Normal Lines Drawn to a Parabola and Geometric Constructions

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Abstract

Consider an arbitrary parabola and a point $P_0(x_{0,}y_0)$ on the plane. Depending on the position of P_0 on the plane, one can draw exactly one, two or three normal lines to the parabola from P_0 (see [9]). Using the discriminant of a certain cubic polynomial, we will obtain a necessary and sufficient condition on the coordinates of P_0 for the above phenomenon to happen. This will lead to the observation that there are only two points A and B on the parabola with the property that exactly two normal lines can be drawn from either A or B to the parabola. Except A and B, from any other point on the parabola, one can draw either exactly one or three normal lines to the parabola. Therefore, in a certain sense, these points A and B serve as special cut-off points on the parabola. We will obtain a theorem to partly reveal the special nature of the points A and B. We have used the computer algebra system *Mathematica* to perform several calculations and also to test and form several conjectures on these normal lines. For the use of *Mathematica* as a conjecture forming tool, the reader can refer to [2], [3], [4], [5], [6], [7] or [8]. Our conjectures lead to several more theorems. One of these theorems provides a beautiful geometric construction of all the normal lines from any point on the plane to an arbitrary parabola. We have also used to *Mathematica* to write a program to generate all the normal lines from a point in the plane to a parabola. Some good references on *Mathematica* are [10], [11] and [12].

1. Introduction

Consider an arbitrary parabola on the plane. One can take a coordinate system OXY, so that the origin O is at the vertex of the parabola, and the X-axis is along the axis of the parabola. With respect to this coordinate system, one can write the equation of the parabola as $y^2 = 4ax$ for some nonzero real constant a. Let $P_0(x_0, y_0)$ be any point on the plane. We are interested in studying the normal lines drawn from P_0 to the parabola.

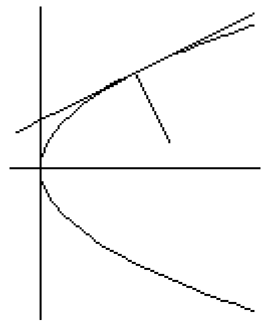

Figure 1. Normal line from a point to a parabola.

Suppose Q is the foot of the normal drawn from P_0 to the parabola. Since Q is on the parabola $y^2 = 4ax$, one can write the coordinates of Q as $(at^2, 2at)$ for some real parameter t. One can differentiate both sides of the equation $y^2 = 4ax$ with respect to x to obtain $2yy' = 4a$, yielding

$$y' = \frac{2a}{y} \tag{1}$$

The slope of the normal line P_0Q is equal to negative reciprocal of the derivative y' evaluated at $Q(at^2, 2at)$. Hence equation (1) yields,

$$\text{slope of } P_0Q = -t \tag{2}$$

However,

$$\text{slope of } P_0Q = \frac{y_0 - 2at}{x_0 - at^2} \tag{3}$$

By equating the equations (2) and (3), and after simplifying, one obtains

$$at^3 + t(2a - x_0) - y_0 = 0 \tag{4}$$

The equation (4) is a cubic equation in t, and has in general three solutions, not necessarily distinct. Its discriminant will exactly reveal the nature of the roots. Recall that the discriminant of the cubic equation $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ is given by $-27(a_0^2a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 - 3a_1^2a_2^2 + 4a_1^3a_3)$ (see [1]). Hence, the discriminant D of equation (4) is given by

$$D = -a\left[27ay_0^2 + 4\left(2a - x_0\right)^3\right] \tag{5}$$

The equation (4) has three distinct real roots, three real roots with *at least* two the same, or one real root and two non real roots which are conjugates, according as the discriminant D is positive, zero or negative. Therefore, we have the following three cases.

Case I: $D > 0$
One can draw exactly three distinct normal lines to the parabola from P_0.

Case II: $D = 0$
First note that this means that the equation (4) has a repeated root of multiplicity two or three. Suppose t_1 is any root of equation (4). Let $f(t) = at^3 + t(2a - x_0) - y_0$. Then $f'(t) = 3at^2 + (2a - x_0)$ and $f''(t) = 6at$. Hence, under the present case, t_1 is a triple root of equation (4) if and only if $t_1 = 0$. However, $t_1 = 0$ is a root of equation (4) if and only if $y_0 = 0$. Also note that from equation (5) that $y_0 = 0$ if and only if $x_0 = 2a$. This discussion means that if $D = 0$, then equation (4) has a triple root if and only if $(x_0, y_0) = (2a, 0)$. For all other points (x_0, y_0) such that $D = 0$, the equation (4) has only two distinct real roots, one with multiplicity one, and the other with multiplicity two.

The above discussion means that if $(x_0, y_0) = (2a,\ 0)$, then one can draw exactly one normal line from $P_0(x_0, y_0)$ to the parabola. On the other hand, if $(x_0, y_0) \neq (2a, 0)$ is such that $D = 0$, then one can draw exactly two distinct normal lines from $P_0(x_0, y_0)$ to the parabola.

Case III: $D < 0$
One can draw exactly one normal line to the parabola from P_0.

The above three cases provide a necessary and sufficient condition on the coordinates of point P_0 in order that the number of normal lines from P_0 to the parabola is either one, two or three.

In order to see the geometric significance of the above three cases, one needs to find out for which points (x, y) on the plane, the quantity $27ay^2 + 4(2a - x)^3$ is negative, zero or positive. Therefore, it is a good idea to draw the following two graphs on the same set of axes:

$$y^2 = 4ax \tag{6}$$

$$y^2 = -\frac{4}{27a}(2a - x)^3 \tag{7}$$

As the following figure illustrates, the parabola given by equation (6) intersects the semi cubical parabola given by equation (7) at exactly two points A and B. One can simultaneously solve the equations (6) and (7) to obtain these points of intersection as $A(8a,\ 4\sqrt{2}\,a)$ and $B\,(8a,\ -4\sqrt{2}\,a)$. Also, observe that the vertex of the semi cubical parabola is given by $C(2a, 0)$.

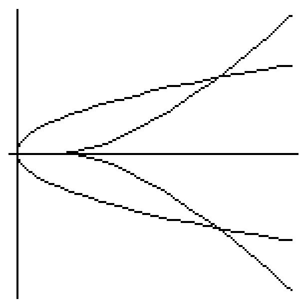

Figure 2. Parabola and a semi-cubical parabola

One can imagine that the semi cubical parabola given by equation (7) divides the XY plane into two regions. For any point (x, y) to the right of the semi cubical parabola, $27ay^2 + 4(2a - x)^3$ is negative, so from those points one can draw exactly three normal line to the parabola $y^2 = 4ax$. On the other hand, for any point (x, y) to the left of the semi cubical parabola, $27ay^2 + 4(2a - x)^3$ is positive, so from those points one can only draw exactly one normal line to the parabola $y^2 = 4ax$. Also above case II implies that for any point (x, y) on the semi cubical parabola other than the vertex $C(2a, 0)$, one can draw exactly two normal lines to the parabola, while from the vertex C itself, one can only draw one normal line.

The following *Mathematica* program draws all normal lines from a point in the plane to a parabola.

Program 1

```
(* First, make sure to invoke the command <<Graphics`ImplicitPlot`  *)
StylePrint["The Normal Lines Drawn From a Point on the Plane to a
Parabola", "Text", FontSize->24, FontFamily->"Times",
FontColor->RGBColor[1,0,0], FontWeight->"Bold", TextAlignment->Center,
Background->RGBColor[0.9,0.75,0] ]
a = ?;  x0 = ?;  y0 =?;
Which[27a*y0^2 + 4(2a - x0)^3 < 0,Print["Three normal lines"],
  27a*y0^2 + 4(2a - x0)^3 > 0,Print["One normal line"],
```

```
  27a*y0^2 + 4(2a - x0)^3 == 0 && {x0,y0} ! = {2a,0},
Print["Two normal lines"], {x0,y0}=={2a,0}, Print["One normal lines"] ]
r  = Cases[ t /. NSolve[a*t^3 + t(2a - x0) - y0 == 0, t], x_/; Im[x] == 0];
ImplicitPlot[{ y^2 == 4a*x, 27a*y^2 == - 4(2a - x)^3}, {x, -1, Max[x0, 9a] },
   PlotStyle -> {{ RGBColor[0.7, 0, 0. 3] },{ RGBColor[0, 0.8, 0.1] } },
   Epilog -> {{ RGBColor[1, 0, 0], PointSize[1/70], Point[{ x0, y0 }] },
   {RGBColor[0, 0, 1], Table[Line[{ {x0, y0}, { a*r[ [ i ] ]^2, 2a r[ [ i ] ]} } ],
                                                       {i, 1, Length[r] } ] } } ]
```

2. Special points on the parabola

The previous discussion implies that the points A and B on the parabola are very special - They are the only points *on* the parabola from which one can draw exactly two normal lines each to the parabola itself. The following theorem partly reveals the special nature of the points A and B.

Theorem 1 Consider an arbitrary parabola and all the circles passing through its vertex V, touching the parabola at a point P, and passing through a third point Q on the parabola, where V, P and Q are all distinct, but arbitrary. Then only two such circles will have PQ as the endpoints of a diameter. One of these circles pass through the fore mentioned point $A(8a,\ 4\sqrt{2}\,a)$ on the parabola, while the other passes through the point $B\,(8a,\ -4\sqrt{2}\,a)$.

Proof Without loss of generality, one can take the equation of the parabola to be

$$y^2 = 4ax \tag{8}$$

so that its vertex V is the same as the origin O. The equation of any circle passing through O can be written as,

$$x^2 + y^2 + 2gx + 2fy = 0 \tag{9}$$

where g and f are some real numbers. We are assuming that this circle touches the parabola at a point P and passes through another point Q, where O, P, and Q are all distinct. By eliminating y between the equations (8) and (9), one can easily see that the y-coordinates of the points of intersections P and Q are given by

$$y^3 + 16a^2y + 8agy + 32a^2f = 0 \tag{10}$$

By considering the first derivative of the left hand side of equation (10) one notices that any repeated root of equation (10) must satisfy

$$3y^2 = -8a(g + 2a) \tag{11}$$

Recall that the solutions of equation (10) are nothing but the y-coordinates of the points of intersection P and Q, and the circle touches the parabola at P and passes through Q. Therefore, one can write the solutions to equation (10) as y_1, y_2 and y_3 where $y_1 = y_2$ and $y_1 \neq y_3$. In other words, y_1 and y_3 are the y-coordinates of the points P and Q respectively. Therefore, by considering the sum of the roots of equation (10) one obtains that $y_1 + y_2 + y_3 = 2y_1 + y_3 = 0$, yielding $y_3 = -2y_1$. However, since the center of the circle (9) is $(-g, -f)$ and since PQ is a diameter of the circle, one can write $-f = (y_1 + y_3)/2 = -y_1/2$. Therefore, $y_1 = 2f$. Plug this back into equation (11) to obtain

$$12f^2 = -8a(g+2a) \tag{12}$$

Also, since equation (10) has a repeated root, its discriminant must be zero. This yields, using the method described immediately following equation (4),

$$27af^2 = -2(g+2a)^3 \tag{13}$$

Further note that $g + 2a \neq 0$, for if $g + 2a = 0$, then equation (12) implies that $f = 0$. This in turn will imply that $y = 0$ is a solution of equation (10), which is a contradiction. Since $g + 2a \neq 0$, one can divide equation (12) by equation (13) to obtain

$$9a^2 = (g + 2a)^2 \tag{14}$$

The equation (14) implies that $g = a$ or $g = -5a$. However, if $g = a$, then equation (13) implies that f cannot be real. So the only possibility is $g = -5a$. Then equation (13) will imply that $f = \pm a\sqrt{2}$. Therefore, there exists only two circles with the given property: one is centered at $(5a, a\sqrt{2})$, and the other centered at $(5a, -a\sqrt{2})$.

For the first circle centered at $(-g, -f) = (5a, a\sqrt{2})$, one calculates $y_3 = -2y_1 = -4f = 4a\sqrt{2}$, and $x_3 = y_3^2/(4a) = 8a$. So this circle passes through the point $A(8a, 4a\sqrt{2})$. Similarly, the circle centered at $(5a, -a\sqrt{2})$ would pass through the point $B(8a, -4a\sqrt{2})$. Hence the theorem. ∎

3. Concurrency of the normals

The next two theorems deal with the concurrency of normals at three distinct points on a parabola (see also [9]).

Theorem 2 Let $P_i = (at_i^2, 2at_i)$, $i = 1, 2, 3$ be three distinct points on the parabola $y^2 = 4ax$. Then the following are equivalent.
I. $t_1 + t_2 + t_3 = 0$.
II. The normals to the parabola at the points P_i, $i = 1, 2, 3$ are concurrent.

III. The circle through the points P_i, $i = 1,2,3$ pass through the vertex of the parabola.

Proof We will show **I** ⇔ **II** and **I** ⇔ **III**.

I ⇔ II
Suppose **I** is true. The equation (1) implies that the slope of the normal line at $P_i = (at_i^2,\, 2at_i)$ is equal to $-t_i$, $i = 1,2,3$. Therefore, the equation of the normal line at P_i is given by, for $i = 1,2,3$

$$xt_i + y = 2at_i + at_i^3 \tag{15}$$

Using *Mathematica*, one can easily obtain the points of intersection of the lines corresponding to $i = 1$ and $i = 2$ as $P = (2a + at_1^2 + at_1t_2 + at_2^2,\ -at_1^2t_2 - at_1t_2^2)$. Another *Mathematica* calculation yields, P lies on the line $xt_3 + y = 2at_3 + at_3^3$ if and only if $a(t_1 - t_3)(t_3 - t_2)(t_1 + t_2 + t_3) = 0$. But this condition holds, since we are assuming that $t_1 + t_2 + t_3 = 0$. This proves **I** ⇒ **II**. Similarly, one can show that **II** ⇒ **I.**

I ⇔ III
Suppose **I** is true. Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the circle passing through the points $P_i = (at_i^2,\, 2at_i), i = 1,2,3$. One can plug in the coordinates of the points P_i in the equation of the circle, to obtain 3 equations in the unknowns g, f and c. Then one can use *Mathematica* to obtain the solutions $g = -a(4 + t_1^2 + t_2^2 + t_3^2 + t_1t_2 + t_2t_3 + t_3t_1)/2$, $f = a(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)/4$, and $c = -a^2t_1t_2t_3(t_1 + t_2 + t_3)$. Since we are assuming that $t_1 + t_2 + t_3 = 0$, one obtains that $c = 0$. Therefore, the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through the origin. This proves **I** ⇒ **III**. The proof of **III** ⇒ **I** is similar. ∎

Theorem 3 Let $P_i = (at_i^2,\, 2at_i)$, $i = 1,2,3$ be any three distinct points on the parabola $y^2 = 4ax$ such that the normals to the curve at P_i are concurrent. Then the point of concurrency is given by $D = (\,a(2 + t_1^2 + t_1t_2 + t_2^2),\ \text{-}\ at_1t_2(t_1 + t_2)\,)$. Further, the orthocenter of the triangle $P_1P_2P_3$ and the circumcenter of the triangle formed by the tangents at P_i are equidistant from the axis of the parabola. Moreover, the orthocenter of the triangle formed by the tangents at P_i and the point of concurrency D are also equidistant from the axis of the parabola.

Proof The coordinates of D were essentially obtained in the proof of the previous theorem, immediately following equations (15). Recall that the orthocenter of any triangle is the point of the concurrency of the perpendiculars drawn from the vertices to opposite sides. Likewise the circumcenter is the point of concurrency of the perpendicular bisectors of three sides. Let H_1 and O_1 be the orthocenter and the circumcenter of the triangle $P_1P_2P_3$ respectively, and let H_2 and O_2 be the orthocenter and the circumcenter of the triangle formed by the tangents at P_i. Using *Mathematica* and the relationship $t_1 + t_2 + t_3 = 0$ one can establish that

$$H_1 = (a(-4 + t_1^2 + t_1t_2 + t_2^2),\ a\,t_1t_2(t_1 + t_2)/2\,) \tag{16}$$

$$O_1 = (a(4 + t_1^2 + t_1t_2 + t_2^2)/2,\ -at_1t_2(t_1 + t_2)/4\,) \tag{17}$$

$$H_2 = (-a,\ -at_1t_2(t_1 + t_2)\,) \tag{18}$$

$$O_2 = (-a(-1 + t_1^2 + t_1t_2 + t_2^2)/2,\ a\,t_1t_2(t_1 + t_2)/2\,) \tag{19}$$

Then one can observe that the points H_1 and O_2 have the same y-coordinates. Likewise the points D and H_2 have the same y-coordinates. Hence the theorem. ■

Next a crucial theorem of the paper: In some sense, it relates Theorem 2 to Theorem 3.

Theorem 4 Consider a circle centered at (u, v) passing through the vertex of the parabola $y^2 = 4ax$, and intersecting the parabola at three more distinct points P_i, $i = 1, 2, 3$ all distinct from the vertex. Then the normals at the points P_i are concurrent at $E(2(u - a), 4v)$.

Proof The equation of the circle centered at (u, v) and passing through the vertex of the parabola, which is the origin, can be written as

$$x^2 + y^2 - 2ux - 2vy = 0 \tag{20}$$

One can write the points P_i as $P_i\,(at_i^2, 2at_i)$ for some real parameters t_i , $i = 1, 2, 3$. Recall that the equations of the normal lines at the points P_i , $i = 1, 2, 3$ are given by equations (15). Therefore, $E(2(u - a), 4v)$ lies on those normal lines if and only if the following holds for $i = 1, 2, 3$:

$$at_i^3 + 4at_i - 2ut_i - 4v = 0 \tag{21}$$

However, the points $P_i\,(at_i^2, 2at_i)$ lie on the circle given by equation (20). Hence, a direct substitution yields,

$$at_i(at_i^3 + 4at_i - 2ut_i - 4v) = 0 \tag{22}$$

for $i = 1, 2, 3$. But, since, $a \neq 0$ and $t_i \neq 0$ for $i = 1, 2, 3$, the equation (22) implies equation (21). Therefore, the normal lines at the points P_i are concurrent at $E(2(u - a), 4v)$. Hence the theorem. ■

Corollary Consider a point $P_0(x_0, y_0)$ on the plane, the parabola $y^2 = 4ax$, and the circle centered at $C(a + x_0/2,\ y_0/4)$ with radius OC. Other than the origin O itself, this circle can intersect the parabola at zero, one, two or at most at three points. If P is any such point of intersection other than the origin, then P_0P is a normal to the parabola at P. If the origin is the only point of intersection, then P_0 lies on the X-axis, and in this case, X-axis is the only normal from P_0 to the parabola.

Proof Follows from the previous theorems. The details are left to the reader. ∎

The above corollary provides a beautiful geometric construction of all the normal lines from a point in the plane to an arbitrary parabola:

Construction Consider any point P on the plane and a parabola with vertex V and focus F. Draw PQ perpendicular to the line l_1 through V and F, intersecting l_1 at Q. Then find the point R on PQ so that $QR = PQ/4$. Draw a line l_2 passing through R parallel to VF. Find the point S on VQ such that $VS = VF + VQ/2$. Draw a perpendicular to VQ at S intersecting the line l_2 at C. Then C is the center of the circle referred to in the previous corollary. Finally construct the circle centered at C with radius CV. In general, this circle intersects the parabola at three more distinct point P_i, $i = 1, 2, 3$, other than the vertex V. Then PP_1, PP_2, and PP_3 are the required normals from point P.

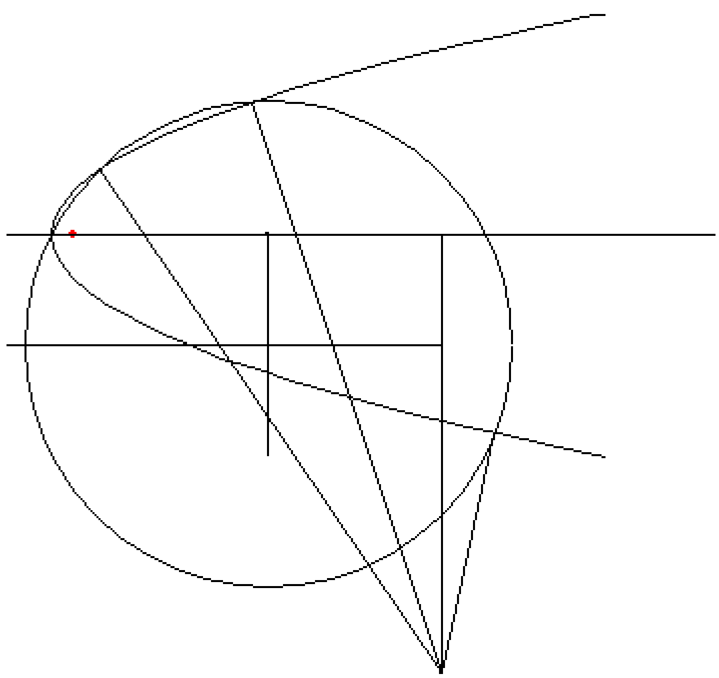

Figure 3. Construction of normals from a given point to a parabola

References

1. Archbold, J. *Algebra*, Pitman Publishing Limited, London, UK, 1978.

2. de Alwis, T. Mathematica and the Power Method, *International Journal of Mathematics Education in Science and Technology*, 1993, 24 (6), 813-824.

3. de Alwis, T. Effective Use of Mathematica - Pattern Recognition, Conjecture Making and Simulations (ed B. Jaworski), pp. 189-196, *Proceedings of the International Conference on Mathematics Teaching*, Birmingham, UK, 1993, University of Birmingham, UK, 1993.

4. de Alwis, T. Mathematica as a Conjecture Making and a Multimedia Tool (ed T. Otttman & I. Tomek), pp. 642-643, *Proceedings of the World Conf. on Educational Multimedia and Hypermedia*, Vancouver, 1994, Association for the Advancement of Computing in Education, Charlottesville, VA, 1994.

5. de Alwis, T. Projectile Motion with Arbitrary Resistance, *College Mathematics Journal*, 1995, 26 (5), 361-367.

6. de Alwis, T. Families of Plane Curves Bounding a Constant Area (ed H. Fong & W. Yang), pp. 300-309, *Proceedings of the First Asian Technology Conference in Mathematics*, Singapore, 1995, The Association of Mathematics Educators, Singapore, 1995.

7. de Alwis, T. The Power of Animation in Visualizing Mathematics, *Proceedings of the World Conference on Educational Multimedia and Hypermedia* (to appear), Calgary, Canada, 1997.

8. de Alwis, T. Families of Plane Curves with a Constant Arc Length (ed V. Keranen), pp. 37-44, *Proceedings of the Second International Mathematica Symposium*, Finland, 1997, Computational Mechanics Publications, Southampton, UK, 1997.

9. Loney, L. *The Elements of Coordinate Geometry*, Macmillan and Company, London, UK, 1895.

10. Maeder, R. *Programming in Mathematica*, Addison-Wesley, Reading, MA, 1991.

11. Wagon, S. *Mathematica in Action*, W. H. Freeman, New York, NY, 1991.

12. Wolfram, S. *The Mathematica Book, 3rd Edition*, Wolfram Media, Champaign, Illinois, 1996.