Calculation system for the dual graph of resolution of the algebraic curve singularities using Risa/Asir

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Abstract

In this paper, we introduce a method of generating the dual graph of the minimal normal resolution of the algebraic curve singularities. Using factorization of polynomials over algebraic extension field, we can calculate the dual graph from the coefficients of the given polynomial exactly. The computing process includes no approximation. We constructed the system to generate the dual graph on Risa/Asir. Users can get the resulting dual graph by the list form and also by the graphical form.

1 Introduction

A resolution of an algebraic curve C at a singular point p of C is said *minimal normal*, if the union of the proper transform of C and the exceptional curve E of the resolution is of normal crossing type in a neighborhood of E, and it is "minimal" among the resolutions having this property. Usually, the geometric configuration of the resolution is represented by a weighted graph, called the "dual graph".

The dual graph of the minimal normal resolution of the algebraic curve singularities is one of the most fundamental invariants in algebraic geometry. At present, there is no computing system for the dual graphs.

Our method computes the dual graphs of the minimal normal resolutions of the algebraic curve singularities at origin defined by polynomials in two variables with coefficients in the algebraic extension field over the rational number field. We can calculate the dual graph from the coefficients of the given polynomial exactly. The computing process includes no approximation.

In our algorithm, Newton Polygon Algorithm is used to obtain Puiseux pairs. Newton Polygon Algorithm needs finding roots of polynomials with coefficients in the algebraic extension field over the rational number field. We implemented this algorithm on Risa/Asir, a computer algebra system developed at FUJITSU LABORATORIES LIMITED ([4]). Risa/Asir is good at computing Gröbner basis and factoring polynomials over algebraic extension fields, which was very suited to implement our algorithm.

In this paper, we show the algorithm generating the dual graph, and introduce our system on Risa/Asir.

2 Blowing Ups and Dual Graphs

We shall assume from now on that M is a non-singular projective algebraic surface over the complex number field and E an algebraic curve on M.

Definition 1 (Normal crossing type) *If each irreducible compornent of E is non-singular and intersect each other at only one point at most, then E is called* of normal crossing type.

Definition 2 (Dual graph) *For a curve E of normal crossing type, we represent each irreducible component of E by a vertex and join the vertices if and only if the corresponding irreducible components intersect each other. We associate to each vertices an integer, called* weight, *equal to the self-intersection number of the corresponding irreducible component on M. The weighted graph thus obtained will be called the* dual graph *of E.*

Let C be an algebraic curve on M passing through a point P. By replacing the point P with $\mathbb{P}^1$ we can construct new algebraic surface M'. This operation is called *blowing up at a point P*. And the curve on M' corresponding to $C \setminus P$ (resp. $\mathbb{P}^1$) is called *proper transform of C* (resp. *exceptional curve*). The resolution of an algebraic curve at a singular point is obtained by finitely many blowing up operations ([2]).

Following lemmas related to blowing up are well-known.

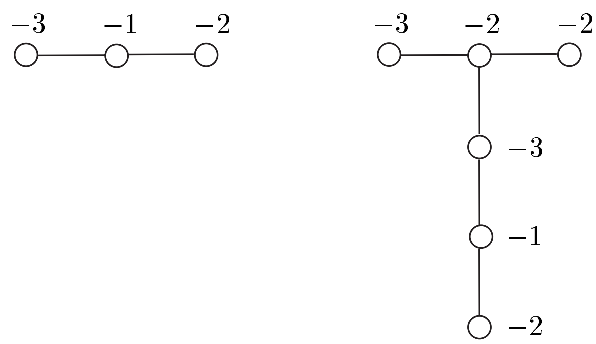
Lemma 1 *The self-intersection number of the exceptional curve obtained by a blowing up is -1.*

Lemma 2 *The self-intersection number N of the exceptional curve E decrease to $N-1$ by a blowing up at a point of E.*

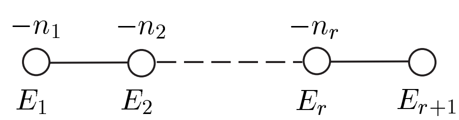
From these lemmas, the dual graph relating a resolution of an algebraic curve at a singular point is obtained.

For example, (See Definition 2)

$f = y^2 - x^3$ $\qquad\qquad$ $f = (y^2 - x^3)^2 + y^5$



Lemma 3 *Let $E_1, E_2, \dots, E_r, E_{r+1}$ be the irreducible components of E and assume that the dual graph is of the following linear type :*



Furthermore assume that there exists a holomorphic function f on a neighborhood U in the following form.

$$(f) = \sum_{i=0}^{r} m_i E_i + m_{r+1} E_{r+1} \cap U.$$

Then,

(1) $m_2, \dots, m_{r+1}$ are all multiple of m_1.

(2) Set $p_i = m_i/m_1$ $(1 \le i \le r+1)$, then (p_{r+1}, p_r) are coprime each other and the following continuous fraction expansion holds.

$$\frac{p_{r+1}}{p_r} = n_r - \cfrac{1}{n_{r-1} - \cfrac{1}{n_{r-2} - \ddots - \cfrac{1}{n_1}}}$$

Proof. Since $(f) \cdot E_i = 0$, we have $m_{i+1} = n_i m_i - m_{i-1}$ for $i = 1, 2, \dots, r$, where $m_0 = 0$. The two assertions of the lemma are immediate consequences of these equations. $\square$

Definition 3 (Intersection matrix) *Let $E_1, E_2, \ldots, E_r$ be the irreducible components of E. We call*

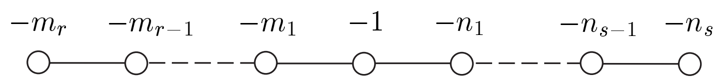
$$I_E = ((E_i \cdot E_j))_{i,j=1,\ldots,r}$$

the intersection matrix *of E.*

The following two lemmas can also be proved by a direct computation.

Lemma 4 *The determinant* $\det(-I_E)$ *is invariant under the blowing ups of the points on E. Namely, if $\tau : M' \to M$ is a blowing up of a point P on E, we have then* $\det(-I_{\tau^{-1}(E)}) = \det(-I_E)$.

Lemma 5 *Assume that the dual graph is of the following type :*



We have then

$$\det(-I_E) = pq - aq - bp,$$

where p, a, q, b are the natural numbers defined by the continuous fractions

$$\frac{p}{a} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \ddots - \cfrac{1}{m_r}}}$$

$$\frac{q}{b} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \ddots - \cfrac{1}{n_s}}}$$

satisfying

$$(p, a) = 1,\ (q, b) = 1,\ 0 < a < p,\ 0 < b < q.$$

3 Newton Polygon Algorithm

Let $f(x, y)$ be a polynomial in two variables with complex coefficients. We can write $f(x, y) = \sum c_{\alpha\beta} x^\alpha y^\beta$.

Definition 4 (Newton polygon) *The* Newton Polygon *of f is the convex closure of set*

$$\bigcup_{c_{\alpha\beta}\neq 0} \{(\alpha,\beta)+\mathbb{R}_+^2\}$$

The Newton boundary *of f is the union of compact faces of the boundary of the Newton polygon of f.*

Notice that the boundary of the Newton polygon differs from the Newton boundary by two non-compact faces parallel to the coordinates axes.

We assume $f(0,0)=0$ and $f(x,y)$ is not divisible by x. We choose one among segments constructing Newton boundary, let Δ_0 be the segment. And let $-\frac{1}{\mu_0}$ be its slope. Then μ_0 is a positive rational, say

$$\mu_0 = \frac{p_0}{q_o}$$

where p_0, q_0 are coprime positive integers. Let $f_{\Delta_0}(x,y)=\sum_{(\alpha,\beta)\in\Delta_0} c_{\alpha\beta}x^{\alpha}y^{\beta}$ and t_0 a nonzero root of $f_{\Delta_0}(1,t)=0$.

Next we shall substitute $x=x_1^{q_0}$ and $y=x_1^{p_0}(t_0+y_1)$ for $f(x,y)$. And we get

$$\begin{aligned} f(x_1^{q_0}, x_1^{p_0}(t_0+y_1)) &= \sum c_{\alpha\beta} x_1^{q_0\alpha+p_0\beta}(t_0+y_1)^{\beta} \\ &= x_1^{q_0\alpha_0+p_0\beta_0} f_1(x_1,y_1) \end{aligned}$$

where $(\alpha_0,\beta_0)\in\Delta_0$ and $f_1(x_1,y_1)$ is a polynomial in x_1 and y_1, not divisible by x_1.

We now repeat the whole process, replacing $f(x,y)$ by $f_1(x_1,y_1)$, and continue indefinitely. We obtain a sequence of positive rationals

$$\mu_0=\frac{p_0}{q_0},\ \mu_1=\frac{p_1}{q_1},\ \mu_2=\frac{p_2}{q_2},\ldots$$

and complex numbers

$$t_0,\ t_1,\ t_2,\ldots$$

(p_i,q_i) is called *Puiseux pair.* Using $\{(p_i,q_i)\}$ and $\{t_i\}$ we can get the *Puiseux expansion* of $f(x,y)$ ([3]). The whole process is called *Newton Polygon Algorithm.*

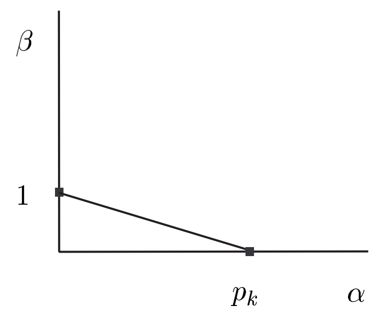
4 Computation of Dual Graph

Let $f(x,y)$ be a polynomial in two variables with coefficients in the algebraic extension field over the rational number field. Moreover we assume that the algebraic curve C defined by $f(x,y)=0$ has an isolated singularity at origin.

4.1 The method by Puiseux Pair

Using Squarefree Decomposition Algorithm we can find multiple factors of f ([1]). Here, we assume that f has not multiple factors.

Using Newton Polygon Algorithm we get Puiseux pairs (p_i, q_i) of f until (i) Newton boundary is parallel to a coordinate axis or (ii) Newton polygon is of following type :



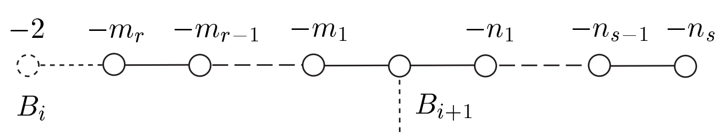
Notice that in case of (ii) the proper transform of C and y_k-axis intersect transversely.

Let (p_0, q_0), (p_1, q_1), ... , (p_{k-1}, q_{k-1}), $(p_k, 1)$ be obtained Puiseux pairs. As shown below, using k Puisuex pairs (p_0, q_0), (p_1, q_1), ... , (p_{k-1}, q_{k-1}) we can get the dual graph for f.

For each Puiseux pair (p_i, q_i) of f if $p_i > 1$ and $q_i > 1$, then (p_i, q_i) coinsides with (p, q) in Lemma 5. From the fact that the determinant $\det(-I_E)$ is equal to 1 when the algebraic curve E is locally irreducible, we get a pair of natural numbers (a, b) such that

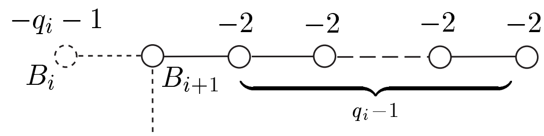
$$p_i q_i - a q_i - b p_i = 1,\ (p_i, a) = 1,\ (q_i, b) = 1,\ 0 < a < p_i,\ 0 < b < q_i\ ,$$

and we can determine $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s$ in Lemma 5 by continuous fraction expansions of p_i/a, q_i/b. Namely, the dual graph corresponding to this Puiseux pair is following type :

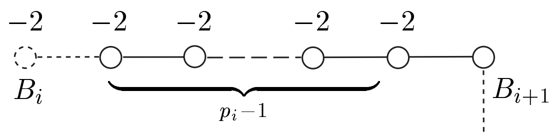


where B_i is a branch vertex such that B_0 is omitted and the weight of B_k is -1.

If $p_i = 1$, then the dual graph is of following type :



If $q_i = 1$, then the dual graph is of following type :



4.2 Another method for locally irreducible case

The algorithm described in 4.1 is not only used in case of locally irreducible but also locally reducible. However, the algorithm needs factoring polynomials over algebraic extension field. In general, this computation is very heavy.

If f is locally irreducible, there exists another algorithm for the computation of dual graph does not need factoring polynomials over algebraic extension field. Here we introduce the another method.

Let $f(x, y)$ be a polynomial in two variables with coefficients in the algebraic extension field L over the rational number field $\mathbb{Q}$. Moreover assume that the algebraic curve C defined by $f(x, y) = 0$ has an isolated singularity at origin and is locally irreducible. Let C' be the proper transform of C obtained by a blowing up, $h(x, s)$ the defining polynomial of C'. Notice that if f is locally irreducible, the exceptional curve obtained by the blowing up intersects C' at only one point. Using this fact, it follows that the root of $h(0, s)$ is in L. Thus we can compute the intersection point without factoring polynomials over algebraic extension field.

Step 1. For the defining polynomial f of C, compute the defining polynomial h of the proper transform C' of C by a blowing up.

Step 2. Compute the intersection point P of C' and the exceptional curve obtained by *Step 1.*

Step 3. If P is in origin, then let $f = h$ and go to *Step 1.* Otherwise, for h do the coordinate transformation such that C' passes through origin, say h'.

Step 4. If the multiplicity of h' is 1, then terminate. Otherwise, let $f = h'$ and go to *Step 1.*

4.3 Implementation and Examples

In this section, we introduce some of functions on Risa/Asir, which were implemented by us in order to compute dual graphs.

4.3.1 `roots`

Let *poly* be a polynomial in one variable with coefficients in the algebraic extension field over the rational number field.

`roots(`*poly*`)` returns the list of all roots of the polynomial *poly*. In this function, the function `af` in the package `'sp'` is used.

Example 1

```
[0] A0=newalg(x^2-x+1);
(#0)
[1] roots(81*x^4+(-18*A0)*x^2+(A0-1));
[(#1),(-#1)]
[2] roots(x^5+A0*x^2-1);
[(#4),(-#4-#3-#2+#0-1),(#3),(#2),(-#0+1)]
```

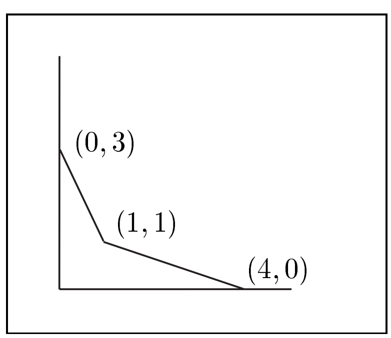
4.3.2 `n_polygon`

Let *poly* be a polynomial in two variables with coefficients in the algebraic extension field over the rational number field.

`n_polygon(`*poly*,*flag*`)` returns the Newton polygon of *poly*. If *flag* is `0`, then the result is outputted by the list form. If *flag* is `1`, by the list form and the graphical form.

Example 2

```
[0] F=3*y^3+x*y^5-2*x*y+x^3*y^2+x^4+x^5+x^6*y^2$
[1] n_polygon(F,0);
[[[3,[0,3]],[-2,[1,1]]],[[-2,[1,1]],[1,[4,0]]]]
[2] n_polygon(F,1);
```



4.3.3 p_pair

Let *poly* be a polynomial in two variables with coefficients in the algebraic extension field over the rational number field.

`p_pair`(*poly*, n) returns n Puisuex pairs of *poly*.

Example 3

```
[0] F=((y^3+x)^2+x^5)^2+x^9*y^3+x^10$
[1] p_pair(F,4);
[[1,3],[9,2],[9,2],[9,1]]
```

4.3.4 dualgraph

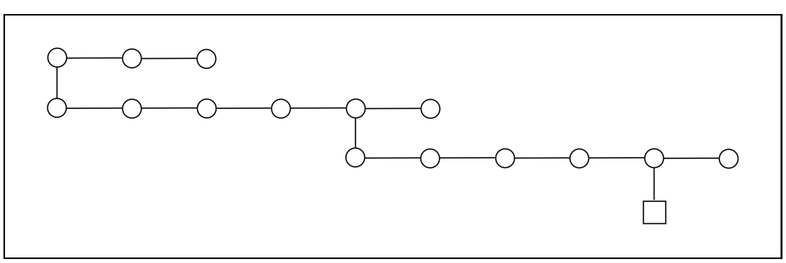
Let *poly* be a polynomial in two variables with coefficients in the algebraic extension field over the rational number field, C the algebraic curve defined by *poly*.

`dualgraph`(*poly*,*flag*) returns the dual graph of the minimal normal resolution of the algebraic curve defined by *poly*. If *flag* is `0`, then the result is outputted by the list form. If *flag* is `1`, by the list form and the graphical form.

In case of locally irreducible, this function use the method of 4.2. Otherwise, this function use the method of 4.1.

Example 4

```
[0] F=((y^3+x)^2+x^5)^2+x^9*y^3+x^10$
[1] dualgraph(F,0);
[[],-2,[-2,-2]]
[[-2,-2,-2,-3],-2,[-2]]
[[-2,-2,-2,-3],-1,[-2]]
[2] dualgraph(F,1);
```



Where the square in the figure is associated with the proper transform of C.

5 Conclusion

We have presented the method for the computation of the dual graph. Furthermore we implemented our algorithm on Risa/Asir. This system can compute the dual graph in both of locally irreducible case and reducible case. In reducible case, factoring polynomials over algebraic extension field is needed. This computation is very heavy. However, since this factoring function is planning to be improved on next version of Risa/Asir, the performance of our system will be also improved.

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