Semantic Nets for Solving Word Problems

(SPA - A computerized scheme-based environment)

Pearla Nesher

The University of Haifa

1. The role of schemes

My paper analyses the role of schemes in the learning of word problems in mathematics. I will report about our attempts to employ schemes in our design of computerized environment for solving word problems (SPA- Schemes for Problem Analysis). In our case SPA serves as a semantic net organized around a scheme that lends itself to a mathematical structure.

Rumelhart writes "…schemata (schemes) truly are the building blocks of cognition. They are fundamental elements upon which all information processing depends. Schemata are employed in the process of interpreting sensory data (both linguistic and non linguistic) in determining goals and sub goals, in guiding the flow of processing the system" (Rumelhart, 1980) pp.33-34).

Fischbein (Fischbein, 1997) thinks that a sceme is also a strategy for solving a certain class of problems .He stresses the behavioral aspect of scheme: for him, it is a plan for action. Using a scheme in solving word problems is, in our view, a mapping between the semantic relations underlying a given situation (described in natural language) and its mathematical structure. The process by which a mathematical structure that model the situation is reached, does not proceed in one direction only, from the linguistic level or the formal mathematical structure, but is rather an ongoing interactive and parallel process. The solver interacts simultaneously on all levels of interpretation available to him. The task is ultimately to be able to relate a pre-assumed structure which is well defined in terms of its arguments and its inter-relation to a diffuse situation described vaguely in natural language.

I will demonstrate this interactive process with the help of a very simple example:

Problem 1:

There are 16 boys and 13 girls in the group. How many children are there in the group?

A situation that calls for one additive binary operation when described in natural language has minimally two complete components that describe the sets and their extensions, and one set that is described by merely its set description. Finding its extension is usually the mathematical task.

The text of Problem 1 consists of 3 components (propositions in the underlying structure):

1) *16 boys in the group* - **A complete component** - it consists of a set description (‘boys’) and its extension (‘13’).

2) *13 girls in the group* - **A complete component**, as above)

Please note that the information presented so far cannot help determine which binary operation to choose. One can now ask various questions depending on one’s focus or interest. For example: “How many more boys than girls are there? “, or “How many different couples of a boy and a girl can be arranged?”. Each of the above questions will lead to a different mathematical model. Thus, it is the question component which is an **incomplete** component (missing the extension) that is necessary in order to model the situation. Actually it is a three argument relation and not a binary relation that we are looking for even in such a simple situation.

The incomplete component is decisive in determining the mathematical operation to be used. Any modeling that employs binary operations is based on the semantic relations among the three sets described in the text and not on their extensions. Understanding the triple relation: ‘boys’, ‘girls’ and ‘the group’ is a necessary condition for modeling it. In this case it is the semantic knowledge that ‘boys’ and ‘girls’ are disjoint sets, and that the ‘boys’ and the ‘girls” are included in ‘the group’. Formally it can be presented as follows: If B stands for ‘boys’ , G for girls and P for the ‘group’, the following formal conditions hold:

1) B and G are disjoint sets.

2) B is a subset of P

3) G is a subset of P

4) Any element belonging to P is either B or G.

It is on the basis of the relation between B, G and P that the modeling is uniquely determined and not on the basis of their extensions. Moreover the triple relation is an additive relation and whether it will call for the addition operation or the subtraction operation is a negligible issue.

It is well established now that by cognitive researchers that difficulties students face in modeling word problems depend on whether the schemes needed for solving them are available to the those students (Carpenter, Moser, & Romberg, 1982; De Corte, 1987; Greer, 1993; Kintsch, 1968; Mayer, ; Nesher, 1976; Nesher, Greeno, & Riley, 1982b; Reusser, 1990; Riley, 1983; Verschaffel, 1993). Because of time limitation I intend to concentrate here on the additive structures, yet similar analysis was performed for the multiplicative structure (Nesher, 1988; Schwartz, 1986a; Vergnaud, 1988).

SPA that will be described later (Hershkovitz, Nesher, & Yerushalmy, 1990) was based on the above analysis and it aims at making the underlying structures which are mathematical elements of modeling explicit to the student. SPA will be contrasted with similar programs that aim at helping the student in modeling by emphasizing the binary operation.

2. Schemes that underlie the modeling competence

Schemes that underlies the modeling competence develop slowly. As the child starts to describe the world of various sets with numbers, he employs mainly the "**predication**" and "**cardinality**" schemes (Nesher et al., 1982b). This kind of schemes enable to solve various types of problems, counting all, each time from the beginning. Later on the child is able to **link** different sets by cause and effect and to anticipate results of actions described in ordinary language. He constructs the **change scheme** that enables him to model additional problems that preserve the events order.
Next, the child is able to construct a **Part-Part-Whole scheme** that can be used to represent set relations with a slot for an unknown quantity for a set that was defined merely by its description. In mathematics at this level, the additive structure is reversible and includes the = sign as denoting an equivalent relation. The Part-Part-Whole scheme is reversible and also incorporates the arithmetic additive relationship which now includes the operations + and — as related to inverse operations operating on the same structure. Additional advanced schemes are further constructed by the student and they enable him to cope with more complicated situations that incorporates more advanced mathematics.

In the description of the above developmental levels we assume that there are at least two sources of knowledge which are involved in modeling : (a) A child's knowledge of the world, and (b) A child's knowledge of logico-mathematical structures. The sources of these two knowledge structures, as was noted by Piaget, are not the same. The logico-mathematical growth of the child cannot, of course, be understood as divorced from his experience with physical objects. Yet the mechanism for that growth is different, as indicated by Piaget's reference to 'simple abstraction' and 'reflective abstraction' (Piaget, 1971 (1967))

I will not elaborate more on the development of early schemes (See, (Nesher et al., 1982b). However, once the scheme of Part-Part-Whole is available to the child, as well as the additive structure among number-triples, partial information of a given scheme can be represented with a slot for the unknown quantity. At this stage it is meaningful to the student.

The ability to solve problems such as <u>Problem 2</u> at this level brings in one of the most powerful predictions of our theoretical analysis.

<u>Problem 2:</u>
Dan had some marbles.
He found 5 more marbles.
Now he has 8 Marbles.

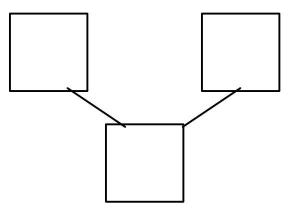
How many marbles did he have to start with?

In this kind of problems, the semantic schemes that originated in the child's experience with ordinary language ***contradict*** the newly learned semantics of addition and subtraction (+ and —). The child's experience with natural language will direct him to add ('found' means 'adding'). Choosing to subtract (for the correct solution) can be achieved only if the semantics of natural language and the mathematical language are differentiated as two autonomous systems, so that each one of them can be further elaborated to reach the necessary coordination between the two systems. Solving problem 2 involves interpreting the 'initial state', the 'change' and the final state' of the above problem in a non-temporal manner as in a part-part-whole relationship. Since one part and the whole are given, finding the second part is achieved by subtraction. Thus, at this level, the student should be able to make the mapping between his natural language knowledge and the mathematical knowledge, *not* on the basis of isolated verbal cues, but rather on the basis of the understanding of the underlying semantics of both languages and coordinating the schemes originated in the two distinct realms.

3. More complex mathematical structures

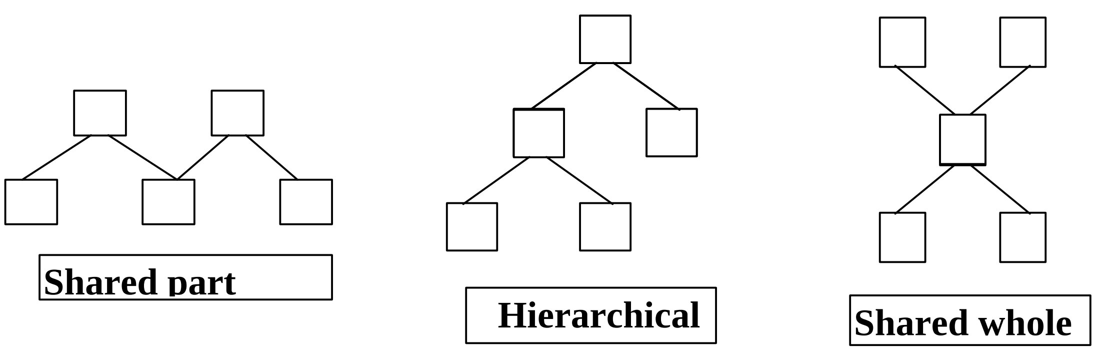
Once the child has reached the Part-Part- Whole scheme, he has already acquired an autonomous mathematical additive structure that will serve him for all the contexts in which addition and subtraction are required. This structure can be depicted in a diagram consisting of three related components (see Fig.1). Note, that regardless the fact that one component is an incomplete component, each component has a defined role is the additive relation (structure). In the diagram the two upper boxes represent the subsets, while the bottom box represents the union of these two subsets.

Figure 1



Having now the above scheme encapsulated as a mathematical object we can construct higher mathematical hierarchies that will serve us as schemes for more complex situations. The following schemes will demonstrate all possible situations for two-step problems (see figure 2).

Figure 2



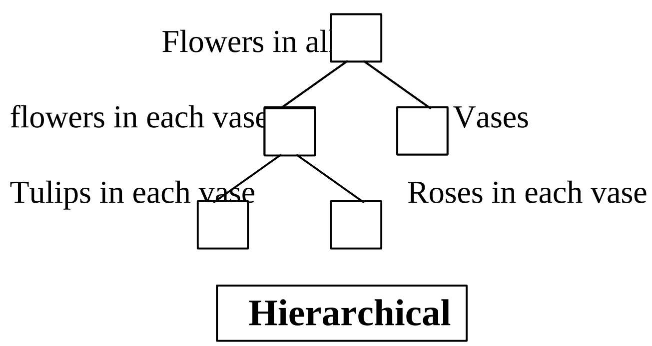
The fact that there are limited number of schemes we employ in mathematics that correspond to many situations in which they can be applied, should guide us to teach modeling in mathematics via general schemes.

I will now give some situations described by the above schemes. In each situation we can ask different question that are derived from the same situation (I)):

> ***Situation I:*** A total of X flowers is distributed equally among Y vases. In each vase there are M tulips and N roses.

About the situation described above we can ask various questions such as: "How many roses are in each vase?"; "How many vases are there?"; "How many flowers are there in all vases?"; or, "How many tulips are there in each vase?".
All the above questions share the same underlying structure. It is actually the same situation. See the following figure:

Figure 3



This was a situation described by am Hierarchical scheme. Here is a situation described by the "Shared Whole" scheme. **Situation II:**
In the class there are X children. Y of them are boys and the rest are girls. They were divided into M equal groups. How many children were in each group?

Situation III will illustrate the case of "Shared Part" scheme. **Situation III:**
X children went to the party. M were boys and the rest were girls. At the end of the party, Y flowers were left and they were given to the girls. Each girl got the same number of flowers. How many flowers did each girl get?

Situations II and III, of course, can be elaborated into other problems, just as I have detailed in Situation I. All the above situations share some common characteristics, that are detailed elsewhere:(Hershkovitz & Nesher, 1996; Hershkovitz & Nesher, 1997; Nesher & Hershkovitz, 1994).

SPA was studied (Hershkovitz & Nesher, 1997) with a comparison to another program **AP** - Algebraic Proposer (Schwartz, 1986a). The AP program assists in making lists of the given information. It is sequential in nature. The solver attends to each given piece of information (the numbers and their descriptions) and makes a list of these in order to operate on them. Each element appears in AP as a line in a table with three headings: 'How many?' in tending to capture the given number; 'What', intending to capture the units of measurement; and 'Notes' that intend to describe the situation's element.
The following problem demonstrates how it works:.

> *All the six graders of a school were divided into 7 groups. In every group there are 17 children of whom 9 are boys. How many girls are there?*

The student starts with filling in the table with the given information.
See 1a.

Table 1a:

How many	What	Notes
A: 7	groups	equal groups
B: 17	children per group	
C: 9	boys per group	

After filling in the table it is possible to define the mathematical operations between any two lines in the table. In our example the first operation will be: B - C . Before performing any numerical operation the program ask to operate first on the units of measurement. In this case the computer will announce that the unit of the result is children per group (adding and subtracting demand the same units). On approving that, the computer will add a new line to the table, namely, D.

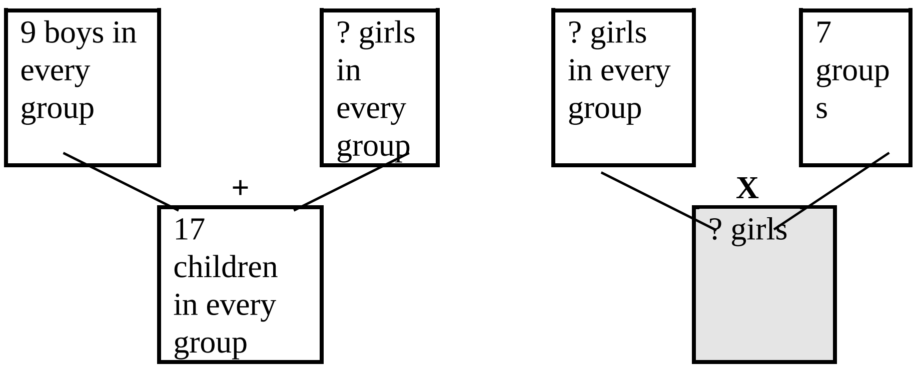
Table 1b:

How many	What	Notes
A: 7	groups	equal groups
B: 17	children per group	boys and girls
C: 9	children per group	boys
D: 8	children per group B-C	girls

The next step will be finding the number of the girls. In order to do it the solver will have to multiply: A x D . In general terms, the progress is achieved by progressing each time by a binary operation on the basis of already known information.

We will illustrate with the above problem how SPA works. After reading the text the student first needs to define the unknown component (? girls) and to select the schemes that are relevant to the situation. (either the additive scheme and/or of the multiplicative scheme). Before filling the schemes slots the user has to decide about the role of each component in the entire scheme (a part, a factor, or a whole). Moreover, he has to fill in all three elements of the scheme, though some numerical information might be missing. In SPA the user cannot declare an operation unless he has comprehended all the 3-place relation depicted by the location of the three arguments in the scheme. The next step, therefore, will be filling in the schemes and marking the unknown location in the scheme.

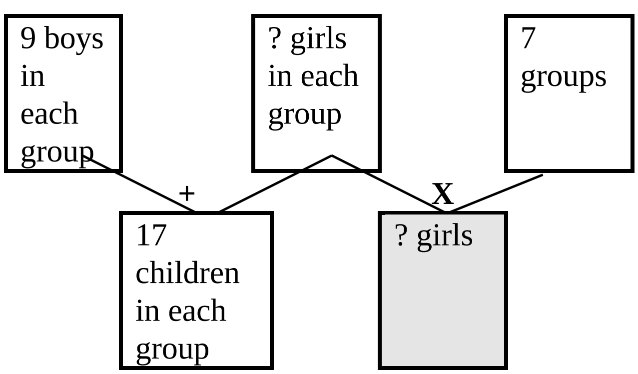
Figure 4



Additive scheme **multiplicative scheme**

Each time the computer will show one particular scheme to the child and will ask him about the relevant units similar to the AP procedure. The schemes will now be merged by the student via their shared component. Note that the shared component was not mentioned by the text, and must be deduced by the student.

Figure 5



The solver can now solve the problem by solving only one exercise:
(17-9) X 7= 56 or by solving two exercises: 17 - 9 = 8; 8 X 7= 56

4. The Characteristics of SPA

I am mainly interested in the characteristics of SPA but will to some extent compare it to AP in order to highlight the innovative aspects of SPA.

One) SPA shares with AP the fact that the information given in a natural language text is depicted in terms of three distinct sub-fields: (i) the numerical value, (ii) The description of the sets (the predication), and (iii) Additional note about the situation described in the text of the problem.

Two) In SPA as in AP the computer relates to the semantic description of the sets and not merely to their extensions (numbers).

Three) In SPA in contrast to AP the building blocks are three-place relation (and not binary operations). The information in SPA is encoded in a graph that has fixed slots for different roles in the entire structure. This can be contrasted with similar tree graphs that use the same graph for binary operations (the upper slots are always for the input and the third is for the output).

Four) Once the analysis of the needed scheme is completed and its arguments filled up, the computer performs the numerical calculations .

Five) Since a scheme (when filled-up) is a close structure, the computer is able to supply feedback in a general mode and not only to pre-solved problems that were fed into it.

SPA is use in Israeli schools and a Ph.D. thesis by Dr. Sara Hershkovitz was devoted to compare two programs SPA and AP that aim to teach word problem solving. This is reported elsewhere (Hershkovitz & Nesher, 1996). The main findings show that students working with SPA develop different strategies that lead them to a better performance. This is especially true for the low-level students.

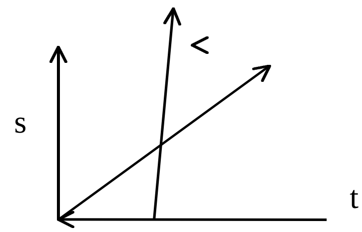
5. General schemes and open problems

To teach via schemes means to teach via the most general cases. I will present one example:

<u>*Example :*</u> *Two cars are traveling from Tokyo to visit another city. Car B is leaving X hours after car A.*

1) Will they meet on their way? Under which conditions?
2) When will they meet?
3) How far from Tokyo will they meet?
4) Make your own questions…

figure 6



In a similar way one can construct a general scheme for cars that are traveling in opposite directions and meet somewhere on the way. We can use the same schemes for other contexts such as work, voltage etc. Once we have generated a general scheme for such problems, we can easily review all the possibilities. We can learn that each problem in our standard textbooks is just one case of many others, that the singular cases are not important, it is the general scheme that counts. I have tried to demonstrate that the ability to solve problems in mathematics is dependent on the level of schemes and structures available to the students. Students can benefit best, if we are aware of the schemes that are needed at each level of learning and if we present them the needed schemes in their most general form.

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