Eight Lines Arrangements on the Real Projective Plane and the Root System of Type E_8

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Abstract

In this report, we discuss a problem of classifying 8 lines arrangements on the real projective plane.

To analyze a geometric feature of N lines arrangements, we propose an algorithm of counting numbers of polygons in any arrangement.

From an experimental calculation by using this algorithm, we give an example of 8 lines arrangement which contains ten triangles but not hexagons, heptagons, octagons.

By analyzing this example, we discuss the relationship between the root system of type E_8 and 8 lines arrangements.

1 Introduction

In this report, we study properties of 8 lines arrangements on the real projective plane. One of our interests is to interpret simple 2-arrangements of 8 lines in terms of the root system of type E_8.

A classification of simple 2-arrangements of 6 lines and 7 lines is well-known (cf. [4]). In the case of 6 lines, J. Sekiguchi and M. Yoshida [9] gave a parametrization of simple 2-arrangements of 6 lines in terms of the root system of type E_6. Furthermore, in the case of 7 lines, J. Sekiguchi showed an *injective* map of the totality of the so-called tetrahedral sets of the root system of type E_7 to certain families of simple 2-arrangements of 7 lines (cf. [7]).

The totality of 8 lines arrangements satisfying some conditions explained in the text forms a configuration space $\mathbf{P}_0(2,8)$ of systems of marked 8 lines on the real projective plane. There is a natural action of the Weyl group $W(E_8)$ of type E_8 on $\mathbf{P}_0(2,8)$. Let $\mathcal{P}_8$ be the set of connected components of $\mathbf{P}_0(2,8)$. Our interest is the determination of $W(E_8)$-orbital structure of $\mathcal{P}_8$.

To analyze a geometric feature of simple 2-arrangements of n lines, we proposed an algorithm counting numbers of polygons included in any simple 2-arrangement [3]. From an experimental calculation by using the algorithm, we obtain an interesting example of simple 2-arrangement of 8 lines which we call AE_8 in this report. The simple 2-arrangement AE_8 contains ten triangles but not hexagons, heptagons, octagons. To each triangle of AE_8, we attach a root of the root system of type E_8. In this manner, we try to interpret the $W(E_8)$-orbit $\mathcal{O}$ of the connected component containing AE_8 in terms of the root system of type E_8 [2]. In particular, we define 8LC sets and 8LC diagrams and discuss the relationship between $\mathcal{O}$ and the totality of 8LC diagrams.

2 Root system of type E_8

We begin this report by introducing the root system of type E_8 [1]. Let E be an 8-dimensional Euclidean space with a standard basis $\{\boldsymbol{e}_j; 1 \leq j \leq 8\}$. Let $\langle \cdot, \cdot \rangle$ be the inner product on E defined by

$$\langle \boldsymbol{e}_j, \boldsymbol{e}_k \rangle = \delta_{jk}.$$

We define the following 120 vectors of E:

$$
\begin{array}{lcll}
\boldsymbol{t}_1 & = & \frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4+\boldsymbol{e}_5+\boldsymbol{e}_6+\boldsymbol{e}_7+\boldsymbol{e}_8) & \\
\boldsymbol{r}_{1j} & = & \boldsymbol{t}_1-\boldsymbol{e}_{j-1}-\boldsymbol{e}_8 & (1<j\le 8) \\
\boldsymbol{r}_{ij} & = & \boldsymbol{e}_{i-1}-\boldsymbol{e}_{j-1} & (1<i<j\le 8) \\
\boldsymbol{r}_{1jk} & = & -\boldsymbol{e}_{j-1}-\boldsymbol{e}_{k-1} & (1<j<k\le 8) \\
\boldsymbol{r}_{ijk} & = & \boldsymbol{t}_1-\boldsymbol{e}_{i-1}-\boldsymbol{e}_{j-1}-\boldsymbol{e}_{k-1}-\boldsymbol{e}_8 & (1<i<j<k\le 8) \\
\boldsymbol{t}_i & = & -\boldsymbol{e}_{i-1}-\boldsymbol{e}_8 & (1<i\le 8) \\
\boldsymbol{t}_{1j} & = & \boldsymbol{e}_{j-1}-\boldsymbol{e}_8 & (1<j\le 8) \\
\boldsymbol{t}_{ij} & = & \boldsymbol{t}_1-\boldsymbol{e}_{i-1}-\boldsymbol{e}_{j-1} & (1<i<j\le 8).
\end{array}
\tag{1}
$$

The totality Δ of $\pm\boldsymbol{r}_{ij}, \pm\boldsymbol{r}_{ijk}, \pm\boldsymbol{t}_i, \pm\boldsymbol{t}_{ij}$ forms a root system of type E_8. It is clear that

$$\boldsymbol{r}_{12}, \boldsymbol{r}_{123}, \boldsymbol{r}_{23}, \boldsymbol{r}_{34}, \boldsymbol{r}_{45}, \boldsymbol{r}_{56}, \boldsymbol{r}_{67}, \boldsymbol{r}_{78}$$

can serve as a system of positive roots; its Dynkin diagram is given as

$$
\begin{array}{ccccccccccccc}
\boldsymbol{r}_{12} & — & \boldsymbol{r}_{23} & — & \boldsymbol{r}_{34} & — & \boldsymbol{r}_{45} & — & \boldsymbol{r}_{56} & — & \boldsymbol{r}_{67} & — & \boldsymbol{r}_{78} \\
 & & & & | & & & & & & & & \\
 & & & & \boldsymbol{r}_{123} & & & & & & & &
\end{array}
$$

Let s_{ij}, s_{ijk} be the reflections on E with respect to $\boldsymbol{r}_{ij}, \boldsymbol{r}_{ijk}$ and let τ_i, τ_{ij} be the reflections on E with respect to $\boldsymbol{t}_i, \boldsymbol{t}_{ij}$. The group generated by the reflections $s_{ij}, s_{ijk}, \tau_i, \tau_{ij}$ is nothing but the Weyl group $W(E_8)$ of type E_8. In the sequel, the symmetric group S_8 is identified with the subgroup of $W(E_8)$ generated by s_{ij} $(i, j = 1, 2, \ldots, 8)$ unless otherwise stated.

3 Configurations of eight lines on the real projective plane

We introduce systems of marked n lines $(l_1, l_2, \ldots, l_n)$ $(n \ge 6)$ on $\mathbf{P}^2(\mathbf{R})$. We give conditions on these n lines:

I. The n lines $l_1, l_2, \ldots, l_n$ are mutually different.

II. No three of $l_1, l_2, \ldots, l_n$ intersect at a point.

III. There is no conic tangent to any six of $l_1, l_2, \ldots, l_n$.

Each connected component of $\mathbf{P}^2(\mathbf{R}) - \cup_{j=1}^n l_j$ is called a polygon. If it is surrounded by p lines, it is called a p-gon. The totality of systems of marked n lines on $\mathbf{P}^2(\mathbf{R})$ with conditions I, II forms the configuration space $\mathbf{P}(2,n)$; the space $\mathbf{P}(2,n)$ is defined by

$$\mathbf{P}(2,n) = GL(3,\mathbf{R})\backslash M'(3,n)/(\mathbf{R}^\times)^n,$$

where $M'(3,n)$ is the set of $3 \times n$ real matrices of which no 3-minor vanishes.

From now on, we focus our attention to the case $n = 8$. In addition to conditions I, II, III, we consider the following:

IV. Let P_j be the point of $\mathbf{P}^2(\mathbf{R})$ dual to l_j $(j = 1, 2, \ldots, 8)$. Then there is no cubic curve C such that C passes through all the points $P_1, \ldots, P_8$ and that one of the points $P_1, \ldots, P_8$ is a cusp point of C.

The totality of systems of marked 8 lines on $\mathbf{P}^2(\mathbf{R})$ with conditions I, II, III and IV forms a subset of $\mathbf{P}(2,8)$ which we denote by $\mathbf{P}_0(2,8)$. Both $\mathbf{P}(2,8)$ and $\mathbf{P}_0(2,8)$ are Zariski open subsets of $\mathbf{R}^8$. Permutations on the 8 lines $l_1, l_2, \ldots, l_8$ induce a biregular S_8-action on $\mathbf{P}(2,8)$ (and also that on $\mathbf{P}_0(2,8)$). Let $\mathcal{P}_8$ be the set of connected components of $\mathbf{P}_0(2,8)$. It is stressed here that the S_8-action on $\mathbf{P}_0(2,8)$ is naturally extended to a birational $W(E_8)$-action (cf. [5], [6]). The $W(E_8)$-action on $\mathbf{P}_0(2,8)$ naturally induces that on $\mathcal{P}_8$. Then it is interesting to attack the problem below:

Problem 1 *Determine the $W(E_8)$-orbital structure of $\mathcal{P}_8$.*

Remark 1 *For the cases of systems of marked 6 lines and those of marked 7 lines, we can define $\mathcal{P}_6$ and $\mathcal{P}_7$ analogous to $\mathcal{P}_8$. Then $W(E_6)$ acts on $\mathcal{P}_6$ transitively (cf. [9]). Also it is provable that $W(E_7)$ acts on $\mathcal{P}_7$ transitively.*

We are going to define the action of $W(E_8)$ on $\mathbf{P}_0(2,8)$ in a concrete manner. Let $(l_j)_{1\le j\le 8}$ be a system of marked 8 lines. We assume that l_j is defined by

$$l_j : a_{1j}\xi + a_{2j}\eta + a_{3j}\zeta = 0, \tag{2}$$

where $(\xi : \eta : \zeta)$ is a homogeneous coordinate of $\mathbf{P}^2(\mathbf{R})$. For the system $(l_j)_{1\le j\le 8}$, we define a 3×8 matrix

$$X = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \end{pmatrix}. \tag{3}$$

By a projective linear transformation, we may take l_1, l_2, l_3, l_4 as $\xi = 0, \eta = 0, \zeta = 0, \xi + \eta + \zeta = 0$. Moreover, we may divide the equation of l_j by a_{1j} $(5 \leq j \leq 8)$. In this manner, it is possible to choose as a representative of any element of $\mathbf{P}_0(2,8)$ a matrix of the form

$$X = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & 1 & 1 & y_1 & y_2 & y_3 & y_4 \end{pmatrix} \tag{4}$$

Therefore $\mathbf{P}_0(2,8)$ is regarded as a Zariski open subset of $\mathbf{R}^8$ by the correspondence

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & 1 & 1 & y_1 & y_2 & y_3 & y_4 \end{pmatrix} \longrightarrow (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4). \tag{5}$$

We introduce the following eight birational transformations $\sigma_0, \sigma_1, \ldots, \sigma_7$ on (x,y)-space:

$$\begin{array}{lcl} \sigma_0 : (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) & \longrightarrow & (\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4}, \frac{1}{y_1}, \frac{1}{y_2}, \frac{1}{y_3}, \frac{1}{y_4}) \\ \sigma_1 : (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) & \longrightarrow & (\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4}, \frac{y_1}{x_1}, \frac{y_2}{x_2}, \frac{y_3}{x_3}, \frac{y_4}{x_4}) \\ \sigma_2 : (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) & \longrightarrow & (y_1, y_2, y_3, y_4, x_1, x_2, x_3, x_4) \\ \sigma_3 : (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) & \longrightarrow & (x'_1, x'_2, x'_3, x'_4, y'_1, y'_2, y'_3, y'_4) \\ \sigma_4 : (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) & \longrightarrow & (\frac{1}{x_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1}, \frac{1}{y_1}, \frac{y_2}{y_1}, \frac{y_3}{y_1}, \frac{y_4}{y_1}) \\ \sigma_5 : (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) & \longrightarrow & (x_2, x_1, x_3, x_4, y_2, y_1, y_3, y_4) \\ \sigma_6 : (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) & \longrightarrow & (x_1, x_3, x_2, x_4, y_1, y_3, y_2, y_4) \\ \sigma_7 : (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) & \longrightarrow & (x_1, x_2, x_4, x_3, y_1, y_2, y_4, y_3), \end{array} \tag{6}$$

where

$$x'_j = \frac{x_j - y_j}{1 - y_j}, \quad y'_j = \frac{y_j}{y_j - 1}, \quad j = 1, 2, 3, 4.$$

The correspondence

$$s_{12} \longrightarrow \sigma_1, \quad s_{123} \longrightarrow \sigma_0, \quad s_{j,j+1} \longrightarrow \sigma_{j-1} \ (j = 3, \ldots, 8) \tag{7}$$

induces a surjective homomorphism $p_{W(E_8)}$ of $W(E_8)$ to the group $\tilde{W}(E_8)$ generated by $\sigma_0, \sigma_1, \ldots, \sigma_7$. In the sequel, we frequently identify $g \in W(E_8)$ with $p_{W(E_8)}(g)$ and subgroups of $W(E_8)$ with their images by $p_{W(E_8)}$ for simplicity.

The first step to attack Problem 1 is to find out a $W(E_8)$-orbit $\mathcal{O}$ of $\mathcal{P}_8$ and parametrize elements of $\mathcal{O}$ in terms of graphs attached with roots of Δ. For this purpose, we first treat the following problem.

Problem 2 *Find out such a system of marked 8 lines $(l_j)_{1\le j\le 8}$ that there is no hexagon for any system of marked six lines constructed from $(l_j)_{1\le j\le 8}$ by taking off two lines.*

Remark 2 *You can easily find systems of marked 8 lines such that there is no hexagon, heptagon, octagon, but that there is a hexagon for the system of marked 7 lines obtained from $(l_j)_{1\le j\le 8}$ by taking off one of the 8 lines.*

For a moment, we identify $\mathbf{R}^2$ with an open dense subset of $\mathbf{P}^2(\mathbf{R})$ by the map $(u,v)\mapsto(1:u:v)$, where (u,v) is a linear coordinate of $\mathbf{R}^2$. We consider the 3×8 matrix

$$X_0=\begin{pmatrix}1&0&0&1&1&1&1&1\\1&1&0&-27/10&-9&3/5&-17/10&-3/2\\1&0&1&12&4&9/5&53/10&21/10\end{pmatrix}. \tag{8}$$

By taking the entries of the vector $(\xi\ \ \eta\ \ \zeta)\cdot X_0=(f_1\dots f_8)$, we obtain a system of marked 8 lines $(l_j^0)_{1\le j\le 8}$, where

$$l_j^0: f_j=0 \quad (1\le j\le 8). \tag{9}$$

By the 8 lines (8), we obtain ten triangles (T_k) $(k=1,2,\dots,10)$ surrounded by the three lines given below:

(T_1)	$l_1^0l_2^0l_3^0$	$\boldsymbol{r}_{123}$
(T_2)	$l_1^0l_4^0l_6^0$	$\boldsymbol{r}_{146}$
(T_3)	$l_1^0l_5^0l_8^0$	$\boldsymbol{r}_{158}$
(T_4)	$l_1^0l_6^0l_7^0$	$\boldsymbol{r}_{167}$
(T_5)	$l_2^0l_5^0l_7^0$	$\boldsymbol{r}_{257}$
(T_6)	$l_2^0l_6^0l_8^0$	$\boldsymbol{r}_{268}$
(T_7)	$l_3^0l_4^0l_5^0$	$\boldsymbol{r}_{345}$
(T_8)	$l_3^0l_7^0l_8^0$	$\boldsymbol{r}_{378}$
(T_9)	$l_4^0l_7^0l_8^0$	$\boldsymbol{r}_{478}$
(T_{10})	$l_5^0l_6^0l_8^0$	$\boldsymbol{r}_{568}$

It is easy to see that the system of marked 8 lines (8) gives an answer to Problem 2, namely, that there are no hexagon, heptagon, octagon for $(l_j^0)_{1\le j\le 8}$. In the sequel, we denote by AE_8 the system (l_j^0). Let C_{AE_8} be the connected component of $\mathcal{P}_8$ containing AE_8.

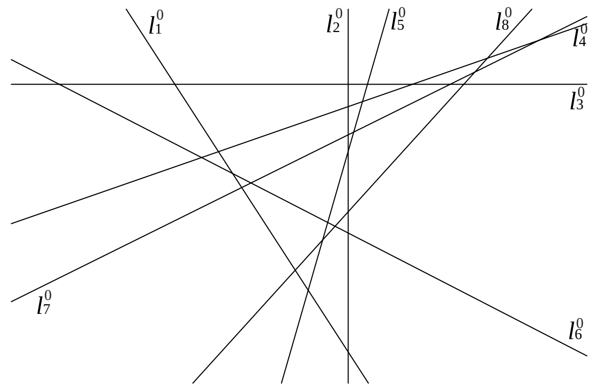

Figure 1: The eight lines configuration AE8

4 8LC sets and 8LC diagrams for the root system of type E_8

In this section, we give an interpretation of the system of marked 8 lines AE_8 in the previous section in terms of the root system Δ. For this purpose, we first introduce the notions of 8LC sets and 8LC diagrams for the root system of type E_8.

Definition 1 *Let a_i $(i = 1, 2, \ldots, 8)$ and b_1, b_2 be roots of Δ. Then the set*

$$\mathsf{A} = \{a_i; i = 1, 2, \ldots, 8\} \cup \{b_1, b_2\} \tag{10}$$

is called an 8LC(= 8 lines configuration) set if the following conditions hold:

(i)	$\langle a_i, a_j \rangle \neq 0$	*if and only if* $i - j \equiv 0$ *or* $\pm 1 \bmod 8$.
(ii)	$\langle b_1, b_2 \rangle = 0$.	
(iii.1)	$\langle a_i, b_1 \rangle \neq 0$	*if and only if* $i = 1$.
(iii.2)	$\langle a_i, b_2 \rangle \neq 0$	*if and only if* $i = 5$.

We would like to visualize each 8LC set by associating a diagram (similar to a Dynkin diagram). Let $\mathsf{A} = \{a_i; i = 1, \ldots, 8\} \cup \{b_1, b_2\}$ be an 8LC set. Then an 8LC diagram for A is a figure consisting of ten circles attached with roots of A and segments constructed in Figure 2. An 8LC diagram for A gives a complete information of orthogonality condition of elements of A. Namely, let c_1, c_2 $(c_1 \neq c_2)$ be roots of A. The $\langle c_1, c_2 \rangle \neq 0$ if and only if the circles corresponding to c_1, c_2 are connected by a segment.

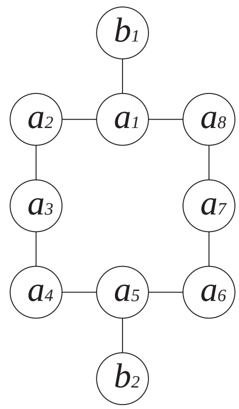

Figure 2: The 8LC diagram for A

For an 8LC set $\mathsf{A} = \{a_i; i = 1, \dots, 8\} \cup \{b_1, b_2\}$, we put

$$\tilde{\mathsf{A}} = \{\pm a_i; i = 1, \dots, 8\} \cup \{\pm b_1, \pm b_2\} \tag{11}$$

and call it an extended 8LC set. Let A' be also an 8LC set. Then A and A' are *equivalent* if and only if $\tilde{\mathsf{A}} = \tilde{\mathsf{A}}'$. In this case, we always identify an 8LC diagram for A and that for A' for simplicity.

Example 1 *The set*

$$\mathsf{U} = \{\boldsymbol{r}_{123}, \boldsymbol{r}_{146}, \boldsymbol{r}_{158}, \boldsymbol{r}_{167}, \boldsymbol{r}_{257}, \boldsymbol{r}_{268}, \boldsymbol{r}_{345}, \boldsymbol{r}_{378}, \boldsymbol{r}_{478}, \boldsymbol{r}_{568}\}$$

is an 8LC set. In particular, the correspondence

$$\begin{array}{llllllllllll}
\boldsymbol{r}_{568} & \longrightarrow & a_1 & \boldsymbol{r}_{123} & \longrightarrow & a_2 & \boldsymbol{r}_{478} & \longrightarrow & a_3 & \boldsymbol{r}_{378} & \longrightarrow & a_4 \\
\boldsymbol{r}_{146} & \longrightarrow & a_5 & \boldsymbol{r}_{167} & \longrightarrow & a_6 & \boldsymbol{r}_{345} & \longrightarrow & a_7 & \boldsymbol{r}_{268} & \longrightarrow & a_8 \\
\boldsymbol{r}_{158} & \longrightarrow & b_1 & \boldsymbol{r}_{257} & \longrightarrow & b_2 & & & & & &
\end{array}$$

induces an 8LC diagram for U.

Put

$$g_1 = s_{16}s_{38}s_{57}\tau_{24}, \quad g_2 = s_{18}s_{27}s_{45}\tau_{36}, \quad g_3 = s_{23}s_{123}s_{45}s_{145}s_{67}s_{167}\tau_8\tau_{18}.$$

Then g_1, g_2, g_3 *generate the isotropy subgroup* $Iso_{W(E_8)}(\tilde{\mathsf{U}})$ *of* $\tilde{\mathsf{U}}$ *in* $W(E_8)$, *where* $\tilde{\mathsf{U}}$ *is the extended 8LC set of* U. *In particular,* $Iso_{W(E_8)}(\tilde{\mathsf{U}}) \simeq (\mathbf{Z}_2)^3$. *Note that* g_3 *is the generator of the center of* $W(E_8)$.

The following lemma is shown by a direct computation.

Lemma 1 *If an 8LC set* A *contains*

$$\boldsymbol{r}_{12},\ \boldsymbol{r}_{123},\ \boldsymbol{r}_{23},\ \boldsymbol{r}_{34},\ \boldsymbol{r}_{45},\ \boldsymbol{r}_{56},\ \boldsymbol{r}_{67},\ \boldsymbol{r}_{78}$$

(these become simple roots of Δ*), then* $\tilde{\mathsf{A}}$ *coincides with*

$$\{\pm\boldsymbol{r}_{12}, \pm\boldsymbol{r}_{123}, \pm\boldsymbol{r}_{23}, \pm\boldsymbol{r}_{34}, \pm\boldsymbol{r}_{45}, \pm\boldsymbol{r}_{56}, \pm\boldsymbol{r}_{67}, \pm\boldsymbol{r}_{78}, \pm\boldsymbol{t}_{18}, \pm\boldsymbol{t}_{8}\}.$$

In virtue of this lemma, the classification of 8LC sets is essentially reduced to that of fundamental systems of roots of Δ, which is well-known, and we get

Proposition 1 *Let* A *and* A' *be 8LC sets. Then there exists* $w \in W(E_8)$ *such that* $w \cdot \tilde{\mathsf{A}} = \tilde{\mathsf{A}}'$.

Conjecture 1 *Retain the notation in Example 1. Then* $g_j \cdot C_{AE_8} = C_{AE_8} (j = 1, 2, 3)$. *Moreover,* $\{g \in W(E_8); g \cdot C_{AE_8} = C_{AE_8}\}$ *coincides with* $Iso_{W(E_8)}(\tilde{\mathsf{U}})$.

5 Experimental results — Generation of arrangements of 8 lines from those of 7 lines

There are 14 S_7-orbits of $\mathcal{P}_7$ which are called of types A, B1, B2, B3, B4, B5, C1, C2, C3, C4, D1, D2, D3, D4 (cf. [7],[8]). We generated 1629 simple 2-arrangements of 8 lines from the 14 simple 2-arrangements of 7 lines by experimental computation. We classified those arrangements into 91 different kinds in the following manner.

To observe the systems of 8 lines with the conditions I, II, we introduce a program counting numbers of polygons and adjacent sides of polygons for any system of n lines. We call the program "`CountPolygon`" which is written by the computer algebra system *Mathematica*. If we input a representative matrix of the system of marked n lines, then "`CountPolygon`" program outputs two kinds of data. One is the list of the numbers of polygons $\{m_3, m_4, \ldots, m_n\}$, where m_p represents the number of p-gons for the system [3]. The other is the matrix $(N_{pq})_{p,q=3,4,\ldots,n}$, where N_{pq} is the number of adjacent sides between p-gons and q-gons.

The result is summarized in Table 1. We explain the contents of the

Type	a	b	c	d	e	f	g	h	i	{3,4,5,6,7,8-gon Number}
(1)	6	24	12							{8,17,4,0,0,0}
(2)	68	34	8	6						{9,15,5,0,0,0}
(3)	11	6	33	54	11	8				{10,13,6,0,0,0}
(4)	41	15	4	12	20	8				{11,11,7,0,0,0}
(5)	18	12	16	1						{12,9,8,0,0,0}
(6)	7									{13,7,9,0,0,0}
(7)	21	16	19							{8,18,2,1,0,0}
(8)	51	31	34	16	21	23	41			{9,16,3,1,0,0}
(9)	36	30	34	54	18	19	25			{10,14,4,1,0,0}
(10)	32	29	12	21	12	15	16			{11,12,5,1,0,0}
(11)	17	17	11	21	13	13	14	10	4	{12,10,6,1,0,0}
(12)	13	7	16	8	14	5				{13,8,7,1,0,0}
(13)	11									{14,6,8,1,0,0}
(14)	6									{8,19,0,2,0,0}
(15)	22	7								{9,17,1,2,0,0}
(16)	12	15	11							{10,15,2,2,0,0}
(17)	12									{11,13,3,2,0,0}
(18)	37	4	19	11						{12,11,4,2,0,0}
(19)	18	16	9							{13,9,5,2,0,0}
(20)	9	6								{14,7,6,2,0,0}
(21)	8									{15,5,7,2,0,0}
(22)	4									{16,4,6,3,0,0}
(23)	31									{8,19,1,0,1,0}
(24)	44	14								{9,17,2,0,1,0}
(25)	15									{10,15,3,0,1,0}
(26)	7	13	16							{12,11,5,0,1,0}
(27)	1									{14,7,7,0,1,0}
(28)	7									{8,20,0,0,0,1}

Table 1: Numbers of arrangements for Types (1)-a,(1)-b, … ,(28)-a.

Table 1. There are 28 different lists of numbers of polygons. The set of arrangements contained in the row (X) (X=1, 2, ..., 28) are divided into families of arrangements by computing the matrices of the numbers of adjacent sides. The number of arrangements of the same matrix are contained in the middle column, say, a, b, ..., i. Our computation is not complete, namely, there are at least two simple 2-arrangements of 8 lines which have the same numbers of polygons and those of adjacent sides but are different with each other as systems of 8 lines with conditions I, II.

Remark 3 *We solved Problem 2 by the experimental computation using "*`CountPolygon`*". In other words, we found the example AE_8 of the system of marked 8 lines treated in section 3 from Type* (3)-a *in Table 1.*

We end this report by noting the difference between systems of 8 lines with conditions I, II and those with conditions I-IV. Two systems of 8 lines with conditions I, II are equivalent provided there exists a one-to-one incidence-preserving correspondence between their polygons. Then any two systems of 8 lines having an octagon are equivalent but the set of systems of 8 lines having an octagon can be decomposed into at least 15 different sets of systems with conditions I-IV (cf. [2]).

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