# On The Leverrier-Faddeev Algorithm

SHUI-HUNG HOU
Department of Applied Mathematics
**The Hong Kong Polytechnic University**
mahoush@polyu.edu.hk

**Abstract**

A novel proof of the recursive formulas that compute the coefficients of the characteristic polynomial of a matrix, as used in the Leverrier-Faddeev algorithm, is given by means of Laplace transform.

## 1 Introduction

Let

$$\begin{aligned} p(s) &= a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \qquad (a_0 = 1) \\ &= (s-\lambda_1)\cdots(s-\lambda_n) \end{aligned} \tag{1}$$

be the characteristic polynomial of an $n \times n$ matrix $A$, and define matrices $B_1, \cdots, B_n$ by

$$\operatorname{adj}(sI - A) = B_1 s^{n-1} + \cdots + B_{n-1}s + B_n,$$

so that

$$(sI - A)^{-1} = \frac{B_1 s^{n-1} + \cdots + B_{n-1}s + B_n}{p(s)}. \tag{2}$$

A well-known algorithm, often attributed to Leverrier, Faddeev and others [1], indicates that the matrices $B_k$ and the coefficients $a_k$ can be obtained in a successive manner by means of the formulas

$$\begin{aligned} B_1 &= I, & a_1 &= -\tfrac{1}{1}\operatorname{tr}(AB_1), \\ B_2 &= AB_1 + a_1 I, & a_2 &= -\tfrac{1}{2}\operatorname{tr}(AB_2), \\ &\vdots & &\vdots \\ B_n &= AB_{n-1} + a_{n-1}I, & a_n &= -\tfrac{1}{n}\operatorname{tr}(AB_n), \end{aligned} \tag{3}$$

with $0 = AB_n + a_n I$ terminating as a check of computation. Here tr stands for the trace of a matrix.

The above algorithm has been covered in most of the books on linear system theory [2]–[5]. However, the majority of them do not give a complete proof with the exception of a few, and the proofs therein are usually rather involved and sometimes incomplete [6, 7]. It is the purpose of this paper to give an alternative derivation using Laplace transform. The approach is more readily accessible to students of engineering and applied sciences.

## 2 Preliminaries

The derivation of the Leverrier-Faddeev algorithm (3) consists of establishing

$$B_1 = I, \quad B_k = AB_{k-1} + a_{k-1}I, \qquad k = 2, \cdots, n, \tag{4}$$

$$0 = AB_n + a_n I, \tag{5}$$

and the coefficient formula

$$a_k = -\frac{1}{k}\,\mathrm{tr}(AB_k), \qquad k = 1, \cdots, n. \tag{6}$$

It is easy to see that (4) and (5) follow immediately from comparing coefficients of like powers of $s$ in the identity

$$(sI - A)(B_1 s^{n-1} + \cdots + B_{n-1}s + B_n) = p(s)I.$$

In order to derive the coefficient formula (6) we observe that (4) implies

$$B_1 = I, \quad B_k = A^{k-1} + a_1 A^{k-2} + \cdots + a_{k-1}I, \qquad k = 2, \cdots, n. \tag{7}$$

Substituting (7) into (6) gives

$$a_k = -\frac{1}{k}\,\mathrm{tr}(A^k + a_1 A^{k-1} + \cdots + a_{k-1}A), \qquad k = 1, \cdots, n. \tag{8}$$

Hence it follows that the desired formula (6) will be established if we can prove (8).

## 3 The Derivation

In order to derive (8) we consider the following rational function of $s$ :

$$G(s) = \frac{\lambda_1}{s-\lambda_1} + \cdots + \frac{\lambda_n}{s-\lambda_n}. \tag{9}$$

Its inverse Laplace transform is given explicitly by

$$\begin{aligned} g(t) &= \mathrm{L}^{-1}[G(s)] \\ &= \lambda_1 e^{\lambda_1 t} + \cdots + \lambda_n e^{\lambda_n t}. \end{aligned}$$

Clearly the derivatives $D^{r-1}g(t)$ have the initial values

$$D^{r-1}g(0) = \lambda_1^r + \cdots + \lambda_n^r = \mathrm{tr}(A^r) = \sigma_r. \tag{10}$$

In addition, we have the following two expressions for $p(s)G(s)$, which are useful in our derivation of (8).

**Lemma 1**

$$p(s)G(s) = sp'(s) - np(s) \tag{11}$$

*with $p'(s)$ denoting the derivative of $p(s)$.*

**Proof** This follows immediately from (1) and the definition of $G(s)$ in (9) by observing that

$$p'(s) = \frac{p(s)}{s-\lambda_1} + \cdots + \frac{p(s)}{s-\lambda_n}.$$

The proof of Lemma 1 is completed.

Next, the Laplace transform of the $k$th derivative $D^k g$ is given by

$$\mathrm{L}[D^k g] = s^k G(s) - s^{k-1}g(0) - s^{k-2}g'(0) - \cdots - sD^{k-2}g(0) - D^{k-1}g(0),$$

which together with $\sigma_r = D^{r-1}g(0)$ from (10) imply

$$s^k G(s) = s^{k-1}\sigma_1 + s^{k-2}\sigma_2 + \cdots + s\sigma_{k-1} + \sigma_k + \mathrm{L}[D^k g]. \tag{12}$$

This enables us to express $p(s)G(s)$ in another way.

**Lemma 2**

$$\begin{aligned} p(s)G(s) = (a_0\sigma_1)s^{n-1} \quad &+ \quad (a_0\sigma_2 + a_1\sigma_1)s^{n-2} \\ &+ \quad \cdots \ + \ (a_0\sigma_n + \cdots + a_{n-1}\sigma_1). \end{aligned} \tag{13}$$

**Proof** From (12) we have

$$\begin{aligned}
G(s) &= \mathrm{L}[g], \\
sG(s) &= \sigma_1 + \mathrm{L}[Dg], \\
s^2G(s) &= s\sigma_1 + \sigma_2 + \mathrm{L}[D^2g], \\
&\cdots \\
s^{n-1}G(s) &= s^{n-2}\sigma_1 + \cdots + s\sigma_{n-2} + \sigma_{n-1} + \mathrm{L}[D^{n-1}g], \\
s^nG(s) &= s^{n-1}\sigma_1 + s^{n-2}\sigma_2 + \cdots + s\sigma_{n-1} + \sigma_n + \mathrm{L}[D^ng].
\end{aligned}$$

Thus,

$$\begin{aligned}
p(s)G(s) &= (a_0s^n + a_1s^{n-1} + \cdots + a_n)G(s) \\
&= a_0s^nG(s) + a_1s^{n-1}G(s) + \cdots + a_nG(s) \\
&= (a_0\sigma_1)s^{n-1} + (a_0\sigma_2 + a_1\sigma_1)s^{n-2} + \cdots + (a_0\sigma_n + \cdots + a_{n-1}\sigma_1) \\
&\quad + \mathrm{L}[p(D)g].
\end{aligned}$$

This produces the desired result (13) by taking into account the fact that $\mathrm{L}[p(D)g] = 0$, because

$$\begin{aligned}
p(D)g &= (D-\lambda_1)\cdots(D-\lambda_n)\left[\lambda_1e^{\lambda_1t} + \cdots + \lambda_ne^{\lambda_nt}\right] \\
&= 0.
\end{aligned}$$

The proof of Lemma 2 is completed.

We now can conclude the derivation of (8). Substituting (11) into (13) we obtain

$$\begin{aligned}
(a_0\sigma_1)s^{n-1} &+ (a_0\sigma_2 + a_1\sigma_1)s^{n-2} + \cdots + (a_0\sigma_n + \cdots + a_{n-1}\sigma_1) \\
&= sp'(s) - np(s) \\
&= -1\cdot a_1s^{n-1} - 2\cdot a_2s^{n-2} - \cdots - n\cdot a_n.
\end{aligned} \tag{14}$$

Equating coefficients of like powers of $s$ in (14) immediately produces

$$-k\cdot a_k = a_0\sigma_k + \cdots + a_{k-1}\sigma_1, \qquad k = 1, \cdots, n,$$

as desired.

## References


[1] D. K. Faddeev and V. N. Faddeeva, *Computational Methods of Linear Algebra*, Freeman, San Francisco, 1963.

[2] T. Kailath, *Linear Systems*, Prentice Hall, Inc., Englewood Cliffs, NJ, 1980.

[3] C. T. Chen, *Linear System Theory and Design*, Holt Reinhart and Winston, New York, 1984.

[4] B. Friedland, *Control System Design*, McGraw Hill, New York, 1986.

[5] M. Jamshid, M. Tarokh, and B. Shafai, *Computer-Aided Analysis and Design of Linear Control Systems*, Prentice Hall, Inc., Englewood Cliffs, NJ, 1992.

[6] L. A. Zadeh and C. A. Desoer, *Linear System Theory, The State Space Approach*, McGraw Hill, New York, 1963.

[7] R. A. DeCarlo, *Linear Systems*, Prentice Hall, Inc., Englewood Cliffs, NJ, 1989.