polynomial equations with the characteristic set method. This method has been used to prove more than 600 geometry theorems.

The area method uses high-level geometric lemmas about geometry invariants such as the area and the Pythagorean difference as the basic tool of proving geometry theorems [2]. The method has been used to produce short, elegant, and human-readable proofs for more than 500 geometry theorems.

The Groebner basis method is also a coordinate-based method [1, 12, 13]. Instead of using the characteristic set method, it uses the Groebner basis method to deal with the polynomial equations.

Vector method is a variant of the area method and is based on the calculation of vectors and complex numbers [2].

The full-angle method is based on the calculation of full-angles. The full-angle method is a rule based method and is not a decision procedure [2]. But this method also has its advantages: all the proofs produced by the method are very short, and it has been used to prove several theorems that all the other methods fail to prove because of very large polynomials occurring in the proving process.

Why do we use more than one methods in the prover? First, with these methods, for the same theorem, the prover can produce a variety of proofs with different styles. This might be important in using GEX to geometry education, since different methods allow students to explore different and better proofs. Second, for a certain class of geometry theorems, a particular method may produce much shorter proofs than other methods.

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