Building Dynamic Mathematical Models with Geometry Expert*
I. Geometric Transformations, Functions and Plane Curves

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1 Introduction

With computers become more and more popular, using computers to assist teaching and learning is changing from theoretical study to practice. There are lots of educational softwares in the market. But most of them are more like "electronic books," at most with sound and animation. They do not accept input from the user, let along solve problems proposed by the user. In our viewpoint, this kind of softwares does not use the full power of modern computers. They actually make little advance from a well-written paper book. Educators are setting higher standards for education softwares [10]. They want a software that can be used by teachers and students to operate and to generate something by themselves.

Geometry Expert (GEX) is built to answer the educators' call for more flexible and powerful educational softwares. GEX is a software for *dynamic geometric drawing* and *automated geometry theorem proving and discovering* [7]. With GEX, we can draw geometry diagrams and figures of functions and plane curves. We can also do geometric transformations and animation with GEX. This feature of GEX provides a powerful and convenient platform for teaching mathematical concepts in an intuitive way.

* This work is partially supported by a grant from Chinese NSF and a grant from US NSF (CCR-9420857).

GEX's theorem prover is based on recent achievements in automated geometry reasoning [12, 3, 4]. With GEX, we have proved hundreds of difficult geometry theorems and generated elegant proofs for them. Furthermore, we can use GEX to discover all properties of a diagram that can be deduced using a set of geometric axioms. With this feature, teachers can easily make exercises and test problems; students can enhance their ability of solving problems by fully exploring the properties of a given diagram. With GEX, students can also learn how to write proofs in a few lines for moderately difficult theorems. The advanced part of the system can be used by geometry experts to solve challenging problems or conjectures.

In this series of papers, we are going to explore how to use GEX to build dynamic models for teaching and learning mathematics. In this paper, we will show how to build dynamic models with geometric transformation and locus construction. These models are useful in teaching plane geometry, analytical geometry, functions, and physics.

Related Work. Symbolic computation softwares like Mathematica [11] and Maple [2] provide powerful tools of solving mathematical problems. They also provide good graphical interfaces. But, solving geometry problems is a weak point of these softwares: they cannot prove geometry theorem efficiently let along generate readable and elegant proofs as GEX does. Also, their graphical editors lack the dynamical ability of GEX. Dynamic geometry softwares Geometer's sketchpad [9] and Gabri are quite similar to the graphical part of GEX. But they cannot prove or discover geometry theorems. MathLab [14] is a series of intelligent softwares which use the same method as GEX and can be used to teach solid geometry, trigonometric functions, analytical geometry, and algebra.

2 Dynamic Models

By *dynamic models*, we mean models built by computer softwares that can be changed dynamically. In this paper, we will consider the following classes of models.

1. Geometric transformations.

2. Loci generated by diagrams constructed using ruler and compass. This class includes conics, functions $y = \frac{\sqrt[n]{f(x)}}{\sqrt[m]{g(x)}}$ where $f(x)$ and $g(x)$ are polynomials and n and m are positive integers, trigonometric functions, various curves defined in polar coordinate systems, etc.

3. Diagrams of functions using the numerical computation facility provided by the C language. This class includes functions of the form: $y = f(x)$ where $f(x)$ could be any "elementary functions"– a^x, x^a, $log(x)$, trigonometric functions – and their arithmetic expressions and compositions. This part is quite similar to most "Graphic Calculator", but is more flexible and powerful.

Dynamic models built with GEX have the following advantages.

Dynamic Transformation We can change diagrams not only by standard geometric transformations like translation, rotation, and dilation, but also by dragging points in the diagram freely and at the same time keeping all the geometric relations in the diagram intact. Using this tool, users may easily get various forms for a diagram and may check conjectures numerically. For instance, when we want to prove that three altitudes of a triangle are concurrent, we may drag one vertex of the triangle to all possible positions and observe that the three altitudes always intersect in a point. This implies strongly that this statement does true. Grünbaum and Shepherd discovered many interesting geometry properties by numerical checking [8]. Dynamic softwares like GEX certainly provide convenient tools for this kind of work.

Animation Through animation, the user may observe the generation process of curves or the figures of functions. For example, by continuously changing the three parameters a, b, c in function $y = a\sin(bx + c)$, we may get an intuitive view of the affect of these parameters on the shape of the sine function. The case for c is particularly interesting: when changing c continuously, we get a wave-like movement of the diagram for the sine function.

Generality With GEX, we can construct diagrams for almost all elementary functions and most of the commonly encountered plane curves. This seems beyond the scope of the usual graphical calculators.

More Participation Using GEX to draw diagram of functions, the users may construct the models by themselves. For one diagram, the user may give different methods to generate it. This flexibility gives students a chance to gain inside properties of a diagram and to enhance their ability of solving problems.

Openness GEX provides an open platform for the user to explore their own problems. For instance, the following problems aree not solved yet.

Problem 1. Design a linkage that can be used to draw a conic passing through five given points.

Problem 2. In a diagram drawn with ruler and compass, what kind of curves can be generated if only one driving point is allowed to move on a line or a circle.

Problem 3. In a diagram drawn with ruler and compass, what kind of curves can be generated if multiple driving points are allowed to move on lines or circles.

3 Geometric Transformations

By transformations, we mean not only standard geometric transformations like translation, rotation, and dilation, but also a powerful tool: dragging points in the diagram freely and at the same time keeping all the geometric relations in the diagram intact.

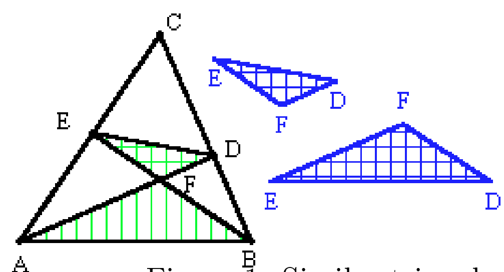

Figure 1. Similar triangles

Transformations have several important educational implications. First, we may use them to teach geometric concepts intuitively. For instance, the statement "two triangles are similar if by doing geometric transformations, they may become congruent triangles" can be made more intuitive by really doing those transformations with GEX. Figure 1 shows how to use geometric transformations to check if triangles ABF and DEF are similar, where D and E are the foot on sides BC and AC of triangle ABC.

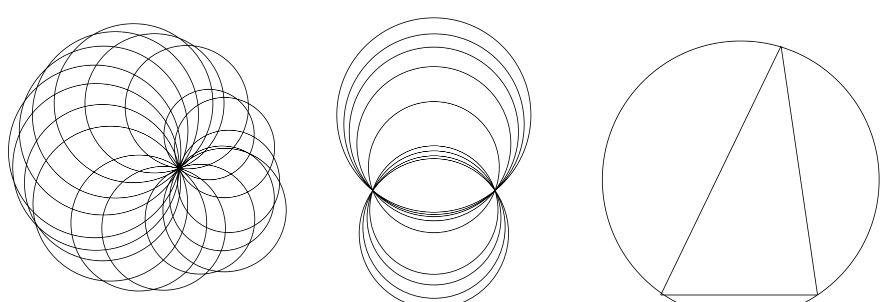

Figure 2. Circles passing through one, two, three points

Second, we may give the concept "number of solutions" an intuitive explanation. For instance, we know that "there are infinite circles passing through one or two given points" and "there is only one circle passing through three non-collinear points." By dragging the points by themselves, students may get a deeper understanding of the concept of infinity (Figure 2).

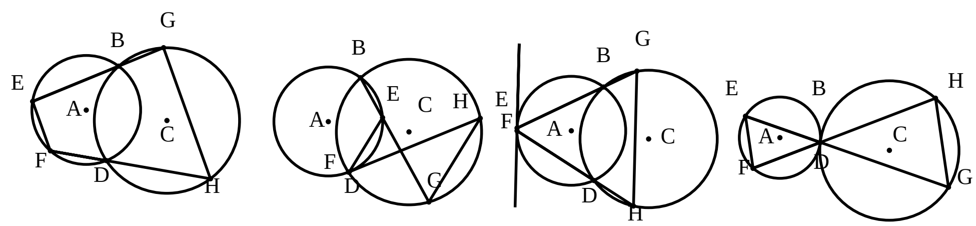

Figure 3. Multiple shapes of one theorem

Third, by dragging points in a diagram we can demonstrate how to get multiple solutions of one problem. An example used by Ms. Lu Jianmin in her classes is as follows: *Two circles A and C intersect in points B, D. Passing through B, D, two lines are drawn meeting circle A and circle C in E, F and G, H. Show that EF is parallel to GH* (Figure 3). This problem has more than ten different forms when we dragging the points in the diagram. For instance, E may become coincident with point B, and the secant EG becomes a tangent line of circle A; points E and F may become coincident and secant EF becomes a tangent line of circle A; furthermore, points B and D may become equal and the two circles are tangent to each other. For each diagram, we get a "new theorem" which is closely connected with the original one, and they also have similar proofs.

4 Loci Generation by Ruler and Compass Construction

Locus generation is a powerful tool of generating diagrams for functions and plane curves. The operation process is quite simple: first we need to draw a diagram with line (ruler) and circle (compass); then we need to select some driving points and assign to each driving point a line or circle; finally we select a point as the locus point. When the driving points move on the lines or circles assigned to them, the locus point will generate a locus.

4.1 Conics Generation

GEX has a builtin operation of generating various conics. But by drawing conics as loci according to their definitions gives students deeper insight of these concepts.

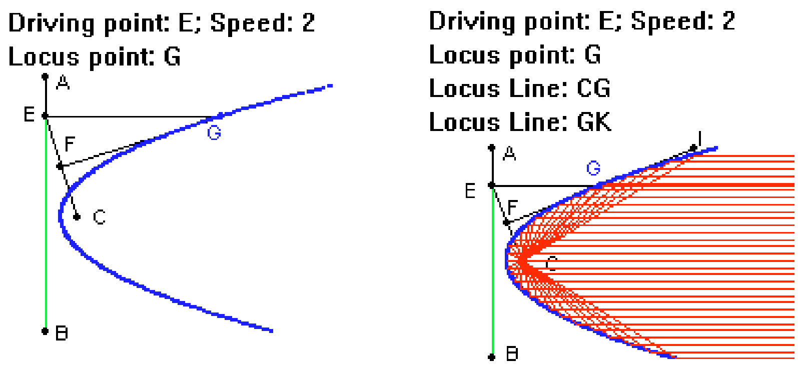

Figure 4. Parabola and its properties

Example 4.1 *A parabola is the locus of points having equal distances to a fixed point and a fixed line. In Figure 4, AB is the fixed line and C is the fixed point. E is a free point on AB; G is the intersection of (TLINE E A B) (the line passing through E and perpendicular to AB) and the perpendicular-bisector of CE. When E moves on line AB, G will give a parabola. We may also show the following physical property of a parabola: "lights coming out from point C and reflecting by the parabola will become parallel."*

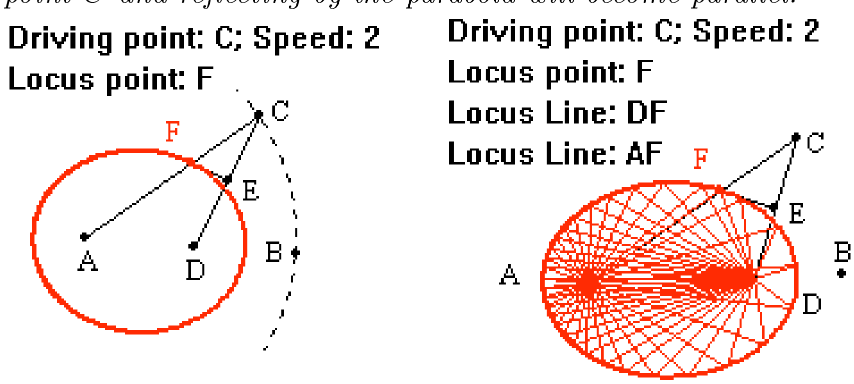

Figure 5. Ellipse and its properties

Example 4.2 *An ellipse (hyperbola) is the locus of points the sum (difference) of distances from which to two fixed points is a constant. In Figure 5, A and D are the two fixed points; C is a free point on circle(A,B); F is the intersection of line AC and the perpendicular-bisector of CD. When C moves on the circle, F will generate an ellipse. When point D is outside circle(A,B), the locus is a hyperbola. Figure 5 also shows the following physical property: "lights coming out from point D and reflecting by the ellipse will pass through point A."*

We can draw conics according to different definitions, such as: draw an ellipse if we know five points on it; draw a parabola if we know its focus point and two tangent lines; etc.

4.2 Horner's Drawing Method and Extensions

Using Horner's drawing method, we can draw the diagram of

$$y = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0.$$

The idea is that by writing the above polynomial as the form $y = (\ldots((a_n x + a_{n-1})x + a_{n-2})\ldots + a_0)$, we need only two operations, multiplication and addition, to construct the diagram of this curve.

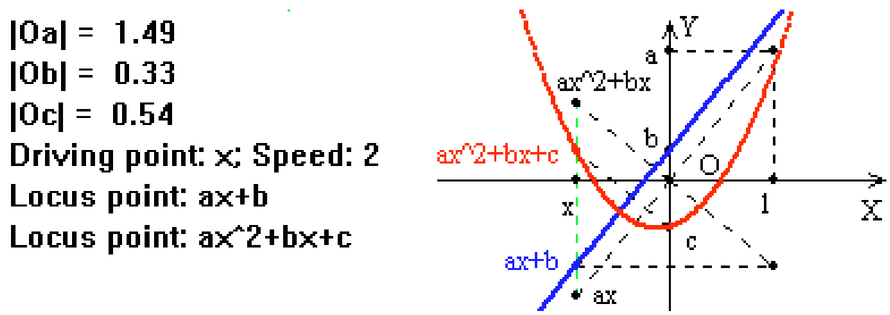

Figure 6. Linear and quadratic functions

In Figure 6, we generate the figures of $y = ax + b$ and $y = ax^2 + bx + c$. The values for a, b, c are represented by segments Oa, Ob, and Oc and hence can be changed continuously to observe the shapes of the two functions. By extending this basic method, we can draw many kinds of functions and curves:

1. Rational functions $y = \frac{f(x)}{g(x)}$. Now we need the operation of division which can be done similarly as multiplication.

2. Inverse function and particularly function $y = \sqrt[n]{f(x)}$. If we have drawn the diagram for $y = f(x)$, then the reflection of this curve with respect to the line $y = x$ is the diagram of the inverse function $y = f^{-1}(x)$.

3. Diagrams for any *rational plane curves* defined by their parametric equations.

$$x = \frac{P(t)}{R(t)}; y = \frac{Q(t)}{R(t)}$$

   where P, Q, R are polynomials in t. What we need to do is first constructing the two rational functions; then use one as the x coordinate and one as the y coordinate to construct a point on the curve.

4. **Interpolation polynomials.** For n distinct points $(x_i, y_i), i = 1, \ldots, n$, draw the diagram of a polynomial $y = f(x)$ such that $y_i = f(x_i)$. The construction is based on the following result.

Aitken Lemma. If $f(x) = g(x)$ for all $x \in T$ and $a \neq b$, then $h(x) = \frac{(b-x)f(x)-(a-x)g(x)}{a-b}$ satisfies: $h(x) = f(x) = g(x)$ for $x \in T$ and $h(a) = f(a), h(b) = g(b)$.

Let $x_1, ..., x_n$ be different, and $f_i(x) = y_i$ be constant functions. Using Aitken's lemma, we can construction $f_{ij}(x)$ which is the line joining points (x_i, y_i) and (x_j, y_j). Figure 7 shows how to construct f_{123} from f_{12} and f_{13} according to Aitken's lemma. It is clear that f_{123} is a quadratic function.

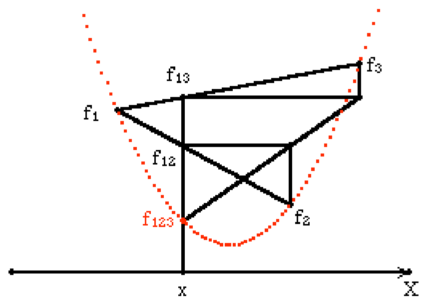

Figure 7. Interpolation

4.3 More Plan Curves

Trigonometric functions. To draw diagrams for trigonometric functions and their inverse functions, we need multiple driving points. Figure 8 shows how to draw the figure of the sine function. A, B, C are fixed points; D is a point in circle(A, B); E is s point on line AC; F is the intersection of TLINE(E, A, C) and PLINE(D, A, C) (i.e., the line passing through D and parallel to AC). When E moves on line AC and D rotates on circle(A, B), point F will generate the diagram for the sine function. We can change the shape of the sine function by changing the speed of points D and E.

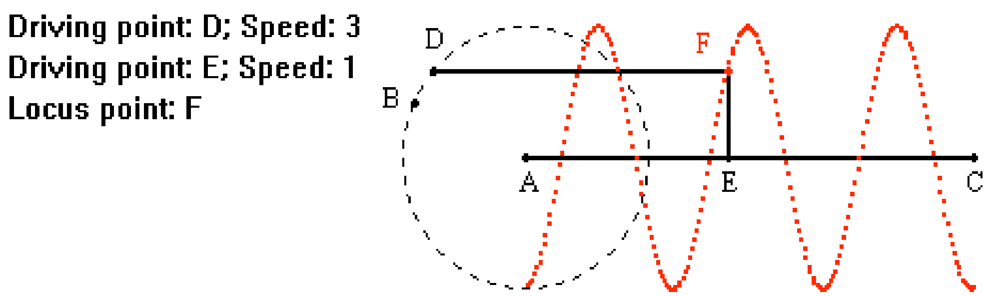

Figure 8. The sine function

Various curves defined in polar coordinate systems can be drawn with GEX using a driving point on a circle [6]. The following diagram is to generate the *locus of the moon* when the moon rotates around the earth on a circle and the earth rotates around the sun on an ellipse.

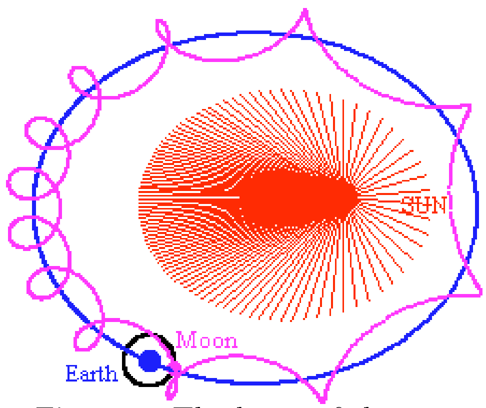

Figure 9. The locus of the moon

5 Figures of Elementary Functions

Using the numerical computation facility provided by C languages, we may draw diagrams for functions of the form: $y = f(x)$ where $f(x)$ could be any "elementary functions" defined below.

- Any real number is an elementary function.
- $f(x) = x$ is an elementary function of x.
- If f and g are elementary functions, $f + g$, $f - g$, $f * g$ f/g, f^g, $log(f)$, $sin(f)$, $cos(f)$, $tan(f)$, $asin(f)$, $acos(f)$, and $atan(f)$ are elementary functions.
- If f and g are elementary functions, $f(g)$ is also an elementary function.

This part is similar to a Graphic Calculator, but is more flexible and powerful.