Approximate Singular-value Decomposition of a Matrix with Polynomial Entries

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Abstract

Singular-values of a matrix play a key role in the theory of linear algebra. Therefore, it is a well-known and very important fact that every constant matrix $M$ has its singular-value decomposition, i.e., $M$ can be decomposed into $M = UVW^T$, where $U$ and $W$ are unitary matrices and $V$ is a diagonal matrix.

When $M$ is a matrix with polynomial entries, it is not so easy to compute such decomposition. However, it is possible to compute polynomial matrices $U^{(k)}, V^{(k)}, W^{(k)}$ which satisfy

$$M \equiv U^{(k)}V^{(k)}(W^{(k)})^T \pmod{(x,y,\ldots,z)^{k+1}}, \quad (1)$$

$$\left(U^{(k)}\right)^T U^{(k)} \equiv E \pmod{(x,y,\ldots,z)^{k+1}}, \quad (2)$$

$$\left(W^{(k)}\right)^T W^{(k)} \equiv E \pmod{(x,y,\ldots,z)^{k+1}} \quad (3)$$

for any integer $k > 0$, where $x, y, \ldots, z$ are variables in $M$ and $V^{(k)}$ is a diagonal matrix. As you can see easily, the decomposition in (1) is the singular-value decomposition of $M$ modulo ideal $(x,y,\ldots,z)^{k+1}$ and we call the decomposition $k$th approximate singular-value decomposition.

In this paper, we use the Hensel lifting technique and give a complete algorithm to compute the above $U^{(k)}, V^{(k)}$ and $W^{(k)}$. The algorithm is quite efficient and the Hensel lifting can be performed with only $2 \times 2$ constant matrix inversions and basic matrix manipulations.

1 Introduction

Given $m \times n$ matrix $M$ of rank $r$, there is $m \times m$ unitary matrix $U$ and $n \times n$ unitary matrix $W$ such that

$$M = UVW^T, \quad (4)$$
where $V$ is an $m \times n$ zero matrix except for the diagonal entries $\sigma_1, \ldots, \sigma_r$ which are the singular-values of $M$. In the case that $m = n$, $V = \text{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$. 

For example, let $M$ be the following

$$
M = \begin{bmatrix}
1 & 2 \\
2 & 3 \\
-1 & 1
\end{bmatrix}.
$$

Then, the singular-value decomposition of $M$ is given by

$$
M = UVW^T,
$$

$$
U = \begin{bmatrix}
0.5251 & 0.0998 & 0.8452 \\
0.8467 & -0.1609 & -0.5071 \\
0.0854 & 0.9819 & -0.1690
\end{bmatrix},
$$

$$
V = \begin{bmatrix}
4.2500 & 0 \\
0 & 1.3920 \\
0 & 0
\end{bmatrix},
$$

$$
W = \begin{bmatrix}
0.5019 & 0.8649 \\
-0.8649 & 0.5019
\end{bmatrix}.
$$

The singular-value is one of the most important properties of a matrix and is closely related to various other properties.

**Relation to eigenvalues:**

The singular-values can be defined with the eigenvalues of a matrix in the following way

$$
\sigma_i(M) = \lambda_i(M^*M) \quad (i = 1, 2, \ldots),
$$

where $\sigma_i(M)$ and $\lambda_i(M)$ denote the singular-values and the eigenvalues of matrix $M$, respectively, and $M^*$ denotes transpose conjugate of $M$.

**Relation to a norm of a matrix:**

Define 2-norm of a matrix $M$ as

$$
||M||^2 = \max ||Mx||^2 ||x||^2,
$$

where $x$ is a vector in adequate size and $||x||^2$ and $||Mx||^2$ denote Euclidean norm. Then, we have

$$
||M||^2 = \sigma_{\max}(M),
$$

where $\sigma_{\max}(M)$ denotes the maximal singular-value of $M$.

Over the past several years, a considerable number of studies have been made to compute the singular-value decomposition of a matrix with numerical entries. However, very little have been done for the computation of
the singular-value decomposition of a matrix with \textit{polynomial} entries. In this paper, we show that the singular-value decomposition of a matrix with polynomial entries in the form of power series, which we call approximate singular-value decomposition, can be computed efficiently and give an algorithm for its computations.

The approximate singular-value decomposition can be used, for example, for \( H_\infty \) theory in the control theory, where the singular-value decomposition of a matrix play a key role. More concretely, given a system with unknown-parameters, the approximate singular-value decomposition enable us to compute the \( H_\infty \) norm of the system in the form of power series.

This paper is organized as follows; In 2, we describe definitions and notations. In 3, we define “approximate singular-value decomposition” and describe an algorithm for its computation. Lastly, in 4, we conclude.

2 Preliminaries

A matrix whose entries are numbers and polynomials are called numerical and polynomial matrix, respectively. With the notation

\[ a \equiv b \pmod{(x, y, ..., z)^{k+1}}, \]

we denote that \( a \) and \( b \) are equal up to the terms with total degree \( k \) of \( x, y, ..., z \). For example, we have

\[
\begin{align*}
\sin(x) & \equiv x + \frac{x^3}{3!} \pmod{x^4}, \\
(1 + x + y)^3 & \equiv 1 + 3x + 3y + 3x^2 + 6xy + 3y^2 \pmod{(x, y)^3}.
\end{align*}
\]

\( M^T \) and \( (M)_{ij} \) denote transpose and \( (i, j) \)th entry of matrix \( M \), respectively. When \( M \) is a polynomial matrix, \( [M]_i \) denotes the numerical matrix which consists of coefficients of \( x^i \). For example, let \( M \) be the following polynomial matrix,

\[ M = \begin{bmatrix}
1 + x & 2 - x + x^2 \\
-1 + x^2 & 3 - x + x^2
\end{bmatrix}.
\]

Then we have

\[ [M]_0 = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \quad [M]_1 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \quad [M]_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.\]

3 Approximate singular-value decomposition

3.1 Definition

\textbf{Definition 1} (Approximate singular-value decomposition)

Let \( M \) be a given polynomial matrix in \( x, y, ..., z \). Then the decomposition
of $M$ in the form of

$$M \equiv U^{(k)} V^{(k)} (W^{(k)})^T \pmod{(x, y, ..., z)^{k+1}}$$  \hspace{0.5cm} (5)$$

is called $k$th approximate singular-value decomposition, where $V^{(k)}$ is a diagonal matrix and $U^{(k)}, W^{(k)}$ are matrices satisfying

$$\begin{align*}
(U^{(k)})^T U^{(k)} &\equiv E \pmod{(x, y, ..., z)^{k+1}}, \quad (6) \\
(W^{(k)})^T W^{(k)} &\equiv E \pmod{(x, y, ..., z)^{k+1}}. \quad (7)
\end{align*}$$

As is obvious from the above definition, diagonal entries $\sigma_i^{(k)}$ ($i = 1, ..., n$) of $V^{(k)}$ are equal to the taylor series expansions of the exact singular-values of $M$ (note that since the exact singular-values of $M$ are algebraic functions of variables in $M$, they have taylor series expansions), up to $k$th total degree of $x, y, ..., z$, i.e.,

$$\sigma_i^{(k)} \equiv \sigma_i \pmod{(x, y, ..., z)^{k+1}},$$

where $\sigma_i$ ($i = 1, ..., n$) are exact singular-values of $M$.

### 3.2 Computation

For simplicity of the algorithm description, we assume that given matrix is uni-variate polynomial matrix in $x$. Our algorithm in this section has a straightforward extension to a multivariate case.

Let $M$ be an $m \times n$ polynomial matrix in $x$. We will compute $k$th approximate singular-value decomposition of $M$, i.e., we find polynomial matrices $U^{(k)}, V^{(k)}$ and $W^{(k)}$ which satisfy the following conditions;

$$\begin{cases}
V^{(k)} \text{ is a diagonal matrix} \\
M \equiv U^{(k)} V^{(k)} (W^{(k)})^T \pmod{x^{k+1}} \\
(U^{(k)})^T U^{(k)} \equiv E \pmod{x^{k+1}} \\
(W^{(k)})^T W^{(k)} \equiv E \pmod{x^{k+1}}
\end{cases}$$  \hspace{0.5cm} (8)$$

We also assume that $m \geq n$ without loss of generality (when $m < n$, it is enough to compute the approximate singular-value decomposition of $M^T$). Note that $U^{(k)}, V^{(k)}$ and $W^{(k)}$ can be expressed in the following form

$$\begin{align*}
U^{(k)} &\equiv U_0 + U_1 x + \cdots + U_k x^k, \\
V^{(k)} &\equiv V_0 + V_1 x + \cdots + V_k x^k, \\
W^{(k)} &\equiv W_0 + W_1 x + \cdots + W_k x^k.
\end{align*}$$
where $U_i, V_i, W_i$ ($i = 0, \ldots, k$) are numerical matrices. Let $M_0$ be the constant part of $M$. Then, $U_0, V_0, W_0$ must satisfy

$$
\begin{align*}
V_0 &\text{ is a diagonal matrix,} \\
M_0 &= U_0 V_0 W_0^T \\
U_0^T U_0 &= E \\
W_0^T W_0 &= E
\end{align*}
$$

which implies that $M_0 = U_0 V_0 W_0^T$ is the singular value decomposition of numerical matrix $M_0$. Hence, $U_0, V_0, W_0$ can be computed with a numerical singular-value decomposition algorithm. Now, we give an algorithm to compute $U_l, V_l, W_l$, assuming that $U_i, V_i, W_i$ ($i = 0, \ldots, l-1$) are already computed. Note that with this algorithm, all of $U_i, V_i, W_i$ ($i = 0, \ldots, l$) can be computed inductively. Our algorithm requires the following two conditions;

\begin{itemize}
  \item [(Cond. 1)] All singular-values of $M_0$ are distinct.
  \item [(Cond. 2)] All singular-values of $M_0$ are non-zeros.
\end{itemize}

We suppose that the two conditions hold. The approximate singular-value decomposition exists even if the two conditions are not satisfied. However, in this case, the approximate singular-value decomposition is not unique and the algorithm for its computation can be complicated. Hence, we deal with the case in the coming paper.

Since condition (8) gives

$$
\begin{align*}
V^{(l)} &\text{ is a diagonal matrix} \\
M &\equiv U^{(l)} V^{(l)} (W^{(l)})^T \pmod{x^{l+1}} \\
(U^{(l)})^T U^{(l)} &\equiv E \pmod{x^{l+1}} \\
(W^{(l)})^T W^{(l)} &\equiv E \pmod{x^{l+1}}
\end{align*}
$$

for any integer $l$ satisfying $(1 \leq l \leq k)$, substituting

$$
\begin{align*}
U^{(l)} &= U^{(l-1)} + U_l x^l \\
V^{(l)} &= V^{(l-1)} + V_l x^l \\
W^{(l)} &= W^{(l-1)} + W_l x^l
\end{align*}
$$

into (12), we obtain

$$
\begin{align*}
M - U^{(l-1)} V^{(l-1)} (W^{(l-1)})^T &\equiv U_l V_0 W_0^T x^l + U_0 V_0 W_0^T x^l + U_0^T W_l x^l \pmod{x^{l+1}}, \\
E - (U^{(l-1)})^T U^{(l-1)} &\equiv U_0^T U_l x^l + U_l^T U_0 x^l \pmod{x^{l+1}}, \\
E - (W^{(l-1)})^T W^{(l-1)} &\equiv W_0^T W_l x^l + W_l^T W_0 x^l \pmod{x^{l+1}}.
\end{align*}
$$
Looking at coefficients of $x^t$ in the above equations, we obtain

\[ \begin{align*}
\tilde{M}_t &= U_t V_0 W_0^T + U_0 V_t W_0^T + U_0 V_t W_0^T, \\
\tilde{P}_t &= U_0^T U_t + U_t^T U_0, \\
\tilde{Q}_t &= W_0^T W_t + W_t^T W_0,
\end{align*} \]

where $\tilde{M}_t, \tilde{P}_t, \tilde{Q}_t$ are numerical matrices defined as

\[ \begin{align*}
\tilde{M}_t &= [M - U^{(t-1)} V^{(t-1)} (W^{(t-1)})^T]_t, \\
\tilde{P}_t &= [E - (U^{(t-1)})^T U^{(t-1)}]_t, \\
\tilde{Q}_t &= [E - (W^{(t-1)})^T W^{(t-1)}]_t.
\end{align*} \]

Multiplying $U_0^T$ and $W_0$ on both sides of (13) from left and right-hand side, we obtain

\[ U_0^T \tilde{M}_t W_0 = U_0^T U_t V_0 + V_t + V_0 W_t^T W_0. \]  \hfill (19)

We define $\tilde{U}_t, \tilde{W}_t$ as

\[ \tilde{U}_t \triangleq U_0^T U_t, \quad \tilde{W}_t \triangleq W_t^T W_0, \]

and (19),(14),(15) give

\[ \begin{align*}
U_0^T \tilde{M}_t W_0^T &= \tilde{U}_t V_0 + V_t + V_0 \tilde{W}_t, \\
\tilde{P}_t &= \tilde{U}_t + \tilde{U}_t^T, \\
\tilde{Q}_t &= \tilde{W}_t + \tilde{W}_t^T.
\end{align*} \]

Note that (21) and (22) give

\[ \left\{ \tilde{U}_t \right\}_{ii} = \frac{1}{2} \left\{ \tilde{P}_t \right\}_{ii}, \quad \left\{ \tilde{W}_t \right\}_{ii} = \frac{1}{2} \left\{ \tilde{Q}_t \right\}_{ii} \quad (1 \leq i \leq n). \]  \hfill (23)

Since $V_0$ is a diagonal matrix, diagonal entries of $U_t V_0$ and $V_0 \tilde{W}_t$ are given by

\[ \left\{ U_t V_0 \right\}_{ii} = \left\{ V_0 \right\}_{ii} \left\{ U_t \right\}_{ii}, \quad \left\{ V_0 \tilde{W}_t \right\}_{ii} = \left\{ V_0 \right\}_{ii} \left\{ \tilde{W}_t \right\}_{ii} \quad (1 \leq i \leq n). \]  \hfill (24)

Therefore, (23) and (24) give

\[ \left\{ U_t V_0 \right\}_{ii} = \frac{1}{2} \left\{ V_0 \right\}_{ii} \left\{ \tilde{P}_t \right\}_{ii}, \quad \left\{ V_0 \tilde{W}_t \right\}_{ii} = \frac{1}{2} \left\{ V_0 \right\}_{ii} \left\{ \tilde{Q}_t \right\}_{ii} \quad (1 \leq i \leq n). \]  \hfill (25)

The above equalities and (20) imply that

\[ \left\{ V_t \right\}_{ii} = \left\{ U_0^T \tilde{M}_t W_0 \right\}_{ii} - \left\{ U_t V_0 \right\}_{ii} - \left\{ V_0 \tilde{W}_t \right\}_{ii} = \left\{ U_0^T \tilde{M}_t W_0 \right\}_{ii} - \frac{1}{2} \left\{ V_0 \right\}_{ii} \left( \left\{ \tilde{P}_t \right\}_{ii} + \left\{ \tilde{Q}_t \right\}_{ii} \right) \quad (1 \leq i \leq n). \]  \hfill (26)
Since right-hand side of the above equation is already known, it is possible to compute \( \{ V_i \}_{ii} \). Since \( V_i \) is a diagonal matrix, this determines whole matrix \( V_i \).

Next, we will determine \( \tilde{U}_i \) and \( \tilde{W}_i \). Diagonal entries \( \{ \tilde{U}_i \}_{ii}, \{ \tilde{W}_i \}_{ii} \ (1 \leq i \leq n) \) can be computed with (23) and non-diagonal entries \( \{ \tilde{U}_i \}_{ij}, \{ \tilde{W}_i \}_{ij} \ (1 \leq i \leq n, \ 1 \leq j \leq n, \ i \neq j) \) can be computed as follows: Since \( V_0 \) is a diagonal matrix, we have

\[
\{ \tilde{U}_i V_0 \}_{ij} = \{ V_0 \}_{jj} \{ \tilde{U}_i \}_{ij}, \quad \{ V_0 \tilde{U}_i \}_{ij} = \{ V_0 \}_{ii} \{ \tilde{W}_i \}_{ij} \ (1 \leq i \leq m, \ 1 \leq j \leq n).
\]

(27)

Notice that (21) and (22) give

\[
\{ \tilde{U}_i \}_{ij} + \{ \tilde{U}_i \}_{ji} = \{ \tilde{P}_i \}_{ij} \ (1 \leq i \leq m, \ 1 \leq j \leq m), \quad (28)
\]

\[
\{ \tilde{W}_i \}_{ij} + \{ \tilde{W}_i \}_{ji} = \{ \tilde{Q}_i \}_{ij} \ (1 \leq i \leq n, \ 1 \leq j \leq n). \quad (29)
\]

Therefore,

\[
\{ \tilde{U}_i \}_{ji} = \{ \tilde{P}_i \}_{ij} - \{ \tilde{U}_i \}_{ij} \ (1 \leq i \leq m, \ 1 \leq j \leq m), \quad (30)
\]

\[
\{ \tilde{W}_i \}_{ji} = \{ \tilde{Q}_i \}_{ij} - \{ \tilde{W}_i \}_{ij} \ (1 \leq i \leq n, \ 1 \leq j \leq n). \quad (31)
\]

Note also that (20) and (27) give, for \( (1 \leq i \leq m, \ 1 \leq j \leq n) \),

\[
\{ U_0^T M_i W_0 \}_{ij} - \{ V_i \}_{ij} = \{ \tilde{U}_i V_0 \}_{ij} + \{ V_0 \tilde{W}_i \}_{ij} = \{ V_0 \}_{jj} \{ \tilde{U}_i \}_{ij} + \{ V_0 \}_{ii} \{ \tilde{W}_i \}_{ij}. \quad (32)
\]

Exchanging index \( i \) and \( j \) in the above equation, we obtain

\[
\{ U_0^T M_i W_0 \}_{ji} - \{ V_i \}_{ji} = \{ V_0 \}_{ii} \{ \tilde{U}_i \}_{ji} + \{ V_0 \}_{jj} \{ \tilde{W}_i \}_{ji} \ (1 \leq i \leq n, \ 1 \leq j \leq m). \quad (33)
\]

Substituting (30) and (31) into the above equation, we obtain, for \( (1 \leq i \leq n, \ 1 \leq j \leq m) \),

\[
\{ V_0 \}_{ii} \{ \tilde{U}_i \}_{ij} + \{ V_0 \}_{jj} \{ \tilde{W}_i \}_{ij} = -\{ V_0^T M_i W_0 \}_{ij} + \{ V_i \}_{ji} + \{ V_0 \}_{ii} \{ \tilde{P}_i \}_{ij} + \{ V_0 \}_{jj} \{ \tilde{Q}_i \}_{ij}. \quad (34)
\]

This and (32) give, for \( (1 \leq i \leq n, \ 1 \leq j \leq n) \),

\[
\begin{bmatrix}
\{ V_0 \}_{jj} & \{ V_0 \}_{ii} \\
\{ V_0 \}_{ii} & \{ V_0 \}_{jj}
\end{bmatrix}
\begin{bmatrix}
\{ \tilde{U}_i \}_{ij} \\
\{ \tilde{W}_i \}_{ij}
\end{bmatrix}
= 
\begin{bmatrix}
\{ U_0^T M_i W_0 \}_{ij} - \{ V_i \}_{ij} \\
-\{ V_0^T M_i W_0 \}_{ji} + \{ V_j \}_{ji} + \{ V_0 \}_{ii} \{ \tilde{P}_i \}_{ij} + \{ V_0 \}_{jj} \{ \tilde{Q}_i \}_{ij}
\end{bmatrix}. \quad (35)
\]
Since we supposed condition (10), for any \( i, j \) satisfying \( i \neq j \), matrix
\[
\begin{bmatrix}
\{V_0\}_{ij} & \{V_0\}_{ii} \\
\{V_0\}_{ii} & \{V_0\}_{jj}
\end{bmatrix}
\]
has its inverse and \( \{\bar{U}_t\}_{ij} \) and \( \{\bar{W}_t\}_{ij} \) (\( 1 \leq i \leq n, 1 \leq j \leq n, i \neq j \)) are given by
\[
\begin{bmatrix}
\{\bar{U}_t\}_{ij} \\
\{\bar{W}_t\}_{ij}
\end{bmatrix} = \frac{1}{(\{V_0\}_{ij} - \{V_0\}_{ii}) (\{V_0\}_{jj} + \{V_0\}_{ii}) - \{V_0\}_{ii}} \begin{bmatrix}
\{V_0\}_{ij} & -\{V_0\}_{ii} \\
-\{V_0\}_{ii} & \{V_0\}_{jj}
\end{bmatrix} 	imes
\begin{bmatrix}
\{M_t M_t W_0\}_{ij} \\
-\{V_0^T M_t W_0\}_{ji} + \{V_0\}_{ii} \{\bar{P}_t\}_{ij} + \{V_0\}_{jj} \{\bar{Q}_t\}_{ij}
\end{bmatrix}
\]
(note that \( \{V_0\}_{ij} = 0, i \neq j \)). Now, we have determined \( \{\bar{U}_t\}_{ij}, \{\bar{W}_t\}_{ij} \) (\( 1 \leq i \leq n, 1 \leq j \leq n \)). Since \( \bar{W}_t \) is an \( n \times n \) matrix, this determines whole \( \bar{W}_t \) and a part of \( \bar{U}_t \). Hence, we only need to compute the rest of \( \bar{U}_t \), i.e., \( \{\bar{U}_t\}_{ij} \) (\( n + 1 \leq i \leq m, n + 1 \leq j \leq m \)). Since we have
\[
\{V_0\}_{ii} = \{V_t\}_{ij} = 0 \quad (n + 1 \leq i \leq m, 1 \leq j \leq n),
\]
(32) gives
\[
\{V_0\}_{jj} \{\bar{U}_t\}_{ij} = \{U_0^T M_t W_0\}_{ij}.
\]
Since we supposed condition (11) (i.e., \( \{V_0\}_{jj} \neq 0 \ (j = 1, \ldots, n) \)), we obtain
\[
\{\bar{U}_t\}_{ij} = \frac{\{U_0^T M_t W_0\}_{ij}}{\{V_0\}_{jj}} \quad (n + 1 \leq i \leq m, 1 \leq j \leq n).
\]
This and (30) determine \( \{\bar{U}_t\}_{ij} \) (\( 1 \leq i \leq n, n + 1 \leq j \leq m \)).

Now only undecided part of \( \bar{U}_t \) are \( \{\bar{U}_t\}_{ij} \) (\( n + 1 \leq i \leq m, n + 1 \leq j \leq m \)). This part of \( \bar{U}_t \) can not be uniquely determined. In fact, every \( \{\bar{U}_t\}_{ij} \) (\( n + 1 \leq i \leq m, n + 1 \leq j \leq m \)) satisfying (28) is enough. Hence, one possible choice of \( \{\bar{U}_t\}_{ij} \) is
\[
\{\bar{U}_t\}_{ij} = \{\bar{U}_t\}_{ji} = \frac{1}{2} \{\bar{P}_t\}_{ij} \quad (n + 1 \leq i \leq m, n + 1 \leq j \leq m).
\]

Now, all entries of matrices \( \bar{U}_t \) and \( \bar{W}_t \) are determined. Then, matrices \( U_t \) and \( W_t \) are computed with
\[
U_t = U_0 \bar{U}_t, \quad W_t = W_0 \bar{W}_t^T.
\]

We describe the above algorithm formally as follows;
Algorithm 1 Computation of the approximate singular-value decomposition

Input: Polynomial matrix $M = M_0 + \cdots + M_r x^r$;
Output: The $k$th approximate singular-value decomposition;
Condition: The singular-value of $M_0$ must be non-zero and distinct;

Step 1 If $m < n$, then put $M \leftarrow M^T$, $t \leftarrow 1$ and exchange $m$ and $n$. Otherwise put $t \leftarrow 0$.

Step 2 Compute singular-value decomposition $M_0 = U_0 V_0 W_0^T$ of $M_0$ with a numerical algorithm. Let $l \leftarrow 1$ and $U^{(0)} \leftarrow U_0, \; V^{(0)} \leftarrow V_0, \; W^{(0)} \leftarrow W_0$.

Step 3 Compute $\tilde{M}_l, \tilde{P}_l$ and $\tilde{Q}_l$ with formula (16), (17) and (18).

Step 4 Compute $\{V_l\}_{ii}$ $(1 \leq i \leq n)$ with formula (26) and determine matrix $V_l$.

Step 5 Compute $\{U_l\}_{ii}$ and $\{W_l\}_{ii}$ $(1 \leq i \leq n)$ with formula (23).

Step 6 Compute $\{U_l\}_{ij}$ and $\{W_l\}_{ij}$ $(1 \leq i \leq n, 1 \leq j \leq n, i \neq j)$ with formula (36).

Step 7 Compute $\{U_l\}_{ij}$ $(n + 1 \leq i \leq m, 1 \leq j \leq n)$ with formula (37).

Step 8 Compute $\{U_l\}_{ij}$ $(1 \leq i \leq n, n + 1 \leq j \leq m)$ with formula (30).

Step 9 Compute $\{U_l\}_{ij}$ $(n + 1 \leq i \leq m, n + 1 \leq j \leq m)$ with formula (38).

Step 10 Put $U_l, W_l$ as

$$U_l = U_0 \tilde{U}_l, \; W_l = W_0 \tilde{W}_l^T$$

and let $U^{(l)}, V^{(l)}, W^{(l)}$ be

$$U^{(l)} \leftarrow U^{(l-1)} + U_l x^l, \quad V^{(l)} \leftarrow V^{(l-1)} + V_l x^l, \quad W^{(l)} \leftarrow W^{(l-1)} + W_l x^l.$$ 

Step 11 If $k < l$, then let $l \leftarrow l + 1$ and go to Step 3.

Step 12 If $t = 1$, then put

$$\text{tmp} \leftarrow U^{(k)}, \; \text{tmp} \leftarrow (W^{(k)})^T, \; V^{(k)} \leftarrow (V^{(k)})^T, \; W^{(k)} \leftarrow \text{tmp}^T.$$ 

Step 13 Return $U^{(k)}, V^{(k)}, W^{(k)}$.

4 Conclusion

The approximate singular-value decomposition of a polynomial matrix was defined and an algorithm to compute the decomposition is given. The algorithm uses only $2 \times 2$ matrix inversions and basic matrix manipulations, hence is quite efficient.
References
