

A Procedural Approach to Explorations in Calculus

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The goal of the workshop “On Tools and Classrooms: New approaches to algebra and calculus using innovative software.” is to explore the mathematical and educational rational for the design of the software and the activities that support the approach of the “Visual Mathematics” algebra and Calculus curriculum¹. In the workshop we will concentrate on three major segments of the curriculum: modeling, algebraic manipulations and procedural approach to calculus. Here we articulate some meaning for the third segment².

1 GRAPHIC AND SYMBOLIC TECHNOLOGY AND THE REFORM OF THE CALCULUS COURSE.

There is something worth noting about the way humans approach mental activity of all sorts. All human languages have grammatical structures that distinguish between noun phrases and verb phrases. They use these structures to express the distinction between objects, and the actions carried out by or on these objects. Mathematics has traditionally dealt with objects such as numbers, shapes, functions, etc. and the actions that it is possible to carry out on and with these objects. In general we can formulate these actions as procedures that may be carried out with mathematical objects of interest. In the past, mathematics education has not always paid great attention to the distinction between mathematical objects and the procedures that are carried out on and with them. Indeed, attention was largely focused on the *results* of standard procedures without adequate regard for the procedures themselves as objects of study. Technology makes possible a dramatic amplification of the possibility of designing both new forms of standard procedures as well as exploring the effects of entirely new procedures. The

¹ The information about software packages, the rational for the design and selection of activities can be found in: <http://www.visual-math.com/>

² More about the approach towards qualitative modeling and Algebra can be found in: Schwartz, J.L., Yerushalmy, M. (1995) “On the need for a Bridging Language for Mathematical Modeling”. For the Learning of Mathematics, 15.

Yerushalmy, M, Gilead, S. (1997) “Solving Equation in a Technological Environment: Seeing and Manipulating” Mathematics Teacher, National Council for Teachers of Mathematics. 90 (2), 156-163.

Yerushalmy, M. (1997) “Reaching the Unreachable: Technology and the Semantics of Asymptotes”, International Journal of Computers for Mathematics Learning.2, 1-25.

programmable calculator has provided us with the opportunity to devise procedures that can be executed on the mathematical object we know as number. The Geometric Supposer³ (and its descendents) allow us to devise and execute procedures on geometric shapes of all sorts. In this paper we present and discuss ideas that arose from the use of a computer environment that allows users to design and explore procedures on the objects of early calculus, i.e., functions of one variable.

We take the position that the subject of calculus is in the main a study of the mathematical objects called functions along with the procedures that may be carried out on and with functions.⁴ It should be pointed that normally the student of calculus is not asked to think about the function as an object in its own right until the study of differential equations.

Appropriate computer environments⁵ provide the opportunity to construct new forms of traditional procedures as well as entirely new procedures that may or may not be mathematically interesting but may well be pedagogically valuable.

2 A FEW EXAMPLES:

2.1. How Does a Function Change: About calculation of the curvature:

In the previous section we describe a problem in which students are asked to investigate the properties of an alternative measure of the rate of change of a function. We illustrate this sort of task here in some detail.

The problem we pose to students is one of defining in a mathematical way a property that is clear to all of them perceptually. The graphs of some functions seem to curve more than others and even within the graph of a single function, the “curvy-ness” of the function can often be seen to change from place to place. Thus we start with a modeling situation that can be dealt with geometrically or algebraically but soon enough lends itself to the study of some of the fundamental ideas of calculus and the need for symbolic language that can be manipulated. Here is a version of the problem we posed to our students:

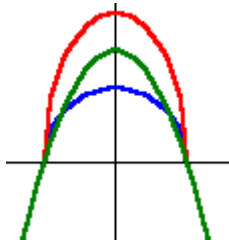
³ The Geometric Supposer is a software geometric construction tool that was developed by Schwartz and Yerushalmy 1985.

⁴ We choose the *function* as the fundamental object of study rather than the *relation* for pedagogic reasons. In most instances it is not possible to provide a single instance of an ordered pair that satisfies a relation. In contrast the value of a function, particularly one of the sort likely to be encountered in a calculus course, can be computed for given values of its argument(s).

⁵ It is clear that powerful general purpose programming languages permit this sort of activity as do a variety of special purpose software environments designed especially for the learning of mathematics in general and algebra and calculus in particular. In most of these environments, however, this kind of activity can be carried out easily by very few users who have mastered the required programming skills.

Here are three curves. Their functional expressions are

$[1-x^2]^{1/2}$	half-circle
$[4-4x^2]^{1/2}$	half-ellipse
$(3/2)[1-x^2]$	parabola



Which of these functions do you regard as the most “curvy”? the least “curvy”? Why?

Do you think that each curve should have a single number to characterize its “curvy-ness” or do you think that the measure of “curvy-ness” should change from place to place along the curve? Why do you think what you think?

We all remember from our own student days that there exists a canonical definition of the curvature of a function. It is also clear that expressions other than $\frac{|F''|}{(1 + F'^2)^{3/2}}$ may

well capture some of the properties of the “curvy-ness” of a function that we are able to detect perceptually. Thus we could imagine students producing a variety of different answers to this problem, each presumably having some merit and possibly some disadvantages.

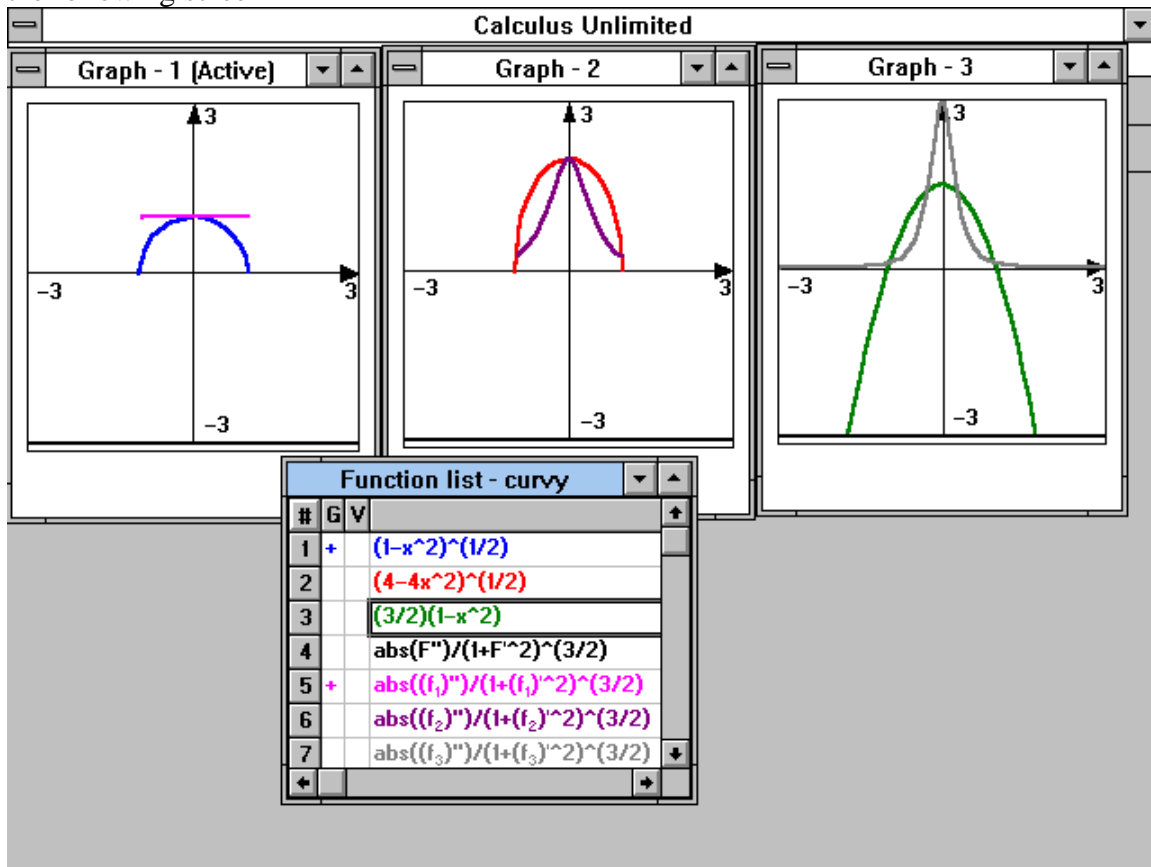
Indeed, we found that students made a variety of different measures of curvy-ness. Some defined curvyness geometrically in terms of the length of a chord compared to the length of arc along the curve joining the two end points of the chord. Needless to say, this notion of a measure of curvyness can be readily translated into symbols and then manipulated symbolically into some convenient form. Some, who felt that curvy-ness was somehow a measure of the rate of change of direction of the curve drew upon the only readily available tool they had for characterizing rate of change, i.e. the derivative with respect to the independent variable (in this case - the variable x).

For these three functions, that share roots and a symmetry line, it might seem that a perfectly reasonable measure of curvyness is the distance from the origin. Such a measure would yield constant curvyness for the half circle and measures of curvyness for the half-ellipse and parabola that vary from place to place.

Analytically, this measure of curvyness can be thought as a procedure that operates on a function $F(x)$ to produce a measure of curvyness, say $C(x)$. The procedure⁶ in this instance is $\left[x^2 + F^2\right]^{\frac{1}{2}}$

Similarly, the canonical definition of curvature can also be thought of as a procedure that takes a function as its argument, in the case of $\frac{|F''|}{(1 + F'^2)^{\frac{3}{2}}}$.

If we now posit an environment in which a function can be dealt with as an entity and not simply evaluated at individual points, then the level of abstraction that the student may reach is higher and the level of understanding is deeper. To illustrate this point consider the following screen⁷



⁶ It is clear, however, using this measure of curvyness any two curves that intersect have the same measure of curvyness at the point of intersection.

⁷ The "Calculus Unlimited" (Schwartz, Yerushalmy 1996) environment which is used in the following screens offers two ways to use functions as an argument of procedures. Procedures can be entered with functions indicated by a capital letter (as in line 4 in the next screen). The software can graph and analyze the procedure only once it gets specific expressions (in x). The other way is to input expressions in x and formulating procedures which its arguments are functions called by the list number (as in line 5 below which uses function #1, f_1 , as the argument of the expression). It is then possible to replace the expression in line #1 with other functions and view the outcome in the graph of line 5.

In lines 1 to 3 of the function list one can see the three functions that the student has been given to work with in the problem. The graph of each of them is plotted in the corresponding graph window, as well as the plot of their curvature.

Line 4 in the function list shows a procedure on functions - in this case the procedure is the one that operates on a function to produce the canonical measure of curvature. This procedure $\frac{|F''|}{(1+F'^2)^{3/2}}$ has been executed (in line 5 which reads `abs(f1'')/(1+f1'^2)^(3/2).`)

on the function on line 1 which is known to the software automatically as f_1 . Similarly, lines 6 and 7 have been generated by executing the same curvature procedure on the other two curves defined in lines f_2 and f_3 .

The various procedures for quantifying curvature that the students offered were all derived from their geometric intuitions. Each of them constitutes an explicit model of the concept of curvature. The models are different, and not consistent with one another. These inconsistencies become apparent in the context of a learning environment that allows the students to express their geometric intuitions formally in symbols while at the same time providing powerful graphical representations of the distinct symbolic formulations.

2.2 Approximating a Function:

Wherever one turns in the calculus one finds that the material can readily be cast in the form of procedures on functions. Consider, for example, the subject of Taylor Series, an important part of any calculus course. A Taylor Series expansion is a strategy for obtaining better and better approximations to a function at a point by constructing a polynomial whose coefficients are successive derivatives of the function at that point. The Taylor expansion can be thought of as a procedure that takes two arguments, the first being the function that is being approximated and the second is the location of the point about which the Taylor expansion is taking place.

The procedure that generates the first several terms in the expansion can be written as

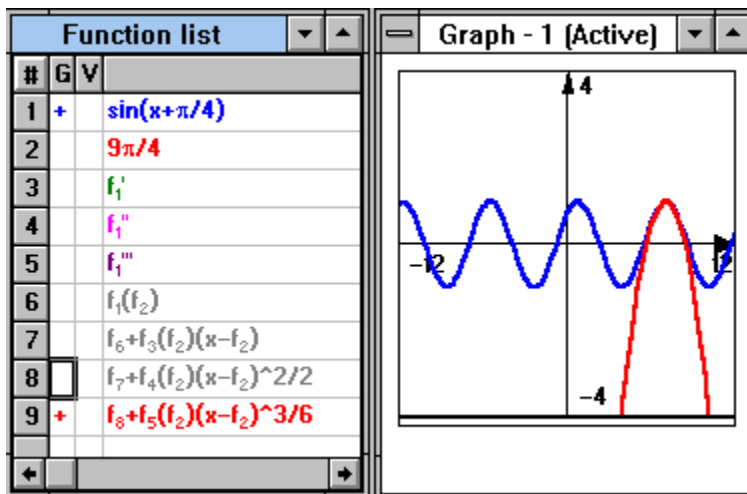
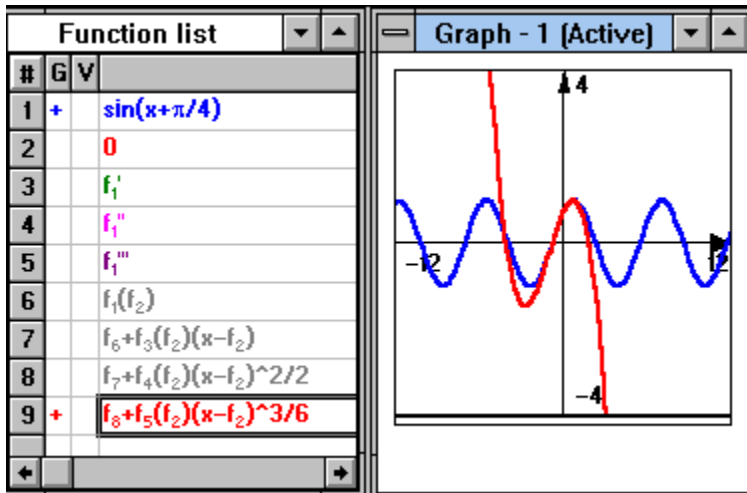
$$F[g(x), a] = g(a) + g'(a)(x - a) + g''(a)\frac{(x - a)^2}{2!} + g'''(a)\frac{(x - a)^3}{3!} + \dots$$

which can be written in the software environment described above in the form

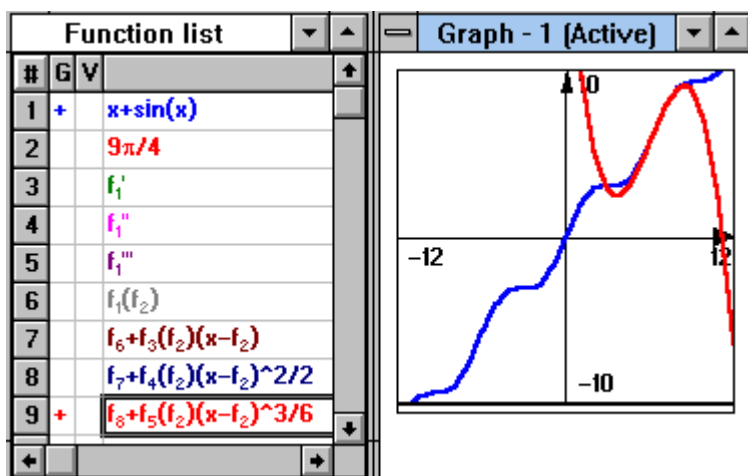
f_1	$\sin(x)$	
f_2	0	
f_3	f_1'	
f_4	f_1''	
f_5	f_1'''	
f_6	$f_1(f_2)$	
f_7	$f_1(f_2) + f_3(f_2)*(x - f_2)$	$= f_6 + f_3(f_2)*(x - f_2)$
f_8	$f_1(f_2) + f_3(f_2)*(x - f_2) + f_4(f_2)*(x - f_2)^2/2$	$= f_7 + f_4(f_2)*(x - f_2)^2/2$
f_9	$f_1(f_2) + f_3(f_2)*(x - f_2) + f_4(f_2)*(x - f_2)^2/2 + f_5(f_2)*(x - f_2)^3/6$	$= f_8 + f_5(f_2)*(x - f_2)^3/6$

The first line being $g(x)$, the second is the value a , the next three lines include intermediate computations and the last four lines are the first four Taylor expansions.

The power of formulating the subject in terms of procedures on functions can now be seen quite explicitly. Consider the following three screens⁸. The first shows the function $\sin(x+\pi/4)$ and the first four terms of the Taylor expansion of this function about the point 0. In the second screen we shift the expansion point to $9\pi/4$ by simply changing the entry on line 2 of the Function list. In the third screen we change the function from $\sin(x+\pi/4)$ to $x+\sin(x)$ by changing the entry in line 1 of the Function list.



⁸ The user would look at multi color screen which helps to visually indicate the function and its approximation.



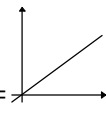
Making a Taylor expansion of a function at a point is a quite common activity in calculus classes. To be sure, such activities are enhanced by both graphing calculators and computer graphing software. What we are suggesting here goes beyond these activities. We are suggesting that a Taylor expansion procedure, with a given number of terms, and about a given point in the domain, will be a “better” approximation for some functions than it will be for others. The activity we propose allows students to inspect and analyze the degree to which a given order Taylor expansion is appropriate for a function. Clearly a four term Taylor expansion is adequate for a cubic polynomial over its entire domain, it might be acceptable for a region that is as wide as π for the function $\sin kx$, if $k=1$ but over how wide a region might one be happy with a four term expansion if $k=100$? We are immediately confronted with the “locality”⁹ of the Taylor expansion. Such considerations of the properties of procedures [in this instance a four term Taylor expansion] can turn approximation into an activity of analyzing, comparing and even inventing new methods of approximation.

2.3. Tools for Syntax and Exploration - Developing insight for solving differential equations

For those who are concerned with students being able to use their mathematics to model the world in which they live, the formulation of calculus in terms of procedures on functions will be a welcome perspective. This is because the models we very frequently build to describe aspects of processes in the natural world require us to relate functions to their rates of change. Thus we tend to formulate our models in terms of differential equations. However, quite aside from the issue of modeling, the formulation of the subject in terms of procedures, in general, and the topic of differential equations, in particular, provides a special opportunity to use the technology for mathematical exploration and the development of insight into the behavior of functions.

⁹ By contrast, approximation procedures that are global in nature, such as Fourier or wavelet analysis provide an interesting way of extending student's insight into the variety of available strategies for approximating a function for different purposes.

A major difficulty with differential equations is the rarity of analytically soluble equations. An instructive analogy to algebraic equations may be drawn. Most algebraic equations are also not analytically soluble. However, when we must have solutions, we devise ways of searching intelligently for the roots that we cannot determine analytically. Root finding strategies of all sorts depend on devising a measure of what it means to approach a root as well as an initial approximation that is consistent with that strategy. Much of this can be automated, but students are almost certainly better served if they are required to decide on a reasonable initial guess as well as a reasonable region of the domain in which to search for a root. Graphic presentation of algebraic equation as a comparison of two functions, either provided by software or later on internalized as a visual thinking process, supports this intelligent search of solutions. Graphic technology that supports the writing and manipulating of procedures on functions permits us to approach the problem of solving differential equations in much the same way. Allowing the student to see the effect of a procedure in the form of a differential operator on various families of functions is an excellent way to build an analogous kind of intuition about what sort of function might be needed to solve a particular differential equation. Let us consider the following example which asks to solve differential equation stated “semi-graphically”.

$$\left[1 + \frac{d}{dx}\right]F = \text{graph of } y = x$$


This problem can be readily stated in terms of procedures on functions. We seek a function that can serve as the argument of the procedure

$$\left[1 + \frac{d}{dx}\right]F \quad \text{see footnote}^{10}$$

and that the result of the procedure operating on this function is a linear function. One might infer from the simple differential equation

$$\frac{dF}{dx} = g(x)$$

that a solution for F is obtained by integrating the function $g(x)$. However there is a whole family of solutions to this simple differential equation as is seen from the fact that one can add an arbitrary constant to the integral.

It is our experience that presented with this differential equation

¹⁰ $\left[1 + \frac{d}{dx}\right]F$ is sometimes written as $F + F'$ or $F' + F$

$$\left[1 + \frac{d}{dx}\right]F = \text{graph of } y=x$$

the particular solution, $F(x)=a(x-1)$ (assuming the right side of the differential equation is of the form ax) is easily found. Is there a family of solutions in this case as well? and if so, what are the other solutions? When presented with this question most people try adding a constant to the solution. This clearly does not work in this case. Stated in terms of procedures we need to find what family of functions when acting as an argument to this procedure always produces zero. In this case the necessary family is Ae^{-x} . The presence of an arbitrary constant A allows us to match a boundary condition and thus to choose one of the family of solutions. It is precisely what we do in the case of a simple integral, i.e.

we seek a family that when it is the argument of the procedure $\frac{dF}{dx}$ produces zero; in that case the family is clearly a family of constants.

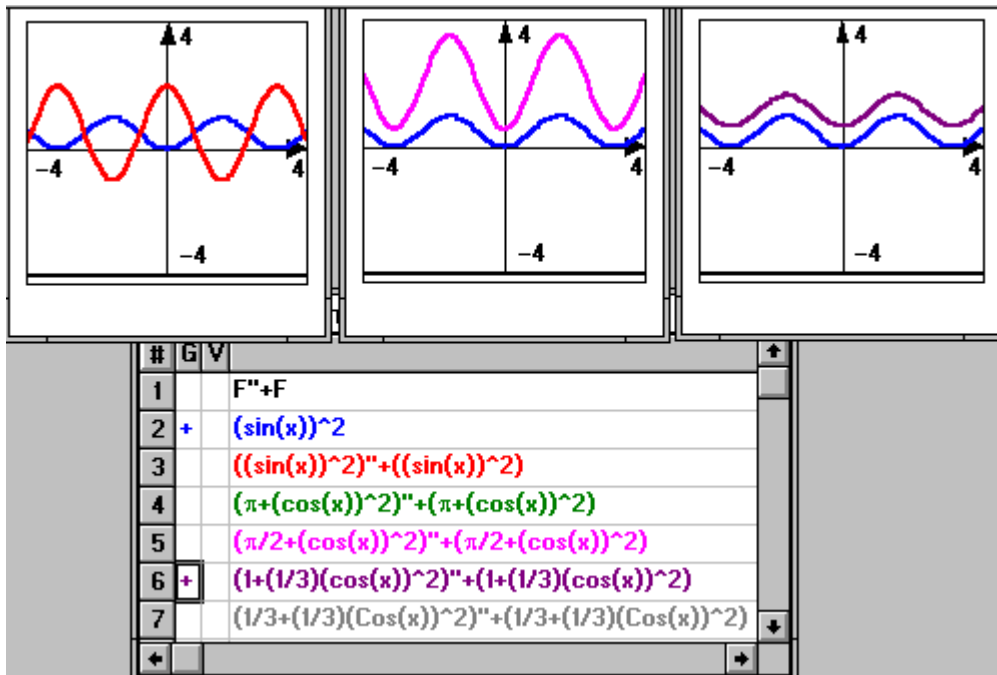
Consider next the differential equation : $\left[\frac{dF}{dx}\right]^2 + 2F\frac{dF}{dx} + F^2 = x^2$.

A bit of exploration leads to the conclusion that the procedure on the left hand side generates a quadratic function when operating on a linear function. Thus a constructive attempt to solve this differential equation might well be to look for a linear particular solution.

What about the solution to the homogeneous equation - i.e. what kinds of functions as arguments of this procedure will lead to an outcome of zero? We may have already solved this problem. In the linear case we discovered that the procedure $\frac{dF}{dx} + F$ leads to a zero result when operating on the family Ae^{-x} . The procedure in the present case looks remarkably like the “square” of the linear procedure. If we can understand the sense in which this is so, we will immediately know the homogeneous solutions to our non-linear differential equation.

As a final example in this section we consider the equation $\frac{d^2 F}{dx^2} + F = \sin^2 x$. The following screen shows an exploration into the behavior of the procedure $F''+F$ which helps approach the particular solution to the equation while developing a visual sense to the effects of the operator $\left[\frac{d^2}{dx^2} + 1\right]F$. In the leftmost graph window we see the result of the procedure operating on the trial function $\sin^2 x$ as well as the function $\sin^2 x$ that we would like the result to be. The result seems to be something like $\cos^2 x$ shifted vertically. The center graph window shows a second attempt to find a function which when operated on by this procedure produces $\sin^2 x$. Using what was learned from the first try we now

look at the effect of the procedure on the function $\frac{p}{2} + \cos^2 x$. This try seems to have moved us “closer” to the desired function. In the rightmost graph window we show the result of the trial function $\frac{p}{2} + \frac{\cos^2 x}{3}$. Here we have diminished the amplitude of variation of the trigonometric term. The result seems to be what we need save for the fact that it is shifted vertically. The particular solution¹¹ to the equation is $\frac{1 + \cos^2 x}{3}$.



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$$F'^2 + F^2$$

$$F'^2 - F^2$$

which lead to the circular and hyperbolic trigonometric functions when we demand that the result of the procedures produce the constant function 1. Another interesting pair of procedures on functions is $[F + F']^2 - [F - F']^2$ and $4FF'$ see footnote¹² and the remarkable fact that these procedures seem to be equivalent. Further, it is worth thinking

¹¹ The full solution requires the addition of the family $A \sin x + B \cos x$.

¹² The reader is cautioned not to identify the first of these procedures with

$$\left[\left[1 + \frac{d}{dx} \right] \cdot \left[1 + \frac{d}{dx} \right] - \left[1 - \frac{d}{dx} \right] \cdot \left[1 - \frac{d}{dx} \right] \right] F$$

long and hard about the possible meaning of the equivalence of two procedures on functions.

Recent approaches to the teaching of differential equations have relied heavily on graphical representations e.g. direction fields. These representations are valuable and can lead to greater insight where applicable. We offer here yet another and more general way of gaining insight into the behavior of differential equations and the systems modeled by them. By formulating the problem in terms of a differential operator on an as yet undetermined family of functions, which can then be applied to a variety of candidate functions, one can test intuitions about the behavior of the system being modeled.

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3. WHAT ROLES DOES TECHNOLOGY PLAY IN THESE EXAMPLES?

We have shown that the subject of calculus can be cast in terms of procedures on functions. To some extent the examples we have shown provide something of an argument for the desirability of doing this. In light of current trends in the reform of calculus education, it is important to consider whether this approach offers a viable and desirable addition and/or alternative to current practice.

In contrast to geometry where there are serious recommendations to change some of the objects of study, in the reform of calculus education the central object of study remains the function. However, lately it seems that we may be seeing two strands of calculus reform with and as a result of technology. One emphasizes modeling that involves differential and integral calculus and that in large measure downplays traditional symbolization. This more informal approach concentrates on the language of calculus presented qualitatively by narratives, graphical icons and simulations. In large measure, the role of technology in these cases is to bring reality into the learning environment and support it with measurement tools and analysis tools¹³. The second trend of calculus reform takes advantage of the tools that can do part of the traditional mathematics for the learner (such as symbolic manipulators and plotting packages) and as a consequence allows students to concentrate on more meaningful symbolic actions and on modeling and application tasks rather than mechanical manipulations¹⁴. We would like to suggest that the perspective described in this paper has a great deal in common with both of these major trends in calculus reform and can play a role in narrowing the apparent gap between them.

Using the calculus to build mathematical models requires the writing of procedures on/with functions that relate quantities and their rates of change in a small region of their variation. In contrast, an algebraic model requires us to understand directly at the outset how a quantity varies over its full range of variation. It is often far simpler conceptually to build the kind of local model that the calculus permits. Using the language of the calculus,

¹³ The approach is documented in Kaput (1997), Yerushalmy (1997), Monk & Nemirovsky (1994).

¹⁴ Two developments in this direction are of the Harvard Calculus (Hughes-Hallet, Gleason 1994) and North Carolina Calculus (1995).

especially when cast in the form of procedures on functions allows us to express the essence of the phenomenon being modeled without requiring us, at first, to be overly specific about details. One may say that this is also the disadvantage of this perspective because learners need the opportunity to deal with formal notations in detail in order to explain and prove. This discrepancy between the intuitions regarding local models and the ability to encapsulate the process with an expression is a major obstacle in learning calculus. Rather than being a springboard to learn calculus, intuitions about real world phenomenon are kept separate from the formal calculus. As Thompson (1994) points out, students and teachers in calculus courses are engaged in "symbol speak" which includes a lot of talk about symbols with no interpretation of the notation itself. We suggest that the need and ability to treat the function as an object and to use procedures on this object in order to model local changes makes a meaningful "symbol speak" natural and important part of the learning of calculus.

Does this perspective really require the use of technology? We suggest here three special contributions that technology has to offer in narrowing the gap between "formal" and "informal" approaches to calculus; all directly related to the procedures on functions approach suggested here.

By virtue of its ability to manipulate symbols formally the technology permits us to provide learners with environments that immediately identify inappropriate and/or incorrect uses of formal notation. Further, users of technological environments are required to state clearly and formally, just what objects they wish to consider and just what actions they wish to carry out on these objects. Moreover, in environments in which users build models in terms of procedures on and with functions, they are obliged to be clear about the issue of whether their procedures are global or local descriptions of the phenomena they are modeling. All of this requires developing a facility with both the syntax and the semantics of the subject.

At the same time, because the function is made into a directly manipulable object that can be translated, squeezed, stretched and reflected, the learner has a rich opportunity to develop intuitions about these objects and the effects of different procedures on them. Finally, we wish to make a point about teaching for understanding: The approach we have presented, with functions as objects of procedures, allows us to build curricula in a powerful and interesting way - namely to pose questions either in terms of known functions and procedures to be determined or in terms of known and well-established procedures and as yet undetermined functions. This kind of learning environment and the use of this kind of pedagogy has been repeatedly called for by mathematics educators. We have come to expect that such environments could be part of the elementary school mathematics classroom where students have intuitions derived from their daily experience. We expect the role of the teacher and the curriculum to be one that creates learning tools and environments that offer ways to construct mathematics knowledge based on these intuitions. In contrast, in the calculus classroom we have never had many tools that can support construction of knowledge in this same way. It is our experience that teaching in the way we have outlined here is the basis for a more suitable calculus curriculum for those students who need mathematics to analyze the world but need to build their understanding on the basis of experience. By making functions into manipulable objects,

we offer them a world of mathematical experiences on which to base intuitions, and, ultimately formal symbolic skill.

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