A New Algorithm Producing Generalized Sturm Sequence for Polynomials with Symbolic Coefficients*

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Abstract

As we know, Sturm sequence plays a very important role in real roots classification of algebraic equations, but it is very difficult to compute the Sturm sequence for polynomials with symbolic coefficients. In [GLRR89], González L. et al introduced the Sturm-Habicht sequence which generalized Habicht's work [H48], the polynomials in this sequence are multiple of polynomials of the Sturm sequence with some changes of signs and repetitions. In [YHZ96], Yang L. et al established a complete discriminant sequence from the principal coefficients of subresultant chain. Both of the above sequences are easier to compute compared with the conventional approach, but these sequences are not the Sturm sequence in usual sense because there might be zero-polynomials in these sequences and some revision of signs are needed. In this paper, for a polynomial with symbolic coefficients, we discuss how to produce its generalized Sturm sequence by subresultant chain aided by computer. The obtained sequence contains no zero-polynomials and no revision of signs is needed, it's clear that our sequence is the Sturm sequence properly. As a main theorem in our paper, the sign relation of generalized Sturm sequence and subresultant chain is found and proved firstly, and an algorithm is given for computing the leading coefficients of the generalized Sturm sequence only by principal subresultant coefficients for the infinite interval case.

1 Introduction

Let I be an integral domain, $I[\mathbf{u}, x]$ be the polynomial ring over I, $\mathbf{u} = u_1, \cdots$. $A, B \in I(\mathbf{u}, x)$ be polynomials of degree m and n respectively with

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$m \geq n$ on x, which can be written as $A = \sum_{i=0}^{m} a_i x^i, B = \sum_{j=0}^{n} b_j x^j$, here $a_0, \cdots, a_m, b_0, \cdots, b_n \in I[\mathbf{u}]$. In this section, we introduce some basic notions and the definition of subresultant chain and principal resultant coefficients, then state two fundamental results about subresultant and pseudo remainder sequence [L82].

Let $A_i = \sum_{j=0}^{n_i} a_{ij} x^j, 1 \leq i \leq k$, be a sequence of polynomials in $I[\mathbf{u}, x]$. Then the $k \times l$ matrix associated with $A_1, \cdots, A_k$ is $\mathrm{mat}(A_1, \cdots, A_k) = (a_{i,l-j})$, where $l = 1 + \max_{1 \leq i \leq k}(n_i)$.

Let M be a $k \times l$ matrix over $I[\mathbf{u}, x]$, $k \leq l$. The *determinant polynomial* $A = \mathrm{detpol}(M)$ is

$$|M^{(k)}| x^{l-k} + \cdots + |M^{(l)}|,$$

where $M^{(j)}$ denotes the submatrix of M consisting of the first $k-1$ columns and the jth column of $M, k \leq j \leq l$.

Definition 1 *Let* $A = \sum_0^m a_i x^i, B = \sum_0^n b_j x^j \in I[\mathbf{u}, x]$ *with* $\deg(A) = m > 0$ *and* $\deg(B) = n > 0$. *For* k, $0 \leq k < \min(m, n)$, *set*

$$\begin{aligned} M_k &= \mathrm{mat}(x^{n-k-1}A(x), \cdots, A(x), x^{m-k-1}B(x), \cdots, B(x)) \\ &= \begin{pmatrix} a_m & a_{m-1} & \cdots & a_0 & & & \\ & a_m & \cdots & & a_0 & & \\ & & \ddots & & & \ddots & \\ & & & a_m & & \cdots & a_0 \\ b_n & b_{n-1} & \cdots & b_0 & & & \\ & b_n & \cdots & & b_0 & & \\ & & \ddots & & & \ddots & \\ & & & b_n & & \cdots & b_0 \end{pmatrix}, \end{aligned}$$

Then we call $S_k = \mathrm{detpol}(M_k)$ *the* k*-th subresultant of* A *and* B, *and call the leading coefficient*[1] $R_k = \mathrm{lc}(S_k)$ *the* k*-th principal subresultant coefficient.*

In particular, if $m = n + 1$, we may extend the definition of $S_{n+1} = A$, $S_n = B$ and call

$$S_{n+1}, S_n, S_{n-1}, ..., S_0$$

the subresultant chain of A and B.

Note also that S_0 is the resultant of A and B, and in general S_k is a polynomial of

$$\deg(S_k) \leq (m + n - k) - (m + n - 2k) = k$$

since M_k has $m + n - 2k$ rows and $m + n - k$ columns. It can be easily seen that $\deg(S_k) = k$ if and only if $\det(M_k^{(k)}) \neq 0$. For convenience we have the following definition.

[1] that is, the coefficient of the highest-degree term of polynomial S_k.

Definition 2 *Let* $A, B \in I[\mathbf{u}, x]$ *be two polynomials of degree* m *and* n*, and* S_k $(0 \leq k < \min(m,n))$ *be the* k*-th subresultant of* A *and* B *defined as in Definition 1. We call* S_k *is defective of degree* k *if* $\deg(S_k) = r < k$ *and regular otherwise.*

Theorem 1 *Let* $S_{n+1}, S_n, \cdots, S_0$ *be a subresultant chain in* $I[x]$ *of* S_{n+1} *and* S_n*. Let* S_{j+1} *be regular and* S_j *be defective of degree* $r < j$ *(with* $\deg(0) = -1$*). Then*

$$S_{j-1} = S_{j-2} = \cdots = S_{r+1} = 0, \quad -1 \leq r < j < n, \tag{1}$$

$$R_{j+1}^{j-r} S_r = \mathrm{lc}(S_j)^{j-r} S_j, \quad 0 \leq r \leq j < n, \tag{2}$$

$$(-1)^{j-r} R_{j+1}^{j-r+2} S_{r-1} = \mathrm{prem}(S_{j+1}, S_j), \quad 0 < r \leq j < n. \tag{3}$$

Where $\mathrm{prem}(S_{j+1}, S_j)$ *stands for the pseudo remainder of* S_{j+1} *and* S_j*.*

The next result gives the general relation of polynomial remainder sequence to the subresultant chain. (See [L82] for the definition of polynomial remainder sequence).

Theorem 2 *Let* $A_1, \cdots, A_r$ *be a polynomial remainder sequence over* $I[\mathbf{u}, x]$ *satisfying for* $e_i, f_i \in I[\mathbf{u}]$*, both non-zero*

$$e_i A_i = Q_i A_{i+1} + f_i A_{i+2} \qquad (1 \leq i \leq r-2)$$

Let $n_1, n_2, \cdots, n_r$ *be the degree sequence and let* $c_1, c_2, \cdots, c_r$ *be the leading coefficient* $\mathrm{lc}(A_i)$ *sequence of* $(A_1, \cdots, A_r)$*. For any* j*,* $2 \leq j \leq r-1$*,*

$$S_k = 0, \qquad 0 \leq k < n_r, \quad n_{j+1} < k < n_j - 1,$$

$$\left\{ \prod_{i=1}^{j-1} e_i^{n_{i+1}-n_j+1} \right\} S_{n_j-1} = \left\{ \prod_{i=1}^{j-1} (-1)^{(n_i-n_j+1)(n_{i+1}-n_j+1)} f_i^{n_{i+1}-n_j+1} c_{i+1}^{n_i-n_{i+2}} \right\} \times c_j^{-n_j+n_{j+1}+1} A_{j+1} \qquad (1 < j < r) \tag{4}$$

$$\left\{ \prod_{i=1}^{j-1} e_i^{n_{i+1}-n_{j+1}} \right\} S_{n_{j+1}} = \left\{ \prod_{i=1}^{j-1} (-1)^{(n_i-n_{j+1})(n_{i+1}-n_{j+1})} f_i^{n_{i+1}-n_{j+1}} c_i^{n_i-n_{i+2}} \right\} \times c_{j+1}^{n_j-n_{j+1}-1} A_{j+1} \qquad (1 < j < r) \tag{5}$$

2 Generalized Sturm Sequence and Subresultant Chain

The goal of this section is to establish a sign relation of generalized Sturm sequence with the subresultant chain of a given polynomial and its derivative. Tarski's concept of generalized Sturm sequence is a very important tool in automated theorem proving, real algebraic geometry and many other applications [T51].

Definition 3 *Let* $A_1, A_2 \in I[\mathbf{u}, x]$. *The* generalized Sturm sequence*(G.S.S. for short) of* A_1 *and* A_2 *is a polynomial remainder sequence* $A_1, A_2, ...$ *satisfying*

$$e_i A_i = Q_i A_{i+1} + f_i A_{i+2}, \quad e_i = c_{i+1}^{2*(n_i - n_{i+1}+1)}, \quad f_i = -1$$

where $Q_i \in I[\mathbf{u}, x]$, $n_i = \deg(A_i)$, $c_i = \mathrm{lc}(A_i)$.

For convenience, we introduce a similar relation for two polynomials as following.

Definition 4 *Let* $A, B \in I[\mathbf{u}, x]$, A *and* B *are called* similar, $A \sim B$, *if and only if there exist non-zero* $a, b \in I[\mathbf{u}]$ *such that* $a^2 A = b^2 B$.

It is clear that similar is an equivalence relation, and satisfies the following two properties:

1. If $A, B, C, D \in I[\mathbf{u}, x]$ and $A \sim B$, $C \sim D$, then $A \cdot C \sim B \cdot D$.
2. If $A, B \in I[\mathbf{u}, x]$ and $A \sim B$, then $lc(A) \sim lc(B)$.

The following is our main result.

Theorem 3 *Let* A_i *be the generalized Sturm sequence defined above,* S_k *be the k-th subresultant of* A_1 *and*A_2. *Then*

$$S_{n_j-1} S_{n_j} \sim (-1)^{n_1 - n_j} \left\{ \prod_{i=1}^{j-2} (c_i c_{i+1})^{n_i - n_{i+2}} \right\} A_j A_{j+1}$$

Proof: Let $\delta_i = n_i - n_{i+1}$. According to the definition of Generalized Sturm Sequence and Theorem 2, we have

$$\prod_{i=1}^{j-1} c_{i+1}^{2(\delta_i+1)(n_{i+1}-n_j+1)} S_{n_j-1} = \prod_{i=1}^{j-1} (-1)^{(n_i-n_j+1)(n_{i+1}-n_j+1)} f_i^{(n_{i+1}-n_j+1)} c_{i+1}^{n_i - n_{i+2}} \times c_j^{-n_j+n_{j+1}+1} A_{j+1}, \tag{6}$$

$$\prod_{i=1}^{j-2} c_{i+1}^{2(\delta_i+1)(n_{i+1}-n_j)} S_{n_j} = \prod_{i=1}^{j-2} (-1)^{(n_i-n_j)(n_{i+1}-n_j)} f_i^{(n_{i+1}-n_j)} c_i^{n_i - n_{i+2}} \times c_j^{n_{j-1}-n_j-1} A_j. \tag{7}$$

It is clear that

$$\prod_{i=1}^{j-1} c_{i+1}^{2(\delta_i+1)(n_{i+1}-n_j+1)} S_{n_j-1} \sim S_{n_j-1},$$

$$\prod_{i=1}^{j-2} c_{i+1}^{2(\delta_i+1)(n_{i+1}-n_j)} S_{n_j} \sim S_{n_j},$$

and therefore

$$\left\{\prod_{i=1}^{j-1} c_{i+1}^{2(\delta_i+1)(n_{i+1}-n_j+1)} S_{n_j-1}\right\} \cdot \left\{\prod_{i=1}^{j-2} c_{i+1}^{2(\delta_i+1)(n_{i+1}-n_j)} S_{n_j}\right\} \sim S_{n_j-1} S_{n_j}.$$

On the other hand, by substituting $f_i = -1$ into the right sides of (6) and (7) and multiplying the results, we have

$$\begin{aligned}
& \left\{\prod_{i=1}^{j-1} (-1)^{(n_i-n_j+1)(n_{i+1}-n_j+1)} (-1)^{(n_{i+1}-n_j+1)} c_{i+1}^{n_i-n_{i+2}} \times c_j^{-n_j+n_{j+1}+1} A_{j+1}, \right\} \\
& \cdot \left\{\prod_{i=1}^{j-2} (-1)^{(n_i-n_j)(n_{i+1}-n_j)} (-1)^{(n_{i+1}-n_j)} c_i^{n_i-n_{i+2}} \times c_j^{n_{j-1}-n_j-1} A_j\right\} \\
= & \left\{\prod_{i=1}^{j-2} (-1)^{(n_i-n_j)(n_{i+1}-n_j+1)} c_{i+1}^{n_i-n_{i+2}}\right\} \cdot (-1)^{n_{j-1}-n_j} c_j^{n_{j-1}-n_j+1} A_{j+1} \\
& \cdot \left\{\prod_{i=1}^{j-2} (-1)^{(n_i-n_j+1)(n_{i+1}-n_j)} c_i^{n_i-n_{i+2}}\right\} \cdot c_j^{n_{j-1}-n_j-1} A_j \\
\sim & \ (-1)^{n_{j-1}-n_j} \left\{\prod_{i=1}^{j-2} (-1)^{2(n_i-n_j)(n_{i+1}-n_j)+n_i+n_{i+1}-2n_j} (c_i c_{i+1})^{n_i-n_{i+2}}\right\} A_j A_{j+1} \\
= & \ (-1)^{n_{j-1}-n_j} \left\{\prod_{i=1}^{j-2} (-1)^{n_i+n_{i+1}} (c_i c_{i+1})^{n_i-n_{i+2}}\right\} A_j A_{j+1} \\
= & \ (-1)^{n_1-n_j} \left\{\prod_{i=1}^{j-2} (c_i c_{i+1})^{n_i-n_{i+2}}\right\} A_j A_{j+1}
\end{aligned}$$

Hence,

$$S_{n_j-1} S_{n_j} \sim (-1)^{n_1-n_j} \left\{\prod_{i=1}^{j-2} (c_i c_{i+1})^{n_i-n_{i+2}}\right\} A_j A_{j+1},$$

as claimed in Theorem 3.

3 Leading Coefficients of G.S.S. and Principal Subresultant Coefficients

In many cases, people only concern about the real root classification of a polynomial in the whole interval $(-\infty, +\infty)$, the case for closed interval $[a, b]$ can also be transformed into infinite interval case. In fact, the Sturm sequence of a polynomial in $(-\infty, +\infty)$ is just the leading coefficients of G.S.S., and the subresultant chain corresponds to principal subresultant coefficients. Let $x \to \infty$ in Theorem 3, we get the following corollary for the leading coefficients of G.S.S. and principal subresultant coefficients.

Corollary 1 *Let $\{R_j\}$ be the principal subresultant coefficients of a given polynomial and its derivative, and $\{c_j\}, \{n_j\}$ be the leading coefficients and degrees of the generalized Sturm sequence respectively, then*

$$R_{n_j-1}R_{n_j} \sim (-1)^{n_1-n_j} \left\{ \prod_{i=1}^{j-2} (c_i c_{i+1})^{n_i-n_{i+2}} \right\} c_j c_{j+1} \tag{8}$$

for all $j \geq 2$.

Corollary 2 *For any $j \geq 2$, if the number of zeros between R_{n_j} and $R_{n_{j+1}}$ in the subresultant chain $R_{n_1}, \cdots, R_0$ is odd, then*

$$R_{n_{j+1}}R_{n_j} \sim (-1)^{n_1-n_j} \left\{ \prod_{i=1}^{j-2} (c_i c_{i+1})^{n_i-n_{i+2}} \right\} c_j c_{j+1}. \tag{9}$$

Proof: By substituting j and r with $n_j - 1$ and n_{j+1} into (2) in Theorem 1, we obtain

$$R_{n_j}^{n_j-n_{j+1}-1} \cdot S_{n_{j+1}} = \mathrm{lc}(S_{n_j-1})^{n_j-n_{j+1}-1} \cdot S_{n_j-1} = R_{n_j-1}^{n_j-n_{j+1}-1} \cdot S_{n_j-1},$$

which implies that

$$R_{n_j}^{n_j-n_{j+1}-1} \cdot R_{n_{j+1}} = R_{n_j-1}^{n_j-n_{j+1}}.$$

It follows immediately that

$$\left(R_{n_j-1}R_{n_j}\right)^{n_j-n_{j+1}} \sim R_{n_j}R_{n_{j+1}}.$$

If the number of zeros between R_{n_j} and $R_{n_{j+1}}$ in the principal subresultant chain is odd, i.e., $R_{n_j-1} \neq 0$, and

$$R_{n_j-2} = \cdots = R_{n_{j+1}+1} = 0, \quad l = (n_j - 2) - (n_{j+1} + 1) + 1 = 1 \pmod 2$$

then $n_j - n_{j+1} = l + 2 = 1 \pmod 2$. Hence, we have

$$R_{n_j-1}R_{n_j} \sim R_{n_j}R_{n_{j+1}}$$

and

$$R_{n_{j+1}}R_{n_j} \sim (-1)^{n_1-n_j} \left\{ \prod_{i=1}^{j-2} (c_i c_{i+1})^{n_i-n_{i+2}} \right\} c_j c_{j+1}.$$

Corollary 3 *If $R_j \neq 0$ and S_j is regular for all $j \geq 2$, then*

$$R_{n_{j+1}}R_{n_j} \sim (-1)^{n_1-n_j} c_j c_{j+1}. \tag{10}$$

Proof: Because all $R_j \neq 0$ and S_j is regular, then $R_{n_j-1} = R_{n_{j+1}}$ and hence $n_j - n_{j+2} = 2$, by Corollary 1, we have

$$R_{n_{j+1}}R_{n_j} \sim (-1)^{n_1-n_j} c_j c_{j+1}.$$

4 Algorithm and Implementation

In this section, we give an algorithm for a polynomial to produce the sequence of $\{c_j\}$ by its non-zero principal subresultant coefficients $\{R_{n_j}\}$.

Before stating our algorithm, we firstly demonstrate how to obtain the sequence of $\{n_j\}$ from the all principal subresultant coefficients $\{R_j\}$. By Theorem 1, it is obvious that any n_j should satisfy $R_{n_j} \neq 0$ and $R_{n_j - 1} \neq 0$. Therefore, all zero elements and all non-zero elements of which successors are zeros in $\{R_j\}$ must be removed, the resulting sequence is just the sequence $\{R_{n_j}\}$. The following example is given to explain above procedure. Suppose the given R_j sequence is:

$$[R_{15} \quad R_{14} \quad R_{13} \quad 0 \quad 0 \quad 0 \quad R_9 \quad R_8 \quad 0 \quad 0 \quad R_5 \quad R_4 \quad R_3 \quad 0 \quad 0 \; 0],$$

in which $R_{12} = R_{11} = R_{10} = R_7 = R_6 = R_2 = R_1 = R_0 = 0$.

Because the successors of R_{13} ,R_8 and R_3 are R_{10}, R_6 and R_2 respectively, we remove above zero elements and R_{13}, R_8 and R_3, get

$$[R_{15} \quad R_{14} \quad R_9 \quad R_5 \quad R_4],$$

and

$$[n_1 \quad n_2 \quad n_3 \quad n_4 \quad n_5] = [15 \quad 14 \quad 9 \quad 5 \quad 4]$$

The following is our algorithm:

Step 0. Input a polynomial $f = a_1 x^{d_1} + \cdots + a_{d_1+1}$,
let $g = f' = d_1 a_1 x^{d_1 - 1} + \cdots + a_{d_1}$, and $c_1 = a_1, c_2 = d_1 a_1$
and $R_{d_1} = a_1, R_{d_1 - 1} = d_1 a_1$.

Step 1 Compute $[R_{d_1}, \cdots, R_0]$, the principal subresultant coefficients of f and g.

Step 2. Remove all zero elements and all non-zero elements of which successors are zeros in $[R_{d_1}, \cdots, R_0]$, get $\{n_j\}$ sequences.

Step 3. Construct all $c_{j+1}, 2 \leq j$ according to the recursive formula given in Corollary 1.

Besides, if $\{R_{n_j}\}$ satisfies the condition of Corollary 2 or Corollary 3, above algorithm can be simplified. For explaining clearly the algorithm in defective case, we give a polynomial with constant coefficients as an example.

Example 1. Consider the following polynomial of degree 16:

$$f(x) = x^{16} + x^{15} - x^{14} - x^3 + 2x - 1$$

Step 1: The principal subresultant coefficients of $f(x)$ and $f'(x)$ are

$$\begin{aligned} & [1, 16, -47, -70, 910, 0, 0, 0, 0, 0, 0, -57101150470, 186542099873397, \\ & \quad -153071917876088452, -5443258205091847394 4, \\ & \quad 92865907763658073193 04, -56734970566507496834 40] \\ \sim \quad & [1, 1, -1, -1, 1, 0, 0, 0, 0, 0, 0, -1, 1, -1, -1, 1, -1] \end{aligned}$$

Step 2: Computing n_j for $j = 1, 2, \cdots$,

$$n_1 = 16, n_2 = 15, n_3 = 14, n_4 = 13, n_5 = 5, n_6 = 4, n_7 = 3, n_8 = 2, n_9 = 1, n_{10} = 0$$

Step 3: Computing c_j recursively by using Corollary 1:

$$\begin{aligned}
\text{Initial:} \quad & c_1 = 1, \\
& c_2 \sim 1; \\
(1): \quad & c_2 c_3 \sim (-1)^{n_1 - n_2} R_{n_2 - 1} R_{n_2} = (-1) \cdot R_{14} R_{15} \sim 1, \\
\Rightarrow \quad & c_3 \sim 1; \\
(2): \quad & c_3 c_4 \sim (-1)^{n_1 - n_3} R_{n_3 - 1} R_{n_3} \cdot (c_1 c_2)^{n_1 - n_3} \sim R_{13} R_{14} \sim 1, \\
\Rightarrow \quad & c_4 \sim 1; \\
(3): \quad & c_4 c_5 \sim (-1)^{n_1 - n_4} R_{n_4 - 1} R_{n_4} \cdot \prod_{i=1}^{2} (c_i c_{i+1})^{n_i - n_{i+2}} \sim (-1) \cdot R_{12} R_{13} \sim 1, \\
\Rightarrow \quad & c_5 \sim 1; \\
(4): \quad & c_5 c_6 \sim (-1)^{n_1 - n_5} R_{n_5 - 1} R_{n_5} \cdot \prod_{i=1}^{3} (c_i c_{i+1})^{n_i - n_{i+2}} \sim (-1) \cdot R_4 R_5 \sim 1 \\
\Rightarrow \quad & c_6 \sim 1 \\
(5): \quad & c_6 c_7 \sim (-1)^{n_1 - n_6} R_{n_6 - 1} R_{n_6} \cdot \prod_{i=1}^{4} (c_i c_{i+1})^{n_i - n_{i+2}} \sim R_3 R_4 \sim -1 \\
\Rightarrow \quad & c_7 \sim -1 \\
(6): \quad & c_7 c_8 \sim (-1)^{n_1 - n_7} R_{n_7 - 1} R_{n_7} \cdot \prod_{i=1}^{5} (c_i c_{i+1})^{n_i - n_{i+2}} \sim (-1) \cdot R_2 R_3 \sim -1 \\
\Rightarrow \quad & c_8 \sim 1 \\
(7): \quad & c_8 c_9 \sim (-1)^{n_1 - n_8} R_{n_8 - 1} R_{n_8} \cdot \prod_{i=1}^{6} (c_i c_{i+1})^{n_i - n_{i+2}} \sim R_1 R_2 \sim -1 \\
\Rightarrow \quad & c_9 \sim -1 \\
(8): \quad & c_9 c_{10} \sim (-1)^{n_1 - n_9} R_{n_9 - 1} R_{n_9} \cdot \prod_{i=1}^{7} (c_i c_{i+1})^{n_i - n_{i+2}} \sim (-1) \cdot R_0 R_1 \sim 1 \\
\Rightarrow \quad & c_{10} \sim -1
\end{aligned}$$

Example 2. Let $p(x)$ be the general polynomial of degree 7:

$$p(x) = x^7 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

Then the principal subresultant coefficients of $p(x)$ and $p'(x)$ are

$$\begin{aligned}
R_7 &= 1, \\
R_6 &= 7, \\
R_5 &= 14a, \\
R_4 &= 20a^3 + 63b^2 - 56ac, \\
R_3 &= -54b^4 + 207b^2ac + 189b^2e - 8b^2a^3 - 420bdc - 80ba^2d + 20ca^4 \\
&\quad -168eca + 60ea^3 + 244c^3 - 136a^2c^2 + 175ad^2,
\end{aligned}$$

R_2 is a polynomial of 51 terms,

R_1 is a polynomial of 159 terms,

R_0 is a polynomial of 320 terms.

In this example we have $n_j = 8 - j$ for $j = 1, 2, \cdots, 8$. The following is the procedure for computing c_j.

$$\begin{array}{ll}
\text{Initial:} & c_1 = 1, \\
& c_2 \sim 1; \\
(1): & c_2c_3 \sim (-1)^{n_1-n_2} R_{n_3} R_{n_2} = (-1) \cdot 14a, \\
\Rightarrow & c_3 \sim -a; \\
(2): & c_3c_4 \sim (-1)^{n_1-n_3} R_{n_4} R_{n_3} = 14a \cdot (20a^3 + 63b^2 - 56ac), \\
\Rightarrow & c_4 \sim -(20a^3 + 63b^2 - 56ac); \\
(3): & c_4c_5 \sim (-1)^{n_1-n_4} R_{n_5} R_{n_4} = (-1) \cdot (20a^3 + 63b^2 - 56ac) \cdot R_3, \\
\Rightarrow & c_5 \sim R_3; \\
(4): & c_5c_6 \sim (-1)^{n_1-n_5} R_{n_6} R_{n_5} = R_2 \cdot R_3, \\
\Rightarrow & c_6 \sim R_2; \\
(5): & c_6c_7 \sim (-1)^{n_1-n_6} R_{n_7} R_{n_6} = (-1) \cdot R_1 \cdot R_2, \\
\Rightarrow & c_7 \sim -R_1; \\
(6): & c_7c_8 \sim (-1)^{n_1-n_7} R_{n_8} R_{n_7} = R_0 \cdot R_1, \\
\Rightarrow & c_8 \sim -R_0.
\end{array}$$

This algorithm has been implemented by symbolic software Maple. Many examples we computed show that the algorithm is efficient, and practical for studying the real-root classification and proving the inequality in real algebra domain [CH91].

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