Approximate Singular-value Decomposition of a Matrix with Polynomial Entries

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June 4, 1998

Abstract

Singular-values of a matrix play a key role in the theory of linear algebra. Therefore, it is a well-known and very important fact that every constant matrix M has its singular-value decomposition, i.e., M can be decomposed into $M = UVW^T$, where U and W are unitary matrices and V is a diagonal matrix.

When M is a matrix with polynomial entries, it is not so easy to compute such decomposition. However, it is possible to compute polynomial matrices $U^{(k)}, V^{(k)}, W^{(k)}$ which satisfy

$$M \equiv U^{(k)}V^{(k)}\left(W^{(k)}\right)^T \pmod{(x,y,...,z)^{k+1}}, \quad (1)$$

$$\left(U^{(k)}\right)^T U^{(k)} \equiv E \pmod{(x,y,...,z)^{k+1}}, \quad (2)$$

$$\left(W^{(k)}\right)^T W^{(k)} \equiv E \pmod{(x,y,...,z)^{k+1}} \quad (3)$$

for any integer k (> 0), where $x, y, ..., z$ are variables in M and $V^{(k)}$ is a diagonal matrix. As you can see easily, the decomposition in (1) is the singular-value decomposition of M modulo ideal $(x,y,...,z)^{k+1}$ and we call the decomposition kth approximate singular-value decomposition.

In this paper, we use the Hensel lifting technique and give a complete algorithm to compute the above $U^{(k)}, V^{(k)}$ and $W^{(k)}$. The algorithm is quite efficient and the Hensel lifting can be performed with only 2×2 constant matrix inversions and basic matrix manipulations.

1 Introduction

Given $m \times n$ matrix M of rank r, there is $m \times m$ unitary matrix U and $n \times n$ unitary matrix W such that

$$M = UVW^T, \quad (4)$$

where V is an $m \times n$ zero matrix except for the diagonal entries $\sigma_1, ..., \sigma_r$ which are the singular-values of M. In the case that $m = n$, $V = \text{diag}(\sigma_1, ..., \sigma_r, 0, ..., 0)$. For example, let M be the following

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 1 \end{bmatrix}.$$

Then, the singular-value decomposition of M is given by

$$\begin{aligned} M &= UVW^T, \\ U &= \begin{bmatrix} 0.5251 & 0.0998 & 0.8452 \\ 0.8467 & -0.1609 & -0.5071 \\ 0.0854 & 0.9819 & -0.1690 \end{bmatrix}, \\ V &= \begin{bmatrix} 4.2500 & 0 \\ 0 & 1.3920 \\ 0 & 0 \end{bmatrix}, \\ W &= \begin{bmatrix} 0.5019 & 0.8649 \\ -0.8649 & 0.5019 \end{bmatrix}. \end{aligned}$$

The singular-value is one of the most important properties of a matrix and is closely related various other properties.

Relation to eignvalues:

The singular-values can be defined with the eigenvalues of a matrix in the following way

$$\sigma_i(M) = \lambda_i(M^*M) \quad (i = 1, 2, ...),$$

where $\sigma_i(M)$ and $\lambda_i(M)$ denote the singular-values and the eigenvalues of matrix M, respectively, and M^* denotes transpose conjugate of M.

Relation to a norm of a matrix:

Define 2-norm of a matrix M as

$$||M||^2 \triangleq \max \frac{||Mx||^2}{||x||^2},$$

where x is a vector in adequate size and $||x||^2$ and $||Mx||^2$ denote Euclidean norm. Then, we have

$$||M||^2 = \sigma_{\max}(M),$$

where $\sigma_{\max}(M)$ denotes the maximal singular-value of M.

Over the past several years, a considerable number of studies have been made to compute the singular-value decomposition of a matrix with numerical entries. However, very little have been done for the computation of the singular-value decomposition of a matrix with *polynomial* entries. In this paper, we show that the singular-value decomposition of a matrix with polynomial entries in the form of power series, which we call approximate singular-value decomposition, can be computed efficiently and give an algorithm for its computations.

The approximate singular-value decomposition can be used, for example, for H_∞ theory in the control theory, where the singular-value decomposition of a matrix play a key role. More concretely, given a system with unknown-parameters, the approximate singular-value decomposition enable us to compute the H_∞ norm of the system in the form of power series.

This paper is organized as follows; In **2**, we describe definitions and notations. In **3**, we define "approximate singular-value decomposition" and describe an algorithm for its compution. Lastly, in **4**, we conclude.

2 Preliminaries

A matrix whose entries are numbers and polynomials are called numerical and polynomial matrix, respectively. With the notation

$$a \equiv b \ (\mathrm{mod}\ (x, y, ..., z)^{k+1}),$$

we denote that a and b are equal up to the terms with total degree k of $x, y, ..., z$. For example, we have

$$\begin{aligned} \sin(x) &\equiv x + \frac{x^3}{3!} \ (\mathrm{mod}\ x^4), \\ (1+x+y)^3 &\equiv 1 + 3x + 3y + 3x^2 + 6xy + 3y^2 \ (\mathrm{mod}\ (x,y)^3). \end{aligned}$$

M^T and $\{M\}_{ij}$ denote transpose and (i, j)th entry of matrix M, respectively. When M is a polynomial matrix, $\lfloor M \rfloor_l$ denotes the numerical matrix which consists of coefficients of x^l. For example, let M be the following polynomial matrix,

$$M = \begin{bmatrix} 1+x & 2-x+x^2 \\ -1+x^2 & 3-x+x^2 \end{bmatrix}.$$

Then we have

$$\lfloor M \rfloor_0 = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \ \lfloor M \rfloor_1 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \ \lfloor M \rfloor_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

3 Approximate singular-value decomposition

3.1 Definition

Definition 1 (Approximate singular-value decomposition)
Let M be a given polynomial matrix in $x, y, ..., z$. Then the decomposition

of M in the form of

$$M \equiv U^{(k)}V^{(k)}\left(W^{(k)}\right)^T \pmod{(x,y,...,z)^{k+1}} \tag{5}$$

is called kth approximate singular-value decomposition , where $V^{(k)}$ is a diagonal matrix and $U^{(k)}, W^{(k)}$ are matrices satisfying

$$\left(U^{(k)}\right)^T U^{(k)} \equiv E \pmod{(x,y,...,z)^{k+1}}, \tag{6}$$

$$\left(W^{(k)}\right)^T W^{(k)} \equiv E \pmod{(x,y,...,z)^{k+1}}. \tag{7}$$

As is obvious from the above definition, diagonal entries $\sigma_i^{(k)}$ $(i = 1, ..., n)$ of $V^{(k)}$ are equal to the taylor series expansions of the exact sigular-values of M (note that since the exact singular-values of M are algebraic functions of variables in M, they have its taylor series expansions), up to kth total degree of $x, y, ..., z$, i.e.,

$$\sigma_i^{(k)} \equiv \sigma_i \pmod{(x,y,...,z)^{k+1}},$$

where σ_i $(i = 1, ..., n)$ are exact singular-values of M.

3.2 Computation

For simplicity of the algorithm description, we assume that given matrix is uni-variate polynomial matrix in x. Our algorithm in this section has a straight-forward extension to a multivariate case.

Let M be an $m \times n$ polynomial matrix in x. We will compute kth approximate singular-value decomposition of M, i.e., we find polynomial matrices $U^{(k)}, V^{(k)}$ and $W^{(k)}$ which satisfy the following conditions;

$$\begin{cases} V^{(k)} \text{ is a diagonal matrix} \\ M \equiv U^{(k)}V^{(k)}\left(W^{(k)}\right)^T \pmod{x^{k+1}} \\ \left(U^{(k)}\right)^T U^{(k)} \equiv E \pmod{x^{k+1}} \\ \left(W^{(k)}\right)^T W^{(k)} \equiv E \pmod{x^{k+1}} \end{cases} \tag{8}$$

We also assume that $m \geq n$ without loss of generality (when $m < n$, it is enough to compute the approximate singular-value decomposition of M^T). Note that $U^{(k)}, V^{(k)}$ and $W^{(k)}$ can be expressed in the following form

$$\begin{aligned} U^{(k)} &= U_0 + U_1 x + \cdots + U_k x^k, \\ V^{(k)} &= V_0 + V_1 x + \cdots + V_k x^k, \\ W^{(k)} &= W_0 + W_1 x + \cdots + W_k x^k, \end{aligned}$$

where U_i, V_i, W_i $(i = 0, ..., k)$ are numerical matrices. Let M_0 be the constant part of M. Then, U_0, V_0, W_0 must satisfy

$$\begin{cases} V_0 \text{ is a diagonal matrix} \\ M_0 = U_0 V_0 W_0^T \\ U_0^T U_0 = E \\ W_0^T W_0 = E \end{cases}, \tag{9}$$

which implies that $M_0 = U_0 V_0 W_0^T$ is the singular value decomposition of numerical matrix M_0. Hence, U_0, V_0, W_0 can be computed with a numerical singular-value decomposition algorithm. Now, we give an algorithm to compute U_l, V_l, W_l, assuming that U_i, V_i, W_i $(i = 0, ..., l-1)$ are already computed. Note that with this algorithm, all of U_i, V_i, W_i $(i = 0, ..., l)$ can be computed inductively. Our algorithm requires the following two conditions;

$$\text{(Cond. 1)} \quad : \quad \text{All singular-values of } M_0 \text{ are distinct.} \tag{10}$$

$$\text{(Cond. 2)} \quad : \quad \text{All singular-values of } M_0 \text{ are non-zeros.} \tag{11}$$

We suppose that the two conditions hold. The approximate singular-value decomposition exists even if the two conditions are not satisfied. However, in this case, the approximate singular-value decomposition is not unique and the algorithm for its computation can be complicated. Hence, we deal with the case in the comming paper.

Since condition (8) gives

$$\begin{cases} V^{(l)} \text{ is a diagonal matrix} \\ M \equiv U^{(l)} V^{(l)} \left(W^{(l)}\right)^T \pmod{x^{l+1}} \\ \left(U^{(l)}\right)^T U^{(l)} \equiv E \pmod{x^{l+1}} \\ \left(W^{(l)}\right)^T W^{(l)} \equiv E \pmod{x^{l+1}} \end{cases} \tag{12}$$

for any integer l satisfying $(1 \le l \le k)$, substituting

$$\begin{cases} U^{(l)} = U^{(l-1)} + U_l x^l \\ V^{(l)} = V^{(l-1)} + V_l x^l \\ W^{(l)} = W^{(l-1)} + W_l x^l \end{cases}$$

into (12), we obtain

$$\begin{aligned} M - U^{(l-1)} V^{(l-1)} \left(W^{(l-1)}\right)^T &\equiv U_l V_0 W_0^T x^l + U_0 V_l W_0^T x^l + \\ &\quad\; U_0 V_0 W_l^T x^l \pmod{x^{l+1}}, \\ E - \left(U^{l-1}\right)^T U^{(l-1)} &\equiv U_0^T U_l x^l + U_l^T U_0 x^l \pmod{x^{l+1}}, \\ E - \left(W^{l-1}\right)^T W^{(l-1)} &\equiv W_0^T W_l x^l + W_l^T W_0 x^l \pmod{x^{l+1}}. \end{aligned}$$

Looking at coefficients of x^l in the above equations, we obtain

$$\begin{aligned} \tilde{M}_l &= U_l V_0 W_0^T + U_0 V_l W_0^T + U_0 V_0 W_l^T, & (13)\\ \tilde{P}_l &= U_0^T U_l + U_l^T U_0, & (14)\\ \tilde{Q}_l &= W_0^T W_l + W_l^T W_0, & (15) \end{aligned}$$

where $\tilde{M}_l, \tilde{P}_l, \tilde{Q}_l$ are numerical matrices defined as

$$\begin{aligned} \tilde{M}_l &\triangleq \lfloor M - U^{(l-1)} V^{(l-1)} \left(W^{(l-1)}\right)^T \rfloor_l, & (16)\\ \tilde{P}_l &\triangleq \lfloor E - \left(U^{(l-1)}\right)^T U^{(l-1)} \rfloor_l, & (17)\\ \tilde{Q}_l &\triangleq \lfloor E - \left(W^{(l-1)}\right)^T W^{(l-1)} \rfloor_l. & (18) \end{aligned}$$

Multiplying U_0^T and W_0 on both sides of (13) from left and right-hand side, we obtain

$$U_0^T \tilde{M}_l W_0 = U_0^T U_l V_0 + V_l + V_0 W_l^T W_0. \qquad (19)$$

We define $\tilde{U}_l, \tilde{W}_l$ as

$$\tilde{U}_l \triangleq U_0^T U_l,\ \tilde{W}_l \triangleq W_l^T W_0,$$

and (19),(14),(15) give

$$\begin{aligned} U_0^T \tilde{M}_l W_0^T &= \tilde{U}_l V_0 + V_l + V_0 \tilde{W}_l, & (20)\\ \tilde{P}_l &= \tilde{U}_l + \tilde{U}_l^T, & (21)\\ \tilde{Q}_l &= \tilde{W}_l + \tilde{W}_l^T. & (22) \end{aligned}$$

Note that (21) and (22) give

$$\left\{\tilde{U}_l\right\}_{ii} = \frac{1}{2}\left\{\tilde{P}_l\right\}_{ii},\ \ \left\{\tilde{W}_l\right\}_{ii} = \frac{1}{2}\left\{\tilde{Q}_l\right\}_{ii}\ (1 \le i \le n). \qquad (23)$$

Since V_0 is a diagonal matrix, diagonal entries of $\tilde{U}_l V_0$ and $V_0 \tilde{W}_l$ are given by

$$\left\{\tilde{U}_l V_0\right\}_{ii} = \{V_0\}_{ii}\left\{\tilde{U}_l\right\}_{ii},\ \ \left\{V_0 \tilde{W}_l\right\}_{ii} = \{V_0\}_{ii}\left\{\tilde{W}_l\right\}_{ii}\ (1 \le i \le n). \qquad (24)$$

Therefore, (23) and (24) give

$$\left\{\tilde{U}_l V_0\right\}_{ii} = \frac{1}{2}\{V_0\}_{ii}\left\{\tilde{P}_l\right\}_{ii},\ \ \left\{V_0 \tilde{W}_l\right\}_{ii} = \frac{1}{2}\{V_0\}_{ii}\left\{\tilde{Q}_l\right\}_{ii}\ (1 \le i \le n). \qquad (25)$$

The above equalities and (20) imply that

$$\begin{aligned} \{V_l\}_{ii} &= \left\{U_0^T \tilde{M}_l W_0\right\}_{ii} - \left\{\tilde{U}_l V_0\right\}_{ii} - \left\{V_0 \tilde{W}_l\right\}_{ii}\\ &= \left\{U_0^T \tilde{M}_l W_0\right\}_{ii} - \frac{1}{2}\{V_0\}_{ii}\left(\left\{\tilde{P}_l\right\}_{ii} + \left\{\tilde{Q}_l\right\}_{ii}\right)\ (1 \le i \le n). & (26) \end{aligned}$$

Since right-hand side of the above equation is already known, it is possible to compute $\{V_l\}_{ii}$. Since V_l is a diagonal matrix, this determines whole matrix V_l.

Next, we will determine $\tilde{U}_l$ and $\tilde{W}_l$. Diagonal entries $\left\{\tilde{U}_l\right\}_{ii}, \left\{\tilde{W}_l\right\}_{ii}$ $(1 \le i \le n)$ can be computed with (23) and non-diagonal entries $\left\{\tilde{U}_l\right\}_{ij}, \left\{\tilde{W}_l\right\}_{ij}$ $(1 \le i \le n,\ 1 \le j \le n,\ i \ne j)$ can be computed as follows; Since V_0 is a diagonal matrix, we have

$$\left\{\tilde{U}_l V_0\right\}_{ij} = \{V_0\}_{jj}\left\{\tilde{U}_l\right\}_{ij},\ \left\{V_0 \tilde{U}_l\right\}_{ij} = \{V_0\}_{ii}\left\{\tilde{W}_l\right\}_{ij}\ (1 \le i \le m,\ 1 \le j \le n). \tag{27}$$

Notice that (21) and (22) give

$$\left\{\tilde{U}_l\right\}_{ij} + \left\{\tilde{U}_l\right\}_{ji} = \left\{\tilde{P}_l\right\}_{ij}\ (1 \le i \le m,\ 1 \le j \le m), \tag{28}$$
$$\left\{\tilde{W}_l\right\}_{ij} + \left\{\tilde{W}_l\right\}_{ji} = \left\{\tilde{Q}_l\right\}_{ij}\ (1 \le i \le n,\ 1 \le j \le n). \tag{29}$$

Therefore,

$$\left\{\tilde{U}_l\right\}_{ji} = \left\{\tilde{P}_l\right\}_{ij} - \left\{\tilde{U}_l\right\}_{ij}\ (1 \le i \le m,\ 1 \le j \le m), \tag{30}$$
$$\left\{\tilde{W}_l\right\}_{ji} = \left\{\tilde{Q}_l\right\}_{ij} - \left\{\tilde{W}_l\right\}_{ij}\ (1 \le i \le n,\ 1 \le j \le n). \tag{31}$$

Note also that (20) and (27) give, for $(1 \le i \le m,\ 1 \le j \le n)$,

$$\begin{aligned}\left\{U_0^T \tilde{M}_l W_0\right\}_{ij} - \{V_l\}_{ij} &= \left\{\tilde{U}_l V_0\right\}_{ij} + \left\{V_0 \tilde{W}_l\right\}_{ij} \\ &= \{V_0\}_{jj}\left\{\tilde{U}_l\right\}_{ij} + \{V_0\}_{ii}\left\{\tilde{W}_l\right\}_{ij}.\end{aligned} \tag{32}$$

Exchanging index i and j in the above equation, we obtain

$$\left\{U_0^T \tilde{M}_l W_0\right\}_{ji} - \{V_l\}_{ji} = \{V_0\}_{ii}\left\{\tilde{U}_l\right\}_{ji} + \{V_0\}_{jj}\left\{\tilde{W}_l\right\}_{ji}\ (1 \le i \le n,\ 1 \le j \le m). \tag{33}$$

Substituting (30) and (31) into the above equation, we obtain, for $(1 \le i \le n,\ 1 \le j \le m)$,

$$\{V_0\}_{ii}\left\{\tilde{U}_l\right\}_{ij} + \{V_0\}_{jj}\left\{\tilde{W}_l\right\}_{ij} = -\left\{V_0^T \tilde{M}_l W_0\right\}_{ji} + \{V_l\}_{ji} + \{V_0\}_{ii}\left\{\tilde{P}_l\right\}_{ij} + \{V_0\}_{jj}\left\{\tilde{Q}_l\right\}_{ij}. \tag{34}$$

This and (32) give, for $(1 \le i \le n,\ 1 \le j \le n)$,

$$\begin{bmatrix} \{V_0\}_{jj} & \{V_0\}_{ii} \\ \{V_0\}_{ii} & \{V_0\}_{jj} \end{bmatrix} \begin{bmatrix} \left\{\tilde{U}_l\right\}_{ij} \\ \left\{\tilde{W}_l\right\}_{ij} \end{bmatrix} =$$
$$\begin{bmatrix} \left\{U_0^T \tilde{M}_l W_0\right\}_{ij} - \{V_l\}_{ij} \\ -\left\{V_0^T \tilde{M}_l W_0\right\}_{ji} + \{V_l\}_{ji} + \{V_0\}_{ii}\left\{\tilde{P}_l\right\}_{ij} + \{V_0\}_{jj}\left\{\tilde{Q}_l\right\}_{ij} \end{bmatrix}. \tag{35}$$

Since we supposed condition (10), for any i, j satisfying $i \neq j$, matrix $\begin{bmatrix} \{V_0\}_{jj} & \{V_0\}_{ii} \\ \{V_0\}_{ii} & \{V_0\}_{jj} \end{bmatrix}$ has its inverse and $\left\{\tilde{U}_l\right\}_{ij}$ and $\left\{\tilde{W}_l\right\}_{ij}$ $(1 \leq i \leq n,\ 1 \leq j \leq n,\ i \neq j)$ are given by

$$\begin{bmatrix} \left\{\tilde{U}_l\right\}_{ij} \\ \left\{\tilde{W}_l\right\}_{ij} \end{bmatrix} = \frac{1}{\left(\{V_0\}_{jj} - \{V_0\}_{ii}\right)\left(\{V_0\}_{jj} + \{V_0\}_{ii}\right)} \begin{bmatrix} \{V_0\}_{jj} & -\{V_0\}_{ii} \\ -\{V_0\}_{ii} & \{V_0\}_{jj} \end{bmatrix} \times \begin{bmatrix} \left\{U_0^T \tilde{M}_l W_0\right\}_{ij} \\ -\left\{V_0^T \tilde{M}_l W_0\right\}_{ji} + \{V_0\}_{ii}\left\{\tilde{P}_l\right\}_{ij} + \{V_0\}_{jj}\left\{\tilde{Q}_l\right\}_{ij} \end{bmatrix} \tag{36}$$

(note that $\{V_0\}_{ij} = 0,\ i \neq j$). Now, we have determined $\left\{\tilde{U}_l\right\}_{ij}, \left\{\tilde{W}_l\right\}_{ij}$ $(1 \leq i \leq n,\ 1 \leq j \leq n)$. Since $\tilde{W}_l$ is an $n \times n$ matrix, this determines whole $\tilde{W}_l$ and a part of $\tilde{U}_l$. Hence, we only need to compute the rest of $\tilde{U}_l$, i.e., $\left\{\tilde{U}_l\right\}_{ij}$ $(n+1 \leq i \leq m,\ n+1 \leq j \leq m)$. Since we have

$$\{V_0\}_{ii} = \{V_l\}_{ij} = 0 \quad (n+1 \leq i \leq m,\ 1 \leq j \leq n),$$

(32) gives

$$\{V_0\}_{jj}\left\{\tilde{U}_l\right\}_{ij} = \left\{U_0^T \tilde{M}_l W_0\right\}_{ij}.$$

Since we supposed condition (11) (i.e., $\{V_0\}_{jj} \neq 0\ (j = 1, ..., n)$), we obtain

$$\left\{\tilde{U}_l\right\}_{ij} = \frac{\left\{U_0^T \tilde{M}_l W_0\right\}_{ij}}{\{V_0\}_{jj}} \quad (n+1 \leq i \leq m,\ 1 \leq j \leq n). \tag{37}$$

This and (30) determine $\left\{\tilde{U}_l\right\}_{ij}$ $(1 \leq i \leq n,\ n+1 \leq j \leq m)$.

Now only undecided part of $\tilde{U}_l$ are $\left\{\tilde{U}_l\right\}_{ij}$ $(n+1 \leq i \leq m,\ n+1 \leq j \leq m)$. This part of $\tilde{U}_l$ can not be uniquely determined. In fact, every $\left\{\tilde{U}_l\right\}_{ij}$ $(n+1 \leq i \leq m,\ n+1 \leq j \leq m)$ satisfying (28) is enough. Hence, one possible choice of $\left\{\tilde{U}_l\right\}_{ij}$ is

$$\left\{\tilde{U}_l\right\}_{ij} = \left\{\tilde{U}_l\right\}_{ji} = \frac{1}{2}\left\{\tilde{P}_l\right\}_{ij} \quad (n+1 \leq i \leq m,\ n+1 \leq j \leq m). \tag{38}$$

Now, all entries of matrices $\tilde{U}_l$ and $\tilde{W}_l$ are determined. Then, matrices U_l and W_l are computed with

$$U_l = U_0 \tilde{U}_l,\ W_l = W_0 \tilde{W}_l^T. \tag{39}$$

We describe the above algorithm formally as follows;

Algorithm 1 Computation of the approximate singular-value decomposition

Input : Polynomial matrix $M = M_0 + \cdots + M_r x^r$;
Output : The kth approximate singular-value decomposition;
Condition : The singular-value of M_0 must be non-zero and distinct;

Step 1 If $m < n$, then put $M \leftarrow M^T,\ t \leftarrow 1$ and exchange m and n. Otherwise put $t \leftarrow 0$.

Step 2 Compute singular-value decomposition $M_0 = U_0 V_0 W_0^T$ of M_0 with a numerical algorithm. Let $l \leftarrow 1$ and $U^{(0)} \leftarrow U_0,\ V^{(0)} \leftarrow V_0,\ W^{(0)} \leftarrow W_0$.

Step 3 Compute $\tilde{M}_l, \tilde{P}_l$ and $\tilde{Q}_l$ with formula (16),(17) and (18).

Step 4 Compute $\{V_l\}_{ii}$ $(1 \le i \le n)$ with formula (26) and determine matrix V_l.

Step 5 Compute $\{U_l\}_{ii}$ and $\{W_l\}_{ii}$ $(1 \le i \le n)$ with formula (23).

Step 6 Compute $\{U_l\}_{ij}$ and $\{W_l\}_{ij}$ $(1 \le i \le n,\ 1 \le j \le n,\ i \ne j)$ with formula (36).

Step 7 Compute $\{U_l\}_{ij}$ $(n+1 \le i \le m,\ 1 \le j \le n)$ with formula (37).

Step 8 Compute $\{U_l\}_{ij}$ $(1 \le i \le n,\ n+1 \le j \le m)$ with formula (30).

Step 9 Compute $\{U_l\}_{ij}$ $(n+1 \le i \le m,\ n+1 \le j \le m)$ with formula (38).

Step 10 Put U_l, W_l as

$$U_l = U_0 \tilde{U}_l, \quad W_l = W_0 \tilde{W}_l^T$$

and let $U^{(l)}, V^{(l)}, W^{(l)}$ be

$$\begin{aligned} U^{(l)} &\leftarrow U^{(l-1)} + U_l x^l, \\ V^{(l)} &\leftarrow V^{(l-1)} + V_l x^l, \\ W^{(l)} &\leftarrow W^{(l-1)} + W_l x^l. \end{aligned}$$

Step 11 If $k < l$, then let $l \leftarrow l + 1$ and go to *Step* 3.

Step 12 If $t = 1$, then put

$$tmp \leftarrow U^{(k)},\ U^{(k)} \leftarrow \left(W^{(k)}\right)^T,\ V^{(k)} \leftarrow \left(V^{(k)}\right)^T,\ W^{(k)} \leftarrow (tmp)^T.$$

Step 13 Return $U^{(k)}, V^{(k)}, W^{(k)}$.

4 Conclusion

The approximate singular-value decomposition of a polynomial matrix was defined and an algorithm to compute the decomposition is given. The algorithm uses only 2×2 matrix inversions and basic matrix manipulations, hence is quite efficient.

References

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