Practical Automated Reasoning on Inequalities: Generic Programs for Inequality Proving and Discovering

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Automated inequality proving has been a difficult topic in the area of automated reasoning for many years. The concerning algorithms depend on real algebra and real geometry, and the computational complexity increases very quickly with the dimension, i.e. the number of parameters. Some well-known algorithms are complete theoretically but inefficient in practice, which cannot verify non-trivial propositions in batches. A dimension-decreasing algorithm presented here can treat radicals efficiently and make the dimensions lowest. Based on this algorithm, a generic program called "BOTTEMA" was implemented on a PC computer. About 500 algebraic and geometric inequalities including more than 100 open problems have been verified in this way. The total CPU time spent for proving 120 basic inequalities from Bottema's monograph, "Geometric Inequalities" on a Pentium/200, was 20-odd seconds only.

Key Words: left/right polynomial, class CGR, critical surface.

1. Algebraic approaches: achievements and difficulties

We have reason to believe that computer will play a much more important role in reasoning sciences in the next century. People will be able to prove theorems class by class instead of one by one. Since Tarski's[14] well-known work, *A Decision Method for Elementary Algebra and Geometry,* published in early 1950's, the algebraic approaches have made remarkable progress in automated theorem proving. Tarski's decision algorithm has only got theoretical significance, that could not be used to verify any non-trivial algebraic or geometric propositions in practice, because of its very high computational complexity. Some substantial improvements were made by Seidenberg, Collins[8] and others afterwards, but it was still far away from mechanically proving non-trivial theorems batch by batch, even class by class. The situation didn't change until Wu Wen-tsün[16][17] proposed in 1977 a new decision procedure for proving geometry theorems of "equality type", i.e. the hypotheses and conclusions of the statements consist of polynomial equations only. This is a very efficient method for mechanically proving elementary geometry theorems (of equality type). S.C. Chou[4] has successfully implemented Wu's method for 512 examples which include almost all the well-known or historically interesting theorems in elementary geometry, and it was reported that for most of the examples the CPU time spent was only few seconds each, or less than 1 second!

The success of Wu's method has inspired in the world the advances of the algebraic approach[11][15] to automated theorem proving. In the past 20 years, some efficient provers have been developed based on different principles such as Gröbner Basis[2][3], Parallel Numerical Method[26] [22], and so on. Especially, J.Z. Zhang and his colleagues gave the algorithms and programs for automatically producing readable proofs of geometry theorems[7][18][27]. The achievement makes the studies in automated proving enter a new stage that the proofs created by machines can compare with those by human being, while the decision problem was playing a leading role before. It has also important applications to mathematics mechanization and CAI.

These packages mentioned above are mainly valid to equality-type theorem proving, however, automated inequality proving has been a difficult topic in the area of automated reasoning for many years. The concerning algorithms depend on real algebra and real geometry, and the computational complexity increases very quickly with the dimension, i.e. the number of parameters. Some well-known algorithms are complete theoretically but inefficient in practice, which cannot verify non-trivial propositions in batches. Recently Chou, Gao et al[5][6] made helpful approaches in this aspect by combining Wu's method with CAD (Cylindrical Algebraic Decomposition) algorithm or others. L. Yang and his colleagues[19][23][20] introduced a strong tool, a complete discrimination system (CDS) of polynomials, for inequality reasoning. By means of CDS, a generic program called "EXPLORER" was also implemented on PC computers that is able to discover new inequalities, without requiring us to put forward any conjectures beforehand. For example, by means of this program, we have re-discovered 37 inequalities in the first chapter of the famous monograph, "Recent Advances in Geometric Inequalities", and found three mistakes at the same page.

The CDS would be able to solve a variety of problems in science, technology and engineering that ask for real solutions, but it would be invalid for automatically proving the theorems of higher dimensions or with more parameters. When the hypotheses contain some algebraic equations, one may think to eliminate some variables to make the dimension lower. In this way, however, usually we have to deal with parametric radicals. A dimension-decreasing algorithm presented here can treat radicals efficiently and make the dimensions lowest. Based on this algorithm, a generic program called "BOTTEMA" was implemented on a PC computer. About 500 algebraic and geometric inequalities including more than 100 open problems have been verified in this way. The total CPU time spent for proving 120 basic inequalities from Bottema's monograph, "Geometric Inequalities" on a Pentium/200, was 20-odd seconds only.

2. An example

For popularity, we show the main point of our algorithm with the following inequality-type proposition.

Proposition 1. Given real numbers u, v, w, x, y, z satisfying the following 9 conditions,

$$\begin{array}{l} u^2 + 6\,xu - y^2 + 2\,yz - 4\,xy - 4\,zx - z^2 + 5\,x^2 = 0, \\ v^2 + 6\,yv - z^2 + 2\,zx - 4\,yz - 4\,xy - x^2 + 5\,y^2 = 0, \\ w^2 + 6\,zw - x^2 + 2\,xy - 4\,zx - 4\,yz - y^2 + 5\,z^2 = 0, \\ u + 3\,x \geq 0, \quad v + 3\,y \geq 0, \quad w + 3\,z \geq 0, \\ x > 0, \quad y > 0, \quad z > 0, \end{array} \tag{1}$$

prove $\quad u + v + w \geq 0.$

To verify this proposition, the basic idea of algebraic decomposition is: decompose the space of parameters u, v, w, x, y, z into a finite number of parts, i.e. some cells with different dimensions, pick out all the parts where the hypothesis of the proposition holds, and then check whether the conclusion $u + v + w \geq 0$ holds over the parts picked out. If so, Proposition 1 is true; otherwise, it's false.

We will face a problem of 6-dimensional space if do a decompositin simply without dimension-decreasing measure. It would be impossible to implememt a nontrivial algebraric decomposion for a 6-dimensional space by means of the current computer softwares and hardwares. In fact, it was said that such a decomposition is very difficult even for spaces of dimensions more than 3. So, we should take some measures to keep the dimension as low as possible.

Noting that the 6 variables are not independent, we can regard x, y, z as papameters and u, v, w as unknowns, and solve the first 3 equations of (1) for u, v, w. We have

$$\begin{array}{rcl} u & = & -3\,x \pm \sqrt{4\,x^2 + 4\,xy + 4\,zx + y^2 - 2\,yz + z^2}, \\ v & = & -3\,y \pm \sqrt{4\,y^2 + 4\,yz + 4\,xy + z^2 - 2\,zx + x^2}, \\ w & = & -3\,z \pm \sqrt{4\,z^2 + 4\,zx + 4\,yz + x^2 - 2\,xy + y^2}. \end{array}$$

Employing the next 3 conditions,

$$u + 3\,x \geq 0, \quad v + 3\,y \geq 0, \quad w + 3\,z \geq 0,$$

we see that the three radicals above are positive, i.e.

$$\begin{array}{rcl} u & = & -3\,x + \sqrt{4\,x^2 + 4\,xy + 4\,zx + y^2 - 2\,yz + z^2}, \\ v & = & -3\,y + \sqrt{4\,y^2 + 4\,yz + 4\,xy + z^2 - 2\,zx + x^2}, \end{array}$$

$$w = -3\,z + \sqrt{4\,z^2 + 4\,zx + 4\,yz + x^2 - 2\,xy + y^2}.$$

Substitute it in the conclusion $u + v + w \geq 0$, we have

$$\begin{aligned} &-3\,x + \sqrt{4\,x^2 + 4\,xy + 4\,zx + y^2 - 2\,yz + z^2} \\ &-3\,y + \sqrt{4\,y^2 + 4\,yz + 4\,xy + z^2 - 2\,zx + x^2} \\ &-3\,z + \sqrt{4\,z^2 + 4\,zx + 4\,yz + x^2 - 2\,xy + y^2} \geq 0. \end{aligned}$$

Thus, Proposition 1 is equivalent the following

Proposition 2. Given real numbers $x > 0, \quad y > 0, \quad z > 0,$ prove

$$\begin{aligned} &\sqrt{4\,x^2 + 4\,xy + 4\,zx + y^2 - 2\,yz + z^2} + \sqrt{4\,y^2 + 4\,yz + 4\,xy + z^2 - 2\,zx + x^2} \\ &+ \sqrt{4\,z^2 + 4\,zx + 4\,yz + x^2 - 2\,xy + y^2} \geq 3(x + y + z). \end{aligned} \tag{2}$$

In comparison with Proposition 1, the number of variables reduces to 3, but 3 radicals occur in hypothesis. To eliminate these radicals, the conventional means is to introduce new variables, that way the problem returns to 6-dimensional again. So, we must find an efficient algorithm not only eliminating the radicals but also keeping the dimensions non-increasing. That is the problem so-called "rationalization for algebraic inequality with radicals".

3. Rationalization for inequality with radicals

For brevity, it is necessary to introduce a few definitions to describe our algorithm.

Definition 1. Assume Φ is an algebraic inequality (or equality) in $x, y, z, \cdots$ etc. $L(T)$ is called a *left polynomial* of Φ, provided

- $L(T)$ is a polynomial in T, and all the coefficients are rational polynomials in $x, y, z, \cdots$ etc.,
- the left hand side of Φ is a zero of $L(T)$,
- amongst[1] all the polynomials satisfying the two items above, $L(T)$ is what has the lowest degree in T.

The *right polynomial* of Φ, namely, $R(T)$, can be defined analogously.

There are different methods to find the left/right polynomial of an algebraic inequality with radicals, e.g. the parametric resultant method[28][21]. For inequality (2) in last section, the left polynomial $L(T)$ is of 8-degree in T as follows:

$$T^8 - 24\,(yz + xy + z^2 + x^2 + zx + y^2)\,T^6 + 144\,(2\,yzx^2 + z^4 + 3\,y^2z^2$$

[1] sometimes, this requirement is unnecessary

$$
\begin{aligned}
&+2\,y^2zx+2\,xy^3+2\,yz^3+2\,z^3x+2\,x^3y+3\,x^2y^2+2\,y^3z+x^4+y^4\\
&+3\,z^2x^2+2\,zx^3+2\,z^2xy)\,T^4-64\,(4\,z^6+4\,x^6+4\,y^6+12\,x^5y-26\,z^3x^3\\
&+12\,z^5x-3\,z^2x^4+12\,zx^5+12\,z^5y-3\,y^4z^2-26\,y^3z^3-26\,x^3y^3-3\,z^4x^2\\
&+12\,y^5z+12\,y^5x-3\,y^4x^2-3\,z^4y^2-3\,x^4y^2+72\,z^4xy+78\,yz^3x^2\\
&+72\,yzx^4+78\,yz^2x^3+78\,y^2z^3x+72\,y^4zx+78\,y^3zx^2+78\,y^2zx^3\\
&+84\,y^2z^2x^2+78\,y^3z^2x)\,T^2+20736\,y^2z^2x^2(x+y+z)^2. \qquad (3)
\end{aligned}
$$

And the right polynomial $R(T)$ is obviously of 1-degree in T:

$$T-3\,(x+y+z). \qquad (4)$$

Thus, Proposition 2 as well as Proposition 1 is equivalent to the following

Proposition 3. Assume the parameters x, y, z in polynomials (3) and (4) take positive values. Show that the greatest real root of (3) is greater than or equal to the greatest real root of (4), regarding (3) and (4) as univariate polynomials in T.

This statement involves a lower dimension. By some parametric transformation, We need only do a planar decomposition, and needn't deal with radicals. The CPU time spent to verify Proposition 3 on a Pentium/200 is about 1.6 seconds, making use of our generic program BOTTEMA.

Such a scheme based on the comparison between the greatest roots of two polynomials, abbreviated as CGR, covers a large class of algebraic and geometric inequalities, that is called class CGR.

BOTTEMA written in MAPLE is aimed at class CGR which most of the propositions in monograph "Geometric Inequalities"[1] belong to. In order to avoid the questions and arguments on the onerous error's estimation for floating-point computation, here is employed the exact computation in every step.

4. More examples in class CGR

The well-known Janous's inequality[10] which was proposed as an open problem in 1986 and solved in 1988 belongs to the class CGR defined as above.

Proposition 4. (Janous's Inequality) By m_a, m_b, m_c and $2\,s$ denote the three medians and perimeter of a triangle. Show that

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}>\frac{5}{s}.$$

The left hand side of the difficult inequality implicitly contains three radicls. BOTTEMA automatically interprets the geometric proposition to algebraic one before proves it. The total CPU time spent for this example on a Pentium/200 is about 80 seconds.

The following open problem appeared as Problem 169 at *Mathematical Communications* (in Chinese) a few years ago.

Proposition 5. By r_a, r_b, r_c and w_a, w_b, w_c denote the radii of the ex-circles and the bisectors of the angles of a triangle, respectively. Prove or disprove

$$\sqrt[3]{r_a r_b r_c} \leq \frac{1}{3}(w_a + w_b + w_c).$$

In other words, *the geometric average of r_a, r_b, r_c is less than or equal to the arithmetic average of w_a, w_b, w_c.*

The right hand side of the inequality implicitly contains 3 radicals. BOTTEMA proved this conjecture on a Pentium/200 with CPU time less than 3 minutes. One more conjecture from the book *Geometric Inequality in China*[13] involves the trigonometric functions:

Proposition 6. By A, B, C, r, R denote the three angles, inradius and circumradius of a triangle, respectively. Prove or disprove

$$\cos\frac{A}{2}\cos\frac{B}{2} + \cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{C}{2}\cos\frac{A}{2} \leq \frac{1+2\sqrt{2}}{2} + (7-4\sqrt{2})\frac{r}{R}. \quad (5)$$

This is apparently beyond class CGR for the right hand side of (5) is not the greatest root of its right polynomial. However, multiplying the both sides of (5) by $\sqrt{2}$, we obtain the following equivalent proposition which belongs to class CGR:

$$\sqrt{2}\,(\cos\frac{A}{2}\cos\frac{B}{2} + \cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{C}{2}\cos\frac{A}{2}) \leq \frac{4+\sqrt{2}}{2} + (7\sqrt{2}-8)\frac{r}{R}. \quad (6)$$

The CPU time spent for proving (6) on a Pentium/200 is about 15 minutes.

The next conjecture proposed by J. Liu[13] was proved on the same machine with CPU time about 8500 seconds and memory 31.8 M.

Proposition 7. By a, b, c, m_a, m_b, m_c and w_a, w_b, w_c denote the side lengths, medians and angle-bisectors of a triangle, respectively. Prove or disprove

$$a\,m_a + b\,m_b + c\,m_c \leq \frac{2}{\sqrt{3}}\,(w_a^2 + w_b^2 + w_c^2).$$

In fact, more than 100 open problems in geometric inequalities have been proved or disproved in a short time using this package.

The following conjecture appeared in the *Research Communications* published informally by the Inequality Reaserch Group in China, was disproved in 3.6 seconds, that means, our prover is not a "yes man".

Proposition 8. By a, b, c, denote the side-lengths of a triangle, respectively. Prove or disprove

$$\frac{1}{6}\left(|b-c|+|c-a|+|a-b|\right)^2 \leq \frac{a\,(b-c)^2}{b+c}+\frac{b\,(c-a)^2}{c+a}+\frac{c\,(a-b)^2}{a-b}. \qquad (7)$$

The computer disproved it by showing a sentence "`no inequality holds`" with the data wherefrom we got a counter-example immediately, say, a triangle with side-lengths $a=7$, $b=8$, $c=13$ which contradicts Proposition 8 obviously so that anybody can check by hands.

Noting the counter-example is an obtuse-triangle, one has reason to guess that (7) would hold for acute-triangles, i.e.

Proposition 9. By a, b, c, denote the side-lengths of an acute-triangle, respectively. Prove or disprove

$$\frac{1}{6}\left(|b-c|+|c-a|+|a-b|\right)^2 \leq \frac{a\,(b-c)^2}{b+c}+\frac{b\,(c-a)^2}{c+a}+\frac{c\,(a-b)^2}{a-b}.$$

With CPU time 82 seconds, actually the screen showed "`the inequality holds`" so probably we have discovered a new theorem if it is.

5. A class of min-max problems

It should be pointed out that class CGR includes a lot of min-max problems which would be of importance in various applications. The next example[12] shows how carefreely the algorithm applies to such problems without inspirations.

Proposition 10. Given real numbers $x>0$, $y>0$, $z>0$, prove

$$x+y+z+\sqrt{x^2+y^2+z^2-xy-yz-zx} \leq 3\max\{x,\,y,\,z\}. \qquad (8)$$

The l.h.s. and r.h.s. are respectively the greatest real roots of $T^2-2(x+y+z)\,T+3(xy+yz+zx)$ and $(T-3\,x)(T-3\,y)(T-3\,z)$, two polynomials in T. So the proposition is equivalent to the following

Proposition 11. Assume the parameters x, y, z take positive values in $T^2-2(x+y+z)\,T+3(xy+yz+zx)$ and $(T-3\,x)(T-3\,y)(T-3\,z)$, which we regard as univariate polynomials in T. Show that the greatest real root of the former is less than or equal to that of the latter.

To prove the proposition by BOTTEMA on a Pentium/200, the CPU time spent is about 0.26 sec.

The next example as well as Proposition 11 belongs to class CGR.

Proposition 12. By m_a, m_b, m_c and h_a, h_b, h_c denote the medians and altitudes, R and r the circumradius and inradius of an acute-triangle, respectively. Show that

$$\max\{m_a - h_a,\, m_b - h_b,\, m_c - h_c\} \geq \frac{1}{2}\,(R - 2\,r).$$

The CPU time for proving the above proposition is about 56 seconds.

The following conjecture[25] seems much more complicated, but it also belongs to class CGR.

Proposition 13. Amongest seven points in a unit square, there always exist three points which form a triangle of area less than or equal to $\frac{1}{12}$.

By $(x_i,\, y_i)$, $i = 1, \cdots, 7$, denote the 7 points, and Δ_{ijk}, $i, j, k = 1, \cdots, 7$, denote the areas of all the triangles formed by the 7 points. So the conjecture is equivalent to the following

Proposition 14. The greatest root of the polynomial

$$\prod_{1 \leq i \leq j \leq k \leq 7} (T + \Delta_{ijk})$$

is greater than or equal to $-\frac{1}{12}$, the greatest root of polynomial $T + \frac{1}{12}$, provided $0 \leq x_i \leq 1$, $0 \leq y_i \leq 1$, for $i = 1, \cdots, 7$.

This requires a decomposition of a 14-dimensional space, that is impossible by the current computer conditions. However, we have seen such a problem is really covered by class CGR, as well as a lot of min-max problems else, so that they are no longer the kind of problems without generic algorithms.

6. A general description of the algorithm

It was showed that class CGR covers a lot of propositions but can't cover all the inequalities in elementary algebraic and geometry. We will describe an algorithm valid in an area more extensive than class CGR.

Definition 2. Assume Φ is an algebraic inequality (or equality) in $x, y, z, \cdots$ etc., $L(T)$ and $R(T)$ are the left and right polynomial of Φ, respectively. By $P(x, y, \cdots)$ denote the resultant of $L(T)$ and $R(T)$ with respect to T, and call the surface defined by $P(x, y, \cdots) = 0$ *the critical surface of* Φ.

To verify a proposition with conclusion and hypothesis formed by algebraic inequlities $\Phi_0,\ \Phi_1,\ \cdots,\ \Phi_s$ in $x, y, z, \cdots$ etc., e.g.

$$\Phi_1 \wedge \Phi_2 \wedge \cdots \wedge \Phi_s \Rightarrow \Phi_0, \tag{9}$$

we take the following procedures.

1. Find the critical surfaces of the inequalities $\Phi_0,\ \Phi_1,\ \cdots,\ \Phi_s$.

2. These critical surfaces decomposes the parametric space into a finite number of parts, $D_1, D_2, \cdots, D_s$. Choose at least one test point in every part, $(x_\nu, y_\nu, \cdots) \in D_\nu,\ \nu = 0, 1, \cdots, s$.

3. We need only check the proposition for such a finite number of test points, $(x_1, y_1, \cdots), \cdots, (x_s, y_s, \cdots)$. The statement is true if and only if it holds over these test values.

The above procedures sometimes may be simplified. When the conclusion Φ_0 is a CGR inequality, what we need in step **3** is to compare the greatest roots of left and right polynomials of Φ_0 over the test values.

According to our record, the numbers of the test points for above examples are listed as follows.

Proposition	2	1	test point
Proposition	4	11	test points
Proposition	5	4	test points
Proposition	6	27	test points
Proposition	7	3	test points
Proposition	8	4	test points
Proposition	9	4	test points
Proposition	10	1	test point
Proposition	12	12	test points

When the conclusion of the proposition to be verified is an inequality of type "$\geq$" or "$\leq$", we can select all the test points from some open sets of the parametric space and thus make all the test values to be rational, that remarkably reduces the computational complexity in the key steps.

7. Conclusion

In last 20 years, the efficiency of automated equality-type theorem proving has increased greatly which became considerably wide apart that of inequality, especially, for geometric theorems. The work reported here is an effort to shorten the distance.

O.Bottema was the first author of the monograph *Geometric Inequalities* which was known as "Bottema's Bible" due to the high frequency of citation. The generic program BOTTEMA was designed mainly aimed at the inequalities which are of class CGR and with 2 freedoms only. Besides a lot of elementary algebraic inequalities, this includes most of the theorems in "Bottema's Bible". A general algorithm was also introduced which is valid for a more extensive class, but the efficiency seems much less than that of BOTTEMA.

In my opinion, one of the key problems to promote the efficiency of an algorithm for inequality proving is how to employ the relevant knowledge such as Arithmetic Mean Inequality and so on. That would be a quite difficult task but the expected achievements will bring the studies to enter another new stage.

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