About the Polynomials in Two Variables

Demonstrating the Characteristics
of Polynomials by Using a Model

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1. Summary and the Purpose of This Study

This research paper presents a new model of polynomials in two variables. By making a computer program of this model we can find many characteristics of polynomials, for example, Pascal's triangle, remainder theorem and many unexpected characteristics of polynomials.

The purpose of this study is as follows:
1)Giving a new model of polynomials
2)Showing many characteristics of polynomials by this model
3)Analyzing an educational effect of using this model

2. A Model of Polynomials in Two Variables

At first we think about a progression $\{a_{i,j}\}$ for the polynomial $\sum a_{i,j} x^i y^j$ and arrange a $\{a_{i,j}\}$ to the flat. For example, coefficients of polynomial $3x^2 y - 4xy^2 + 2xy + 4$ are arranged as follows.(fig.1) Axis of abscissa means degree of x and axis of ordinates means degree of y.
In the case of symmetrical polynomial, we see numbers are arranged symmetrically like in fig.2 and symmetrical axis is diagonal line.(i.e. y=x) In regard to the alternating polynomials, each sign of coefficients is opposite between the two sides which is separated by the line y=x.
(fig.3)

fig.1

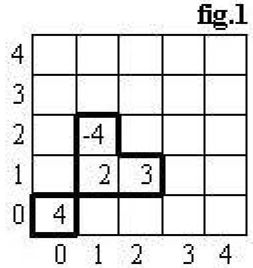


fig.2

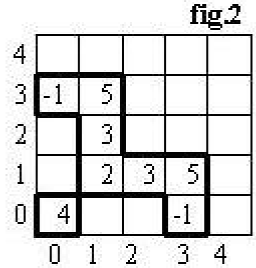
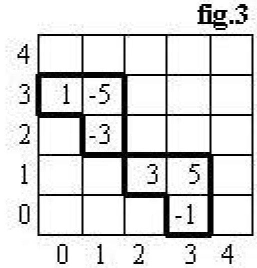


fig.3



2-1. Multiplying Monomials

For example, $3x^2 y - 4xy^2 + 2xy + 4$ multiplied by $x^2 y$ is $3x^4 y^2 - 4x^3y^3 + 2x^3 y^2 + 4x^2 y$. We can explain this process by using this model. In this model $3x^2 y - 4xy^2 + 2xy + 4$ moves 2 steps toward the axis of abscissa and moves 1 step toward the axis of ordinates.(fig.4(2)) In other

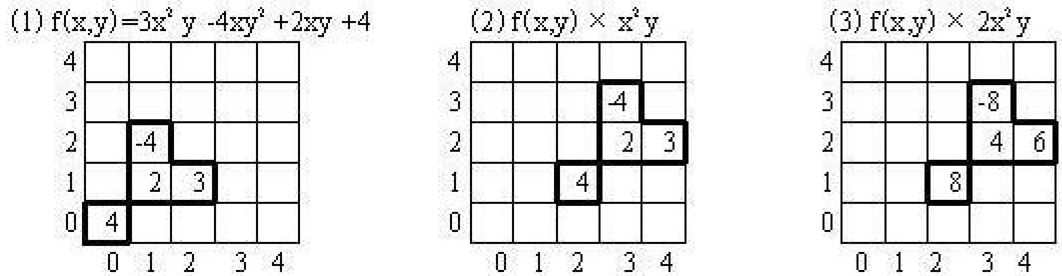


fig.4

words, multiplying a monomial in which coefficient is one means a parallel translation of these numbers. Multiplying $ax^i y^j$ means multiplying a_{ij} to the coefficients after parallel translation.
Fig.4(3) is in the case of a=2.

2-2. Multiplying Polynomials

For example, fig.6 means $x^2 y + x^2 + x + 1$ multiplied by x^2 and fig.7 means $x^2 y + x^2 + x + 1$ multiplied by xy. By using these models we can get an answer of $x^2 y + x^2 + x + 1$ multiplied by $(x^2 + xy)$ from fig.8. In this model multiplying a polynomial means the sum of some moved numbers.

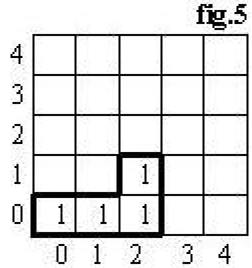


fig.5

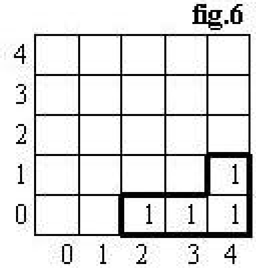


fig.6

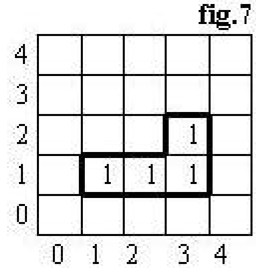


fig.7

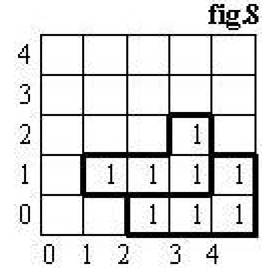
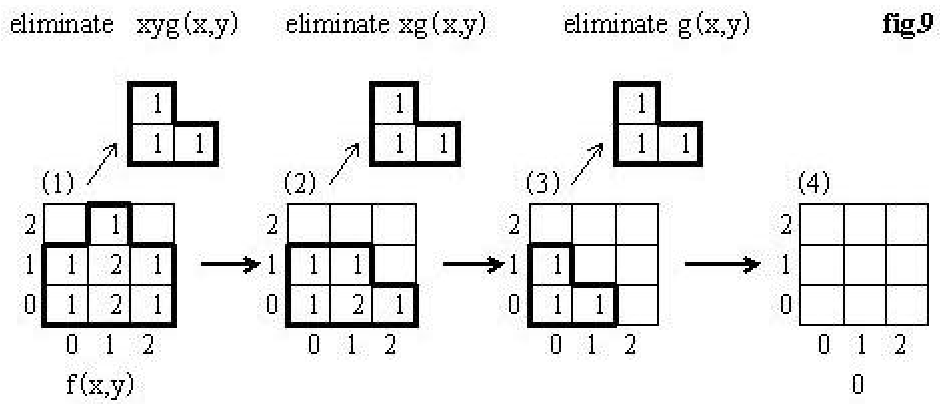


fig.8

2-3. Division of polynomials

In fig.9 we think about f(x,y)/g(x,y). At first we eliminate xyg(x,y)from f(x,y). Then we can get a polynomial of (2). Next we eliminate xg(x,y) from the polynomial of (2). Then we can get a polynomial of (3). Lastly we eliminate g(x,y) from the polynomial of (3) Then we can get a polynomial of (4). (i.e.0) By using expression this process is represented as follows.



f(x,y)-xyg(x,y)-xg(x,y)-g(x,y)=0
•ˆf(x,y)=(xy+xg+1)g(x,y)

This means the quotient of f(x,y) divided by g(x,y) is xy+x+1 and the rest is 0. Generally, dividing f(x,y) by g(x,y) means eliminating some shifted g(x,y)s.

Next we think about a division f(x,y)/g(x,y) in which the rest is not 0. For example, let f(x,y) equal $x^2y+xy^2+x^2+xy+y^2+x+y+1$ and g(x,y) equal x+y+1. We can think of two ways for eliminating shifted g(x,y)s.(fig.10,11) In fig.10 we eliminate shifted g(x,y)s so that the degree of y may be low and in fig.11 we do the same so that the degree of x may be low. Well, an operation in the case of fig.10 means a following division (1) which is arranged by y. And the other means a following division(2) which is arranged by x. These results will present that the quotient and the rest of division of porinomials in two variables are not unique.

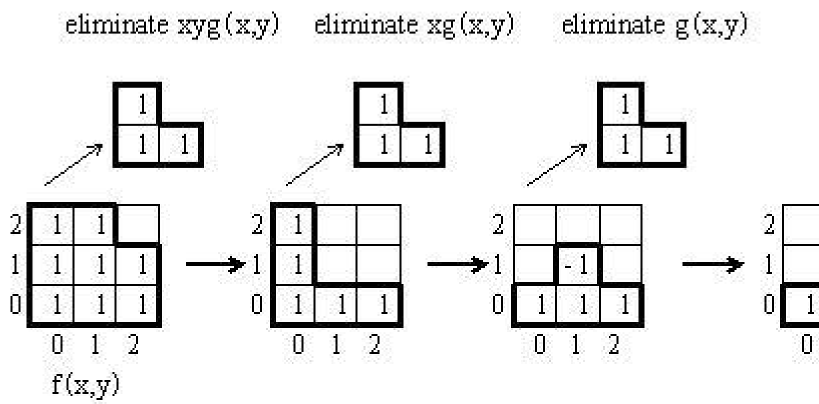


fig.10

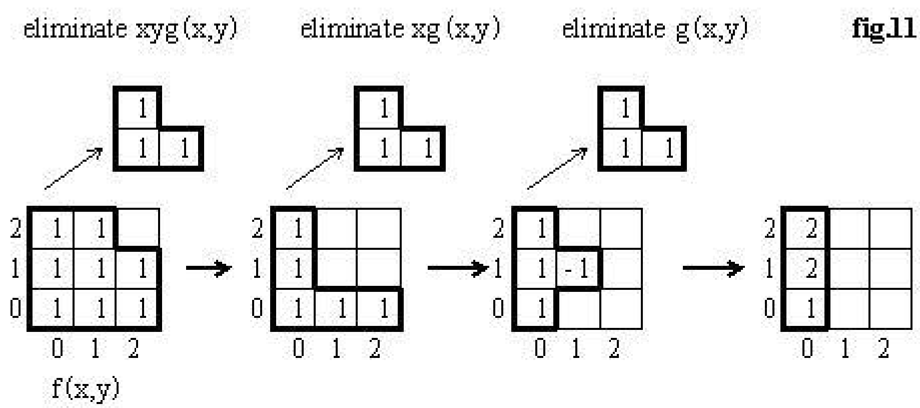


fig.11

division(1)

$$
\begin{array}{r|l}
 & (x+1)y-x \\
\hline
y+(x+1) & (x+1)y^2+(x^2+x+1)y+(x^2+x+1) \\
 & (x+1)y^2+(x^2+2x+1)y \\
\hline
 & -xy+(x^2+x+1) \\
 & -xy+(-x^2-x) \\
\hline
 & 2x^2+2x+1
\end{array}
$$

division(2)

$$
\begin{array}{r|l}
 & (y+1)x-y \\
\hline
x+(y+1) & (y+1)x^2+\ (y^2+y+1)x+(y^2+y+1) \\
 & (y+1)x^2+\ (y^2+2y+1)x \\
\hline
 & -yx+(y^2+y+1) \\
 & -yx+(-y^2-y) \\
\hline
 & 2y^2+2y+1
\end{array}
$$

2-4. Pascal's Triangle

Fig.12 presents a result of calculations x+y, $(x+y)^2$, $(x+y)^3$ and $(x+y)^3$. For example, process (2)to (3) is presented by fig.13(See the middle part). An arrangement of {1,2,1} separates into two directions($x(x+y)^2$ and $y(x+y)^2$) and generates an arrangement {1,3,3,1}. In the same way we can think of changes of arrangement {1,3,3,1} to {1,4,6,4,1}. So this model gives a new interpretation to Pascal's Triangle and we can think of Pascal's Triangle from a new point of view by this method.

fig.12

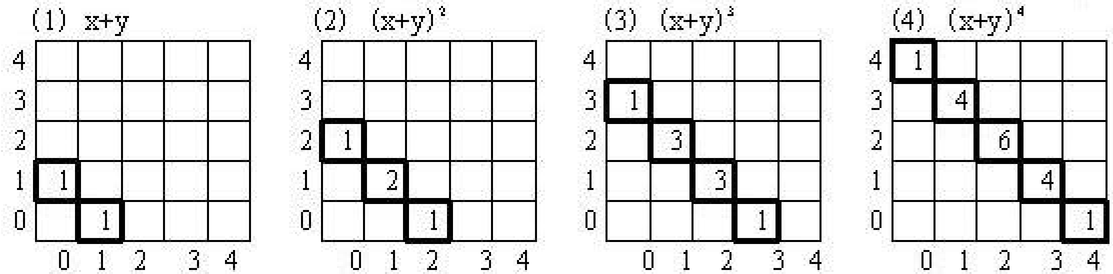
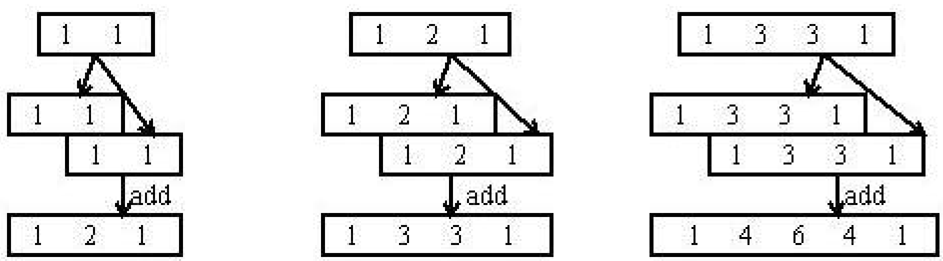


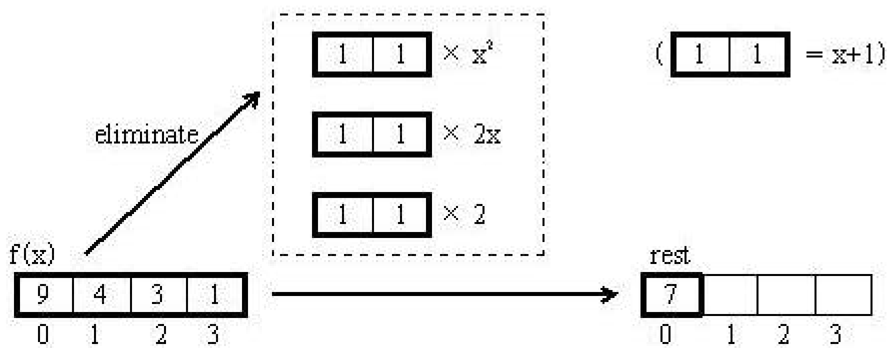
fig.13



2-5. Remainder Theorem

Of course we can apply this model to the polynomials in one variable. For example, let f(x) equal x^3+3x^2+4x+9 and g(x) equal x+1. Then we can think of the result of dividing f(x) by g(x) as follows. (fig.14)

fig.14

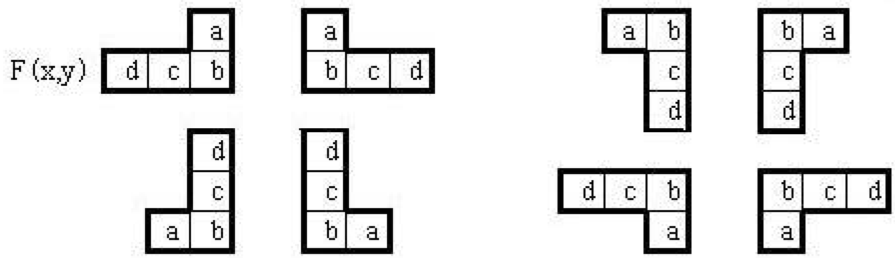


So if we substitute -1 for x then the value in the place above surrounded by the dotted line will be 0 and we can get the rest of division without division. (i.e.7) This is a new account of the remainder theorem.

2-6. "Conjugate Polynomial"

At fig.15 below there are polynomials in which figures and the arrangement of numbers are the same. We will call these polynomials "conjugatepolynomial" for the time being. The eight polynomials below cannot be put upon each other. Generally, there are eight different types of conjugate polynomials which are conjugate with the F(x,y).

fig.15



There are common characteristics in the polynomials which are conjugate with each other. At

fig.16

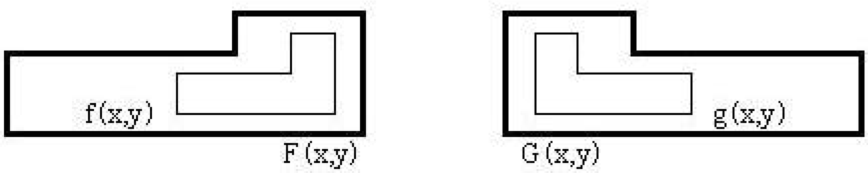


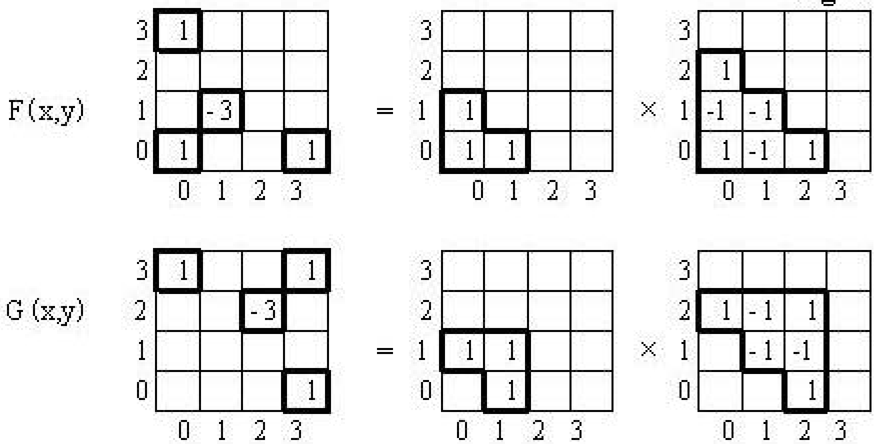
fig.16, let F(x,y) be conjugate with G(x,y). If F(x,y) has a factor f(x,y), then G(x,y) also has a factor g(x,y) which is conjugate with f(x,y). So we can understand the following propositions clearly.

Proposition 1
Let F(x,y) not have the factor x and y, and be conjugate with G(x,y). If F(x,y) is reducible, then G(x,y)is reducible too.

Proposition 2
Let G(x,y) not have the factor x and y, and be conjugate with F(x,y). If F(x,y) is irreducible, then G(x,y) is irreducible too.

For example, $F(x,y)=x^3+y^3-3xy+1$ is factorized into $(x+y+1)(x^2+y^2-xy-x-y+1)$ and $G(x,y)=x^3y^3-3x^2y^2+x^3+y^3$ is factorized into $(xy+x+y)(x^2y^2-x^2y-xy^2+x^2+y^2-xy)$. In this case F(x,y) is conjugate with G(x,y) We can see that $x+y+1$ is conjugate with the $xy+x+y$ and $x^2+y^2-xy-x-y+1$ is conjugate with $x^2y^2-x^2y-xy^2+x^2+y^2-xy$.(fig.17)

fig.17

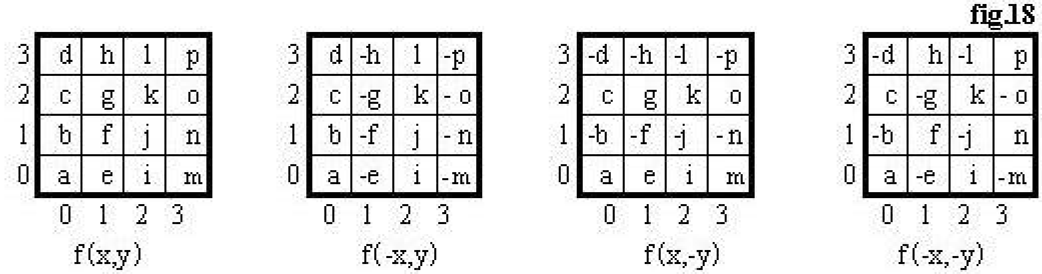


2-7. f(x,y) and ± f(± x,± y)

We know that if f(x,y)is irreducible, then f(-x,y) is also irreducible. Similarly, f(x,-y), f(-x,-y), -f(x,y), -f(-x,y), -f(x,-y) and -f(-x,-y) are also irreducible.(fig.18)

So if f(-x,y) is irreducible, then seven polynomials which are conjugate with f(-x,y) are also irreducible. Generally if f(x,y) is irreducible, then $63(8 \times 8 - 1)$types of polynomials are also irreducible.

fig.18



2-8. "Line Polynomial"

We will call the polynomial in which coefficients are arranged in line "line polynomial".

Then the product of line polynomials which are paralleled each other is also line polynomial. (fig.19) But the product of line polynomials which are not paralleled each other is not line polynomial. (fig.20)

fig.19

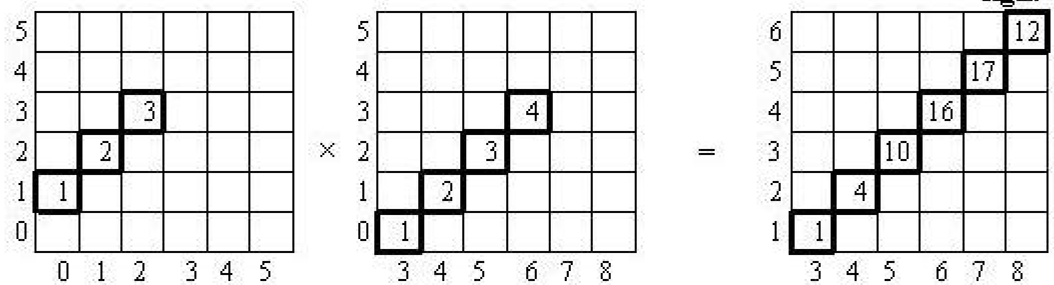
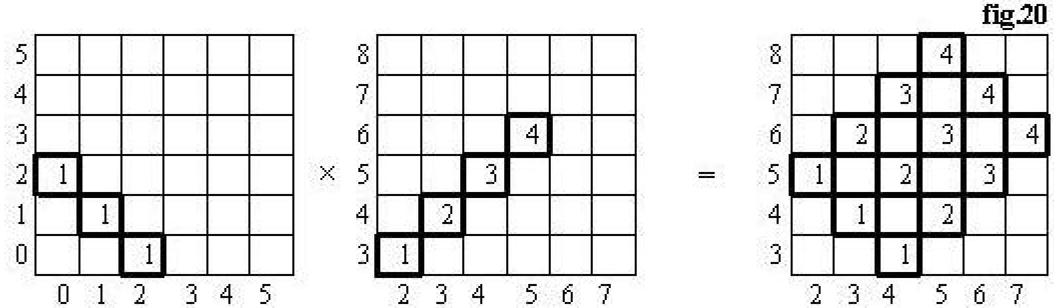


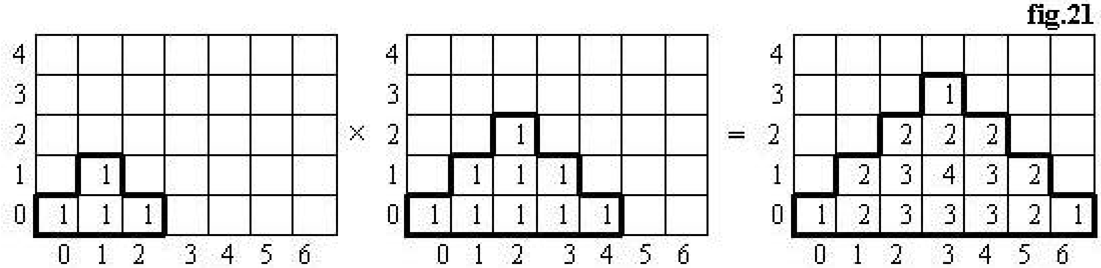
fig.20



2-9. "Triangular Polynomial"

We will call the polynomial in which coefficients are arranged in the figure of triangle "triangular polynomial". Then the product of triangular polynomials in which each side is parallel is also triangular polynomial.(fig.21)

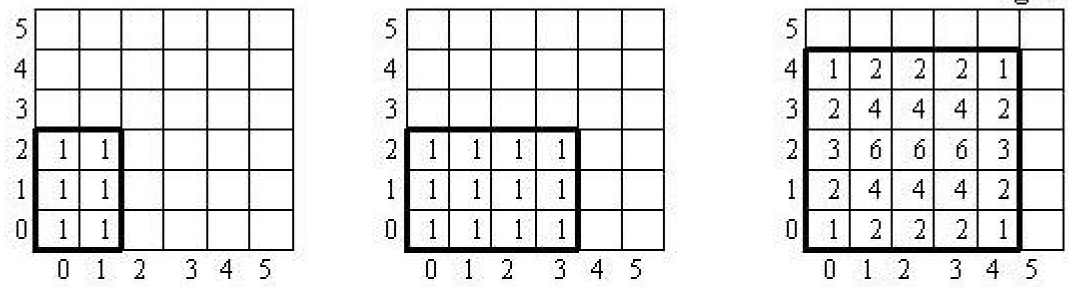
fig.21



2-10. "Square Polynomial"

We will call the polynomial in which coefficients are arranged in the figure of square "square polynomial" Then the product of square polynomials in which each side is parallel is also square polynomial.(fig.22)

fig.22



2-11. "Line Symmetrical Polynomial"

We will call the polynomial in which coefficients are arranged symmetrically like in fig.23 "line symmetrical polynomial". Then the product of line symmetrical polynomial in which axis is parallel each other is also line symmetrical polynomial.(fig.24) We know that the product of symmetrical polynomials is symmetrical and axis is a diagonal line.(i.e. y=x). So we can see symmetrical polynomial is the special case in the line symmetrical polynomials

fig.23

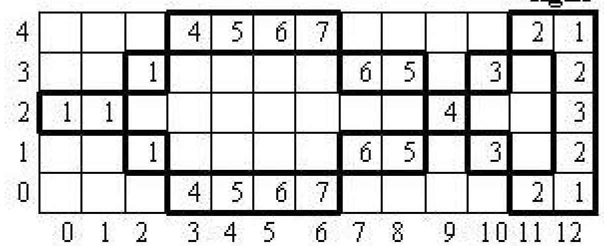
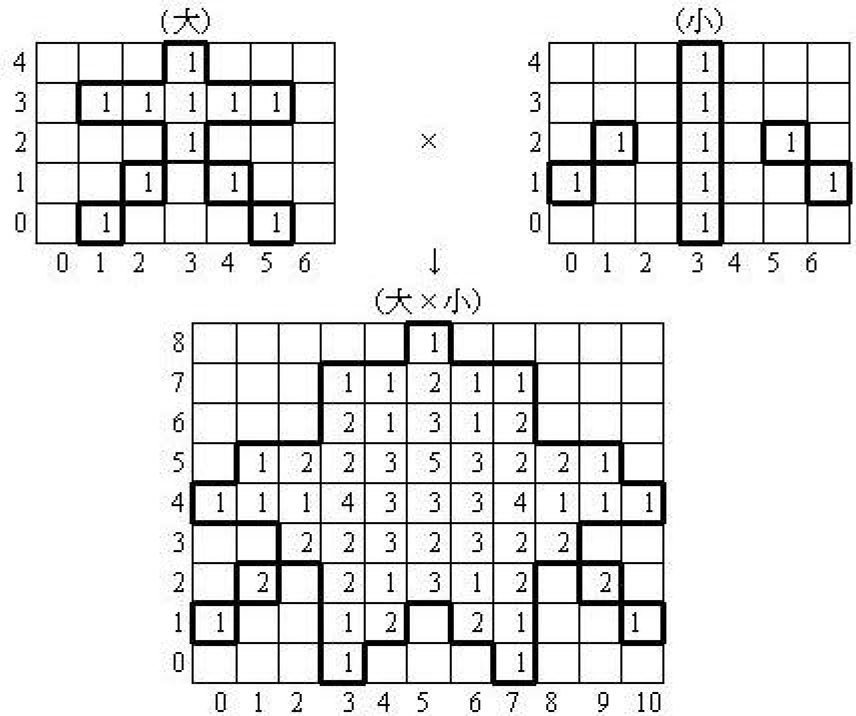


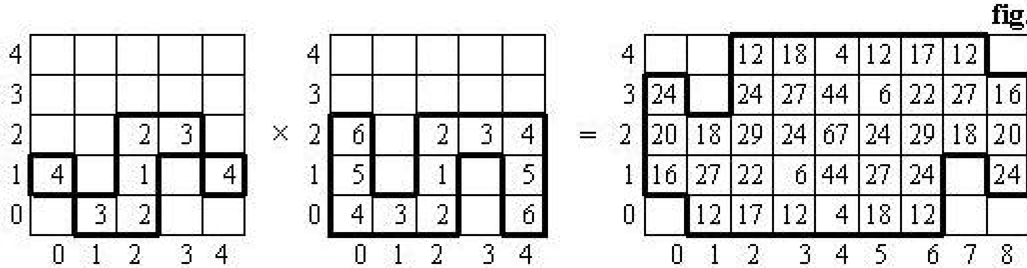
fig.24



2-12. "Point Symmetrical Polynomial"

We will call the polynomial in which coefficients are arranged symmetrically like in fig.25 "point symmetrical polynomial" Then the product of point symmetrical polynomial is also point symmetrical polynomial.

fig.25



2-13. Product of Polynomials Which Are Located Line Symmetrically

Each Other

Fig.26 presents that the product of polynomials which are located line symmetrically is line symmetrical polynomial. Similarly the product of polynomials which are located point symmetrically is point symmetrical polynomial.(fig.27)

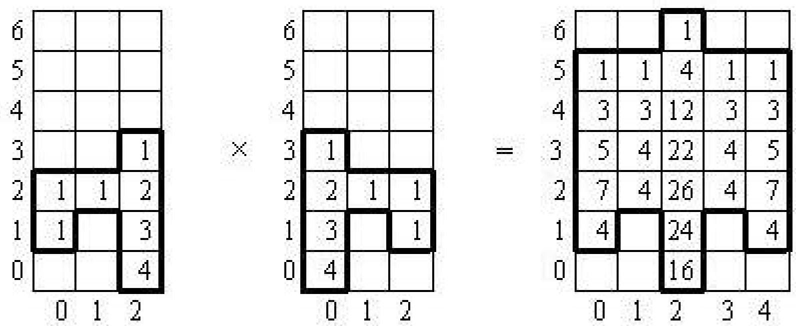
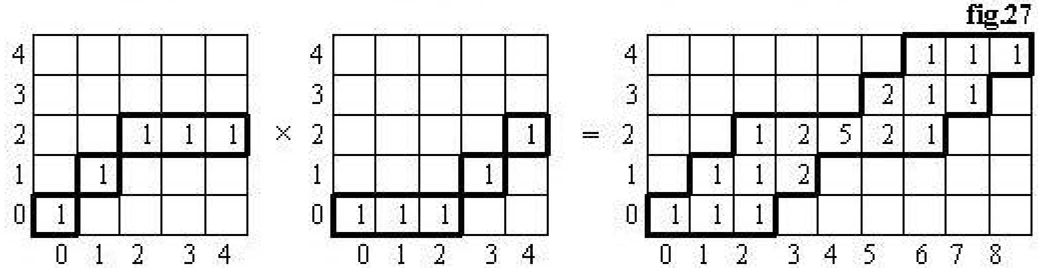


fig.26

fig.27



3. Instruction

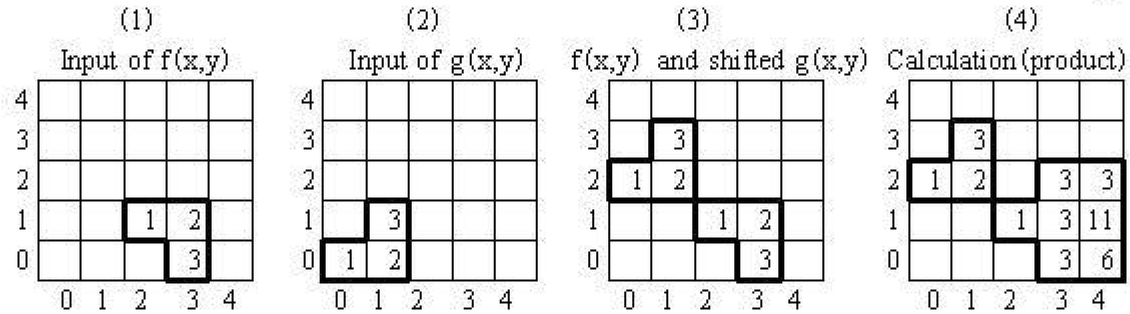
I taught the students in my class the polynomials by using this model. In this chapter we consider an instruction and an educational effect.

3-1. CAI(Computer Assisted Instruction)Program

I wrote a computer program in BASIC. In this program we can input coefficients of two polynomials(f(x,y) and g(x,y)) separately in the form of this model by using a mouse. (fig.28 (1),(2)) The upper limit degree is 30 in both x and y. After these inputs these two polynomials are shown on the computer display in different colors and we can move polynomial g(x,y) toward the axis of abscissa or toward the axis of ordinates by key operation.(fig.28(3))
When the numbers of f(x,y) and g(x,y) are doubled in the same frame, then these numbers are shown together in one frame.

Next we can select one operation from addition, subtraction and product between f(x,y) and shifted g(x,y). Then the computer gives an answer of calculation of these polynomials in this

fig.28



model form. (fig.27(4)) After this the computer redefines the result of calculation as f(x,y), and shifted g(x,y) is shown just as it is. Then we can calculate the operation of f(x,y) and shifted g(x,y) one after another.

By using this program we can see above-mentioned characteristics of polynomials. Some characteristics cannot be found on the paper calculation.

3-2. Instruction by Using This Program

I taught the students(tenth garader) the polynomials by using this program. The students were interested in finding unexpected characteristics of polynomials. Some students input coefficients of polynomials in the figure of letter "A" and calculated "A"^{2}.(fig.29) The abstract of there impressions of this lesson were as follows.

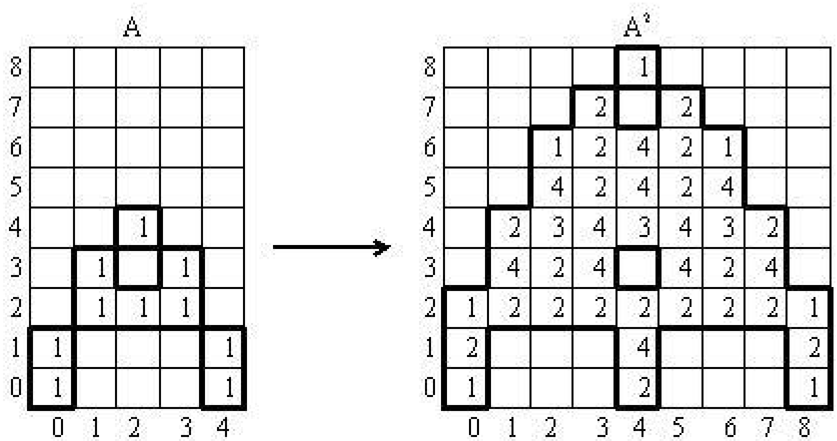


fig.29

·I was surprised that there are some mathematical regulations which are found only by the computer.

·It is a beneficial lesson for me. I think mathematics is a profound science. It was a wonderful experience.

·It is interesting that polynomials are represented such as rectangles. It is like a puzzle.
I think mathematics has an infinite expanse.

·Calculation of polynomials is troublesome. But I can calculate easily by using this program.
By using this program, students who dislike mathematics can understand mathematics.

·I could find many unexpected characteristics of polynomials. I am interested in using a model in mathematics.

·Mathematics is an interesting science! I have had a new appreciation of it. I have gained a lot of new mathematical knowledge by using this model.

·It was a delightful time for me. I think this program can promote our understanding. This is a good program!

·I had a wonderful experience. Now I am more and more interested in mathematics. Thank you!

4. Conclusion

We have seen many characteristics of polynomials so far. They cannot be found only by paper calculation. This model brings some unknown characteristics of polynomials.
In my instruction there was an educational effect on the students by using this model.
Students' impression shows that a proper mathematical model can promote many interests.