Learning Opportunities with Graphing Calculators:

The Case of Asymptotes

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Abstract

The findings reported in literature though are generally in favour of the potential of elevating learning mathematics by technology, the availability of technology never promises a panacea. In fact, an effective use of technology in mathematics classrooms depends on how students interact between the peers, the teacher and the technology. This reliance on interaction will imply a radical change in the culture of the Hong Kong mathematics classrooms, which are famous for demonstrating an expository style. Hong Kong, on the one hand, has to learn from the west. On the other hand, it has to be cautious to look into the feasibility of western models in a region of different culture. This study attempted to introduce a cognitive model whilst introducing the use of graphing calculators. The model has been developed in the Cognitive Acceleration in Mathematics Education (CAME) project in UK and has applied both the constructivists' and 'social constructivists' views of learning. The key features are concrete preparation, cognitive conflict, construction, metacognition and bridging – imbedded in a 'mediation' style of teaching. The trial was carried out in a secondary-6 class on the topic of asymptotes. Episodes of the lessons are used to illustrate how cognitive conflicts could be captured in students' work with the graphing calculators and how the teacher, playing the role of a mediator, could change cognitive conflicts into situations supporting dynamic students' construction.

Introduction

Despite that reports from literature are generally in favour of the potential of elevating learning mathematics by technology, the availability of technology never promises a panacea. An effective use of technology in mathematics classrooms depends on how students interact between the peers, the teacher and the technology (Kieren, 1998). This reliance on interaction will imply a radical change in the culture of the Hong Kong mathematics classrooms, which are famous for demonstrating an expository style. As far as the use of technology is concerned, Hong Kong has hardly made any investigation in the usage of more powerful technology (such as graphing calculators) in mathematics classrooms. Thus, Hong Kong, on the one hand, has to learn from the west. On the other hand, it has to be cautious to look into the feasibility of western models in a region of different culture.

The basic assumption in the current study is that handy graphing tools can provide students an alternative learning environment in which students could have more opportunities for exploration in some related topics in the existing curriculum, thus strengthening their awareness of the graphical images of functions. In order to promote the interaction between the students, teacher and technology, the study has attempted to introduce a cognitive model whilst introducing the use of graphing calculators in Hong Kong classrooms. The model has been developed in the Cognitive Acceleration in Mathematics Education (CAME) project in UK and has applied both the constructivists' and 'social constructivists' views of learning. The key features in the model are concrete preparation, cognitive conflict, construction, metacognition and bridging – imbedded in a 'mediation' style of teaching. Although the model is theoretically grounded and supported by empirical evidence (Adhami, Johnson, and Shayer, 1998a, 1998b), teachers in Hong Kong found it difficult to visualize how these cognitive features could be realized in real lessons and how to carry out the mediating role. Thus, the current study carried out a small-scale trial in a secondary-6 class on the topic of asymptotes aiming to look for exemplary cases.

In the next section, the idea of cognitive conflict and mediated learning as they are used in the cognitive model will be explained. Then, episodes are used to illustrate how cognitive conflicts could be captured in students' work with the graphing calculators and how the teacher, playing the role of a mediator, could change cognitive conflicts into situations supporting dynamic students' construction.

Cognitive Conflict and Mediated Learning

The notion of cognitive conflict is that if children find a problem/task in which their 'lower-level' schemas (achieved or unconscious strategies) appear not to work or to yield a contradiction, then the mental conflict may challenge them to produce a higher-level strategy which does work. In the case of school mathematics the discipline itself has features which provoke the challenge, e.g., through the idea of 'consistency' and the fact that recognising an inconsistency has the potential for provoking the conflict. Here the notion of the **mediator**, and of **mediated learning** is all important. The teacher must tread a tight-rope between dropping the pupils so far in the deep end that they drown in the difficulty of the task, and dropping back into the instruction mode.

The notion of mediated learning comes from Vygotsky. Vygotsky's description of the Zone of Proximal Development (ZPD) is used both in psychological testing, where the child's potential is to be estimated, and also to explain the dynamics of development. Each child is assumed to have at any one time, in addition to sets of completed skills and strategies which enable her to succeed on conventional test items, a spectrum of half-formed or potential strategies which can be revealed by the technique of dynamic assessment, and which the child may turn into complete or successful skills either by chance, by spontaneous effort, or by the mediating influence of an adult. But on Vygotsky's account, it is a mistake to locate the ZPD within each child's head. It exists just as much in the social space which the child shares with his age-peers. The extra half-skill the child may need to knit to his own to create a completed skill may will come from what another child says and does as from his own behaviour.

But for cognitive development to take place only each child's own effort is effective: it cannot be 'taught' by any recognised skill of instructional teaching. So a major skill of intervention teaching is managing the class so that teacher's mediating role is to maximise the opportunities each pupil has to make her next cognitive jump, and a valuable part of this is the whole-class discussion where the incipient ideas and strategies of each, in her own words, are shared in a 'space' form which each can take what she needs.

The Trial Lessons

The students were a class of 30 grade-12 students (secondary 6) of average standard in Hong Kong. 27 of them had got a minimum of C grade in their O-level mathematics and 6 had studied additional mathematics which equipped them with more experience with calculus. They were now studying the AS-level mathematics (between O-level and A-level). The topic asymptotes had been taught briefly before they had the sessions with the graphic calculators. According to their teacher, they were able to calculate the asymptotes for simple stereotype functions and had some ideas of vertical and horizontal asymptotes and they had no experience of working with graphing calculators in advance. There were 4 sessions (about 70 minutes each) for them to work with the graphing calculators. The data in this paper is extracted from the third session.

After careful discussion with the teacher concerning the variation in the students' mathematics background, it was decided to design two different tasks. The students with little experience of calculus would explore the horizontal shifts of the function $f(x)=1/(x+a)$. The 6 students who knew more calculus would explore for the oblique asymptotes.

Exploring the Function $f(x)=1/(x+a)$

The lesson began by the teacher explaining why they would be working on different worksheets and dividing them into groups. The teacher also reminded briefly the use of some function keys (table, zoom, graph). Then the students started working on the worksheet (appendix 1) in 13 groups.

The first part of the worksheet asked the students to predict the graphs of four functions (f(x)=1/(4.5+x), f(x)=1/(4.5-x), f(x)=1/(2.5+x), f(x)=1/(2.5-x)) and then checked their prediction with their calculators. Then, they were asked to predict the geometrical relation between the graphs of the functions f(x)=1/(x-a) and f(x)=1/(a-x).

The group discussion continued for about 30 minutes, then the teacher invited some students to come out to talk about their work.

Describing the Shift of the Curves

Amy and Wan were the first group who came out to show their work. On their worksheet, they had sketched the four graphs, with labels and asymptotes.

1. Amy: When x gets bigger, this is further away.
2. T: x? x getting bigger?
3. Amy: Yes. This.
4. T: O.K. This is getting bigger. Which graph are you referring to? State more clearly. Which one. Try again. There are two types of graphs. Two types.
5. Amy: Err. When the number (referring to 4.5 in 1/(4.5-x)) gets bigger, then the vertical asymptote gets further away.
6. T: Yeah.
7. Amy: Getting further away. That is, further away from the y-axis.
8. (After a few probing, the teacher was aware that the student's description was only referring to the cases for positive parameter (a in 1/(a-x)) as the only examples were the cases of 2.5 and 4.5. He tried to direct the class attention to negative values of the parameter and compare the graphs of 1/(4.5-x) and 1/(-2.5-x).)
9. T: Further to this side. That is. We pay attention to here. We have just looked at the cases on the worksheet. What happens if the numbers 2.5 and 4.5 become negative? Any prediction? Say, -2.5, 1/(-2.5)-x. Where will be the graph? Amy, what do you think?
10. Amy: Err. Then the x will (pointing to the graphs on the board).
11. T: Good. That the line (the asymptote) will become this, on this side. Then the graph, it will follow. Do you agree? Then, this one, the green line, the one also shift to here. Yeah, you have just said that when the numbers were bigger, the vertical asymptote would be further away from the y-axis. How about this?(The class rustled and some murmured.) Think carefully. If 4.5 gets larger, where is the graph?
12. Class: Further away.
13. T: Which way? You said further away. This way. O.K. Not this. That is, if the vertical asymptote moves to this side, towards the right. Okay?

14. Class: (Laughed and showed agreement.)

From their work in the worksheet, it seemed that Amy and Wan were quite successful in predicting the position of the graphs for given equations. However, their own descriptions ("getting further away", line 1-7) in the first part of their presentation were not precise.

This kind of ambiguity is not unexpected and there are plausible factors accounting for Amy's ambiguous descriptions. First, she might not be used to describe mathematics in her own words. Secondly, there has always been difficulty in describing mathematics precisely in ordinary daily life language. Thirdly, the student was probably not aware that the implication of her suggestion when it was applied to the cases of negative parameters which moved the asymptotes towards the left side of the y-axis, thus contradicting her description. Consequently, her answer was incomplete and inaccurate. At this point, we see that the teacher had seized this opportunity and made it into a situation for deeper thought by introducing an additional function f(x)=1/(-2.5-x) (line 9-13). Naturally this action led Amy as well as the rest of the class to see the inadequacy of their own description and the need for a more precise version by adding "to the right". This was not easy but it was important to maintain a support of what students' had said in order to encourage more active participation afterwards.

Symmetry: From One Mirror to Two Mirrors

The teacher then led the class to discuss the second part of the worksheet which asked the students to predict the geometrical relation between the graphs of the functions f(x)=1/(x-a) and f(x)=1/(a-x). Ying and Cheung were first invited to talk about their work. They compared the graphs of 1/(5-x) and 1/(x-5). They noticed the symmetry about the vertical asymptote but failed to notice the symmetry about the x-axis (line 22).

This type of incomplete answers often happen when students close their answers quickly. Yet, the different observations made by different students in a collective discourse may sometimes be complementary. In the case, the teacher played the mediating role and let their classmate point out the mistake and the teacher himself only provided a hint to help the student reviewing the situation.

15. Ying: We assume that 'a' is five. Draw the two lines. You can look at one first.
16. T: You can use this (pointing to the graphs with a pointer). Which are you referring to?
17. Ying: This and that.
18. T: 'y' which? Okay, y5 and y6. One is 1/(5-x), one is 1/(x-5). Okay. Remember this. y5 is 1/(5-x), y6 is 1/(x-5).
19. Ying: We first produce 1/(x-5) (using their graphing calculator).

20. Class: Which is 1/(x-5)?
21. T: Okay. First you produce 1/(x-5) only.
22. Ying: Then, we also produce the other, 1/(5-x) (using their graphing calculator).
23. T: (Waited till the two graphs were shown.) So, what do you want to say about this?
24. Ying: (Reading their own writing on their sheet.) They reflect each other and have the same vertical asymptote which serves as a mirror. (Stop reading, continued in their own words) That is, vertical asymptote acts as a mirror and reflects, so the two sides are symmetrical. Then, they are laterally inverted and not vertically inverted.
25. T: Laterally inverted? Not?
26. Ying: That is, left and right reversed, but not reverse upward and downward.
27. T: Not reverse upward and downward. Left and right reverse. Not reverse upward and downward. (Instead of pointing out that the student was wrong in saying that the graphs were not vertically inverted. The teacher directed the question to the class.) Do you understand what she said?
28. T: Karen did not understand. You explain what you meant by "left and right reversed, but not reverse upward and downward". Yeah. You noticed that this graph, this plane and that are the same graph. Okay. Then, this and that. Well? (The teacher referred to the graphs on the board with the pointer.)
29. Ying: It reverses upward and downward, and also reverse left and right. Upward and downward, left and right, all reverse. Then, the vertical asymptote and the horizontal asymptote act like mirrors. That is, they are symmetrical.

Overlapping Graphs

When the symmetry was settled, Ying and Cheung continued to present what they thought about the effect of the parameter 'a'. When they sketched 1/(x-4) on the board, their sketches crossed each other. When they finished their sketches, it was near the end of the lesson. In this case, no classmates noticed the problem and the teacher needed to direct them to check and see that the curves should not intersect by using zooming function.

30. Ying: (Reading their writing on their sheet) When 'a' increases, the distance between the vertical asymptote and the y-axis increases.
31. T: Okay. (To the class) Do you understand? (To Ying) Any graphs to show us?
32. Ying: Let's draw them. (Ying and Cheung discussed and drew 1/(4-x) on the board. Upon the teacher's request, they labelled their curves with blue and green colours. Nevertheless, their sketches intersected and did not predict the real case. The teacher had to correct this.)

33. T: Pay attention to here. We did not have sufficient time. Have the blue and green graphs any intersections? (Paused for the students to think.) Your drawing seems that they cross each other. Should they?
34. Class: (murmuring.)
35. T: Just guess. You may not have seen it in the calculators.
36. S1: They overlapped.
37. S2: They should not.
38. T: (Demonstrated with the calculator.) I have drawn a few graphs. Let's see. Y=1/(x+2.5). Then, y=1/(x+4.5). Then, we checked the minus. Y=1/(x-2.5). Then, y=1/(x-4.5). If you draw them, you will notice something like this. Alright. You notice that they are very crowded. Then, you zoom in. I don't know whether you have tried this. You will see that they don't intersect at all.

Oblique Asymptotes: Making Their Own Discovery

This section reported the work of the 6 additional mathematics in the third and the fourth sessions. They were working by themselves in a group under the guidance of an alternative worksheet (appendix 2). In their own exploration, they realized that their original conception of asymptotes was incomplete and eventually they found the oblique asymptotes of the function f(x)=(x^3-1)/(x^2-1) by themselves and agreed on an alternative way of defining the asymptotes by themselves.

Their Starting Point

At the beginning, they listed x=1 or x=-1 as the possible asymptotes for f(x). Their answers reflected very clearly that they obtained the equations of the asymptotes by observing when the denominator became zero. Or in their own words, they got the asymptotes "by calculation".

Seein*g* a Conflict: x=1 is Not an Asymptote

Question 1 in the worksheet invited the students to look at the behavior of f(x) near x=1 and x= -1. Their work recorded that they checked the values of f(x) near x=-1 and also used the zoom function to check the graph. With the help of the calculator, they confirmed that x=-1 was an asymptote.

In the later part of the question, they were requested to discuss the differences between the two cases x=1 and x=-1 and whether the concept of asymptote could apply. From the students' work, we could see that with the help of the calculator they founded x=1 was not an asymptote. As a result of this realization, they found two possible consequences when the function was undefined: "discontinue at the point" and "tends to infinity". The latter, "y tends to infinity", thus became an important attribute for them to decide whether a line was an asymptote and helped them to correct their earlier assumption that "undefined" would lead to an asymptote.

Seeking a New Definition for Asymptotes

The second question in the worksheet asked them to investigate the behavior of f(x) for very large and very small values. Similar to what they had done for question 1, they listed, in a table, the values of f(x) for very large values of x (999, 999.01, etc.) and very small values of x (-1000, -999.9, etc.). From these data, they observed that "x values = y values", "[the curve] becomes a straight line", "the slope $\Rightarrow$ [becomes] constant" and "there is no horizontal asymptote".

When they applied their earlier concept of asymptote, they modified their definition of asymptotes to accommodate this new case. And similar conclusion was given to the case when x was very small but negative. The students then concluded that there were not only horizontal and vertical asymptotes and that the horizontal asymptote was a special case in which the slope was zero. As a result, their new definition for asymptotes was based on the trend of the slope of the graph. They wrote:

> "From this case, when $x \rightarrow +\infty$, the slope $\neq 0$ but tends to constant slope. When x is getting larger and larger, the x values is more likely equal to y values.
>
> When x is getting larger and larger, the x values is more likely equal to y values.
>
> When x is getting smaller, the x values is more likely equal to y values.
>
> $\therefore$ There are not only horizontal and vertical asymptotes.
>
> Definition of asymptotes: $x \rightarrow \infty$, the slope of the graph $\rightarrow$ a constant. The line with that constant slope is the asymptote.
>
> $\therefore$ It does not depend on the values, but the slope."

Summary

'Exploration' and 'construction' appeared to be alien in classes which were used to an expository style of teaching. At the initial stage of the project, we shared similar sentiments. However, we gradually learned more about these concepts as we attempted to put them in practice and looked closely into what the students could produce.

The episodes reported in this paper demonstrate possible cognitive conflicts which students might come across when they were guided to explore by their teacher or tried the exploration themselves. The splitting of the students helped us to learn more how different conflicting situations could constitute a learning opportunities. The results of the students' work on the oblique asymptotes were encouraging. In this trial, they made their own prediction, verified and checked their own work with the graphing calculators, found their own mistakes and resolved the conflict by creating a new definition. For this group, the mediation was in fact imbedded in the teacher's design of the worksheet in advance. For the students with less mathematical experience, they might need more help from their teacher. Conflicts often occurred when students' answers were incomplete or wrong. Instead of telling correct answers directly, the teacher or the mediator had a range of possible remedy. For example, as demonstrated in this lesson, the teacher suggested an example to let students see the incompleteness of their answer, sought consensus from the class, reflected upon the graphical images, encouraging students to guess and check with the graphing calculators.

In either case, the mediating role is extremely important. To recapitulate, the teacher must be able to look ahead on behalf of the pupil. Whilst leading students to face new challenges, the teacher has to provide help, yet refraining from dropping back into the instruction mode.

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Acknowledgments

The project is supported by the University of Hong Kong Committee on Research and Conference Grants (CRCG).

Appendix 1

1. (a) Without the help of calculator, sketch the graphs of the following functions in the same x-y coordinates plane. (Your graphs should clearly show all the important features of the functions.)

(i) $f(x) = \dfrac{1}{x+2.5}$ (ii) $f(x) = \dfrac{1}{2.5-x}$

(iii) $f(x) = \dfrac{1}{x+4.5}$ (iv) $f(x) = \dfrac{1}{4.5-x}$

(b) Use the calculator to plot the graphs of the four functions in (a) above in the same graph window. Check to see if your sketches in (a) are correct. If not, please use a red pen to make necessary correction(s) to your sketches.

(c) (i) Predict the behaviour of the functions $f(x)=\dfrac{1}{x-a}$ and $g(x)=\dfrac{1}{a-x}$ where a can be any real number. (You can illustrate your predictions by sketching graphs for the functions with different values of a.)

(ii) What is the geometrical relation between the graphs of $f(x)$ and $g(x)$?

(iii) As the value of a changes, how do the graphs of $f(x)$ and $g(x)$ change?

Appendix 2

1. Consider the function $f(x)=\dfrac{x^3-1}{x^2-1}$.

(a) List all possible vertical asymptotes for $f(x)$.

(b) (i) Use the calculator (set xres=1 in ♦[WINDOW]) to investigate the behavior of $f(x)$ near $x=1$ and $x=-1$.

(Zoom in the graph a few times and use Trace toexplore how $f(x)$ behaves near these points.) Record your findings.

(ii) Discuss the differences and similarities between the two cases. Does the concept of asymptote apply in these cases? Explain.

2. Use the same function $f(x)$ as in 1. above.

(a) Use the calculator (set xres=1 in ♦[WINDOW]) to investigate the behavior of $f(x)$ for very large value of x and very small value of x. (Zoom out the graph a few times and use Trace to explore how $f(x)$ behaves in these cases.) Is there any horizontal asymptote? Record your findings.

(b) Discuss the differences and similarities between the two cases. Does the concept of asymptote apply in these cases? Explain.

3. Use what you've discovered so far, give a definition of an asymptote to a function.