Multigrid Solution With Rotated High-order Discretizations for Solving Poisson Equation

Mohamed bin Othman*
Abdul Rahman bin Abdullah†

Department of Industrial Computing
University Kebangsaan Malaysia
43600 UKM Bangi Selangor D.E.
Malaysia

Abstract

This paper described the halfsweeps multigrid method with the rotated high-order discretization for solving the two dimensional Poisson equation with the Dirichlet boundary conditions. This method along with several ordering strategies shows a relatively good in accuracy and drastic improvement in execution time. The numerical results of test problem are given.

Keywords: Multigrid method, Poisson equation, High-order discretization, Gauss-Seidel relaxation scheme.

C.R. Categories: G.1.8.

1 Introduction

Consider the following two dimensional Poisson equation,

$$\begin{aligned} \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} &= f(x,y), \qquad (x,y) \in \Omega, \\ u(x,y) &= g(x,y), \qquad (x,y) \in \partial\Omega, \end{aligned} \tag{1. 1}$$

with the Dirichlet boundary conditions. When the partial derivatives of equation (1.1) are solved by the finite differences formulae, the most commonly used approximation is the standard five-points stencil ($\mathcal{SFPS}$) as,

$$v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1} - 4v_{i,j} = h^2 f_{i,j}, \tag{1. 2}$$

where the widths between mesh points $\Delta x = \Delta y = \frac{1}{N} = h$ are uniform and the solution domain can be denoted as Ω^h. The $v_{i,j}$ is an approximation to the exact solution $u_{i,j}$ at any mesh points (x_i, y_j) for all $i, j = 1, 2, \ldots, N-1$. This stencil has a truncation error of order $O(h^2)$. An approximation of order $O(h^4)$ can also be used to solve equation (1.1),

*mothman@cs.upm.edu.my
†ara@pkrisc.cc.ukm.my

$$4[v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1}] + v_{i+1,j+1} + v_{i-1,j-1} + v_{i+1,j-1} + v_{i-1,j+1} - 20v_{i,j} = \frac{h^2}{2}[f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} + 8f_{i,j}]. \quad (1.\ 3)$$

This standard nine points stencil is called the *Mehrstellenverfahren* due to L. Collatz, 1960 and it has been known for many years, see [2]. In recent years, several authors have derived the high-order discretization to solve various partial differential equations and they ended up with the stencil (1.3) for the case of Poisson equation. More information about this discretization can be found in [5,6].

Another type of approximation so-called the rotated discretization, can be achieved by rotating the i- and j-plane axis clockwise by 45°. Thus, the discretized form of equation (1.1) at the mesh point (x_i, y_j) with the finite differences formulae will leads the rotated five points stencil ($\mathcal{RFPS}$) as,

$$v_{i+1,j+1} + v_{i-1,j-1} + v_{i+1,j-1} + v_{i-1,j+1} - 4v_{i,j} = 2h^2 f_{i,j}. \quad (1.\ 4)$$

This stencil is order of $O(h^2)$ and has been shown by G. Dahlquist *et. al.*, 1974, see [3]. This stencil combined with the stencil (1.2) are extensively used to develop the four points explicit decoupled group (EDG) method and the halfsweeps multigrid method, see [1,9].

An approximation of high order can be employed to approximate equation (1.1),

$$v_{i+2,j} + v_{i-2,j} + v_{i,j+2} + v_{i,j-2} + 4[v_{i+1,j+1} + v_{i-1,j-1} + v_{i+1,j-1} + v_{i-1,j+1}] - 20v_{i,j} = h^2[f_{i+1,j+1} + f_{i+1,j-1} + f_{i-1,j+1} + f_{i-1,j-1} + 8f_{i,j}]. \quad (1.\ 5)$$

This equation is called a rotated nine points stencil ($\mathcal{RNPS}$) and more details of this stencil are described in [10].

In §2, we describe the multigrid methods which use the standard high order discretization. While in §3 and §4 show the implementation of the halfsweeps multigrid method with the rotated high order discretization and the numerical experiments on the specific Poisons equation, respectively. §5 offers a summary.

2 Multigrid Method

The multigrid method is a fast and one of the most efficient iterative method for solving a wide class of partial differential equations. This method is used in many areas of scientific computing and engineering, see [7].

A simple $V(\nu_1, \nu_2)$-cycle multigrid algorithm can be described as follow,

Algorithm MGV(i, A^h, v^h, f^h)
{
If $(i == 0)$ coarsest grid, then solve $A^h e^h = r^h$ directly.
else {
  Smooth ν_1 times on $A^h v^h = f^h$ on Ω^h
  Compute the residual $r^h \leftarrow f^h - A^h v^h$
  Set $e^{2h} \leftarrow 0$ and restrict $r^{2h} \leftarrow \mathcal{R}_h^{2h} r^h$
  Get $e^{2h} \leftarrow$ MGV$(i-1, A^{2h}, e^{2h}, r^{2h})$
  Compute prolongation and errors (corr.) $v^h \leftarrow v^h + \mathcal{P}_{2h}^h e^{2h}$
  Smooth ν_2 times on $A^h v^h = f^h$ on Ω^h
  }
Return v^h as the approximate solution
}

Algorithm 1: The recursive multigrid algorithm with $V(\nu_1, \nu_2)$-cycle scheme.

The restriction and prolongation are two processes of transferring mesh points from fine to coarse grid ($\mathcal{R}_h^{2h} : \Omega^h \rightarrow \Omega^{2h}$) and from the coarse to fine grid ($\mathcal{P}_{2h}^h : \Omega^{2h} \rightarrow \Omega^h$), respectively. The computed residuals r^h are injected to the coarse grid defined as $r^{2h} \leftarrow \mathcal{R}_h^{2h} r^h$ and there are several restriction operators available such as full-weighting- and half-weighting-injections, see [7]. While the most popular prolongation operator is bilinear interpolation which will be used in this research works.

Many research works have been done in the multigrid method, see [4,7,8,11,12,13]. For instance in [8], the authors used the standard high order discretization scheme with multigrid method to solve three dimensional Poisson equation in cylindrical coordinates. They employed the S- and V-cycles in their experiments and found the S- cycle is more preferable due to the small number of smoothing rate. The standard high order was only used in calculating the residual on the finest grid while the standard 2nd order was used to do the relaxations on all levels and calculate the residual on the coarse levels.

The standard high order discretization was used by P. M. de Zeeuw and E.J.van Asselt, 1985 to develop a *Black Box multigrid solver* for the general linear second order 2-D Elliptic partial differential equations. The solver employs the S-cycle, matrix-dependent grid transfers and incomplete line LU relaxation scheme, see [12].

The nine-points discretization of the Poisson equation was analyzed extensively by Stüben and U. Trottenberg, 1982 and they have shown that for the ($\mathcal{SNPS}$) and four colors has the smallest smoothing factor. Unfortunately, there are no report in practical application due to M.M. Gupta *et. al.,* 1995, see [4]. In the same paper, M.M. Gupta, *et. al.,* 1995 compared extensively the multigrid methods using the five- and nine-points stencils with the half- and full-weighting restriction operators respectively on vector and serial machines. They used the red black Gauss-Seidel relaxation scheme for their pre- and post-smoothing schemes. From the experiments, they found that the multigrid solution using the ($\mathcal{SNPS}$) combined with full-weighting restriction operator is more accurate and efficient than using the five-points stencil ($\mathcal{SFPS}$) on both machines. Nevertheless, both methods used all the mesh points in the process of iteration and it can be denoted as *fullsweeps multigrid method.*

3 Implementation of Halfsweeps Multigrid Method with Rotated High Order Discretization

The halfsweeps multigrid method due to M. Othman and A.R. Abdullah, 1997 needs the solution domain at any levels in the form of red $\circ$ and black $\square$ points labelling. They employed both the standard and rotated 2nd order discretizations and the horizontal zebra line Gauss-Seidel relaxation scheme for their pre- and post-smoothing schemes. From the experiments, they claimed that the method as the *fastest Multigrid Poisson Solver* and details of this method can be found in [9].

The same idea is applied to develop the halfsweeps multigrid method with the rotated high order discretization. In this case, a group of red points will be iterated until the convergence criteria is met, then the other group of points i.e. black points will be executed at once using the (1.2) stencil. All the red points next to the boundaries will be iterated using the (1.4) stencil (i.e. in dotted boxes, see Figure 1) while all the internal red points use the (1.5) stencil. It shows that the stencils (1.4) and (1.5) can be implemented by only involving points of type $\circ$. In other words, the iterative evaluation of points type $\circ$ only require contribution from points type $\circ$ and the same happen for points type $\square$. Therefore, the implementation of these two stencils can be carried out independently.

Those residuals needed to be injected are calculated and then transfer into the respective red points at the coarser grid. Since all the red points involved in the iterative evaluation, the following restriction operator is requires,

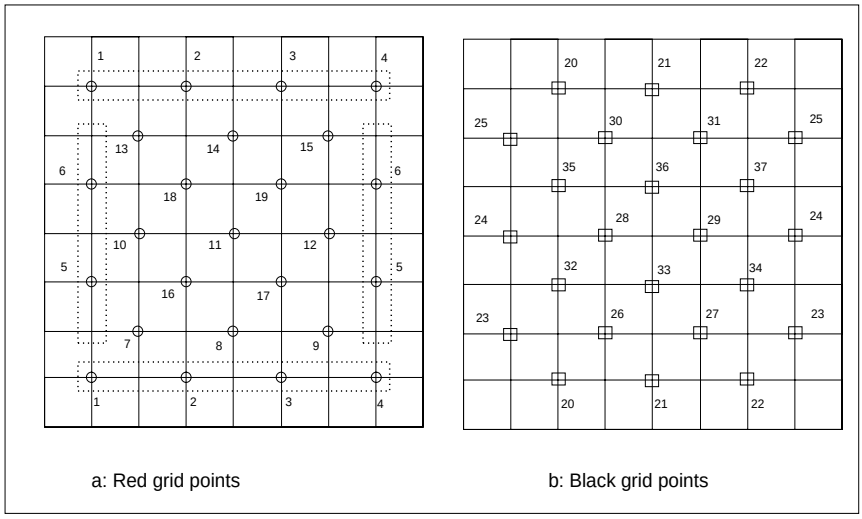

Figure 1: The solution domain in the red ∘ black □ labelling. The numbers indicate the horizontal zebra line- (HZL) ordering strategy on the red mesh points.

$$\mathcal{R}_h^{2h} = \frac{1}{8}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \tag{3. 1}$$

On the other hand, the linear prolongation is used to transfers the red points from the coarser grids to the red points at the finer grids as follows,

$$v_{2i,2j}^h = \quad v_{i,j}^{2h}, \quad \text{for all } i,j \text{ both even or odd.} \tag{3. 2}$$

While the bilinear interpolation is applied to interpolate the red points on the fine grid and stated as follows,

$$\begin{array}{ll} v_{4i-2,4j}^h & = \frac{1}{2}(v_{4i-2,4j-2}^{2h} + v_{4i-2,4j+2}^{2h}), \\ v_{4i,4j-2}^h & = \frac{1}{2}(v_{4i-2,4j-2}^{2h} + v_{4i+2,4j-2}^{2h}), \quad \text{for all } i,j = 1,2,\ldots,N-1, \end{array} \tag{3. 3}$$

and

$$v_{i,j}^h \quad = \quad \frac{1}{4}(v_{i-1,j-1}^{2h} + v_{i+1,j-1}^{2h} + v_{i-1,j+1}^{2h} + v_{i+1,j+1}^{2h}) \text{ for all } i,j \text{ odd.} \tag{3. 4}$$

The most popular smoothing method is the Gauss-Seidel iteration scheme and its a more effective smoother than Jacobi's scheme. These due to the fact that the new updated values are used to calculate the next value as they become available. It is very important that the residuals are well smoothed before they can be transferred to the coarser grids.

Several ordering strategies were employed in the pre- and post-smoothing schemes and they were applied only on the red points at each levels. The optimum ordering strategy so-called the *horizontal zebra line* (HZL) strategy starts from red points $7, 8, 9, \ldots, 14, 15$ and followed by $16, 17, \ldots, 19$, as illustrated in Figure 1a. While red points next to the boundaries used the natural ordering. All the red points will undergo the iteration evaluation using the (1.4) and (1.5) stencils until the convergence criteria is met and then the other half of points are obtained directly once using (1.2) stencil as shown in Figure 1b.

h^{-1}	Fullsweeps Multigrid with $\mathcal{SNPS}$				Halfsweeps Multigrid with $\mathcal{RNPS}$			
	Ordering	$V(1,1)$	Time	Max error	Ordering	$V(1,1)$	Time	Max error
8	H	11	0.1622	4.60×10^{-6}	H	9	0.0728	2.59×10^{-4}
	RB	10	0.1491	1.81×10^{-6}	HZL	8	0.0661	2.59×10^{-4}
	4C	9	0.1341	4.60×10^{-6}				
16	H	12	0.7682	1.16×10^{-7}	H	11	0.3546	2.21×10^{-5}
	RB	10	0.6443	1.16×10^{-7}	HZL	10	0.3237	2.21×10^{-5}
	4C	10	0.6425	1.16×10^{-7}				
32	H	12	3.2106	7.28×10^{-9}	H	12	1.5661	1.76×10^{-6}
	RB	10	2.6861	7.28×10^{-9}	HZL	11	1.4364	1.76×10^{-6}
	4C	10	2.6801	7.28×10^{-9}				
64	H	12	13.3846	4.55×10^{-10}	H	12	6.4841	1.25×10^{-7}
	RB	10	11.1992	4.55×10^{-10}	HZL	11	5.9568	1.25×10^{-7}
	4C	10	11.1900	4.55×10^{-10}				
128	H	13	61.4929	2.83×10^{-11}	H	13	31.9713	8.35×10^{-9}
	RB	10	48.3782	2.72×10^{-11}	HZL	11	27.3852	8.35×10^{-9}
	4C	10	48.1860	2.84×10^{-11}				
256	H	13	256.7041	2.59×10^{-12}	H	13	138.3894	5.44×10^{-10}
	RB	11	222.0144	1.74×10^{-12}	HZL	12	130.0475	5.44×10^{-10}
	4C	10	204.9203	2.14×10^{-12}				

Table 1: The numerical results of fullsweeps- and halfsweeps-multigrid methods with the $\mathcal{SNPS}$ and $\mathcal{RNPS}$, respectively.

4 Numerical Experiments

The multigrid methods with the high order discretizations are tested on the following two dimensional Poisson equation,

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = (x^2+y^2)e^{xy}, \qquad (x,y) \in \Omega = [0,1]\times[0,1], \tag{4. 1}$$

with subject to the Dirichlet boundary conditions and satisfying the exact solution,

$$u(x,y) = e^{xy}, \quad \text{for } (x,y) \in \partial\Omega.$$

The initial and boundary conditions are defined so as to agree with the exact solutions.

In the experiments, several parameters are reported such as the number of $V(1,1)$-cycle, execution time (seconds) and maximum of errors. The methods run on Sequent S27 system with different size of grids $\Omega^h, \Omega^{2h}, \ldots, \Omega^{256h}$. The fullsweeps multigrid method is employed the Gauss-Seidel relaxation scheme along with the horizontal (H), red black (RB) and four colors (4C) ordering strategies. On the other hand, the halfsweeps method is used the same smoothing scheme as described in the previous section. Both methods will stop when the respective mesh points which undergo the process of iterative evaluation at the finest grid are less than $\epsilon = 10^{-10}$ where ϵ is the error tolerance.

After several experiments, all the numerical results are reported in the Table 1. The maximum errors- and optimum execution time- versus grids size were plotted and shown in Figures 2 and 3, respectively.

5 Summary

The numerical results obtained in Figures 2-3 show that the halfsweeps multigrid method with the rotated high order discretization is relatively good in accuracy compared to the fullsweeps multigrid method with the standard high order in all observed cases. This is due to the fact that both methods used the respective stencils of $O(h^4)$. In execution time, the halfsweeps multigrid method

is faster than the fullsweeps multigrid method, reduce nearly 40-50%, this because of only half of the points in the solution domain undergo the process of iteration. While in Table 1, we found that the fullsweeps multigrid method with the four colors Gauss Seidel relaxation scheme is better than the other two strategies which agreed with K. Stüben and U. Trottenberg, 1982. This result contradicted with the one found by M.M. Gupta *et. al.*, 1995.

Overall, on the large size of grids, it can be summarized that the halfsweeps multigrid method with the rotated high order discretization will improve both the accuracy and execution time.

Although at the moment our results may not convince the researchers to move the high order formula immediately, it lays down the foundation for further research in this direction. We believe that high order discretizations are promising in multigrid method.

References

[1] A.R. Abdullah, 1991, *The Four Points Explicit Decoupled Group (EDG) Method: A Fast Poisson Solver*, Int. J. Comp. Math., **38**, pp:61-70.

[2] L. Collatz, 1960, *The Numerical Treatment of Differential Equations*, Springer-Verlag, Berlin.

[3] G. Dahlquist and A. Bjorck, 1974, *Numerical Methods*, Englewood Cliffs, Prentice Hall, New York.

[4] M.M. Gupta, J. Kouatchou and J. Zhang, 1995, *Comparison of 2nd and 4th Order Discretizations for Multigrid Poisson Solvers*, a preprint of this paper is available by anonymous ftp from **casper.cs.yale.edu** in the subdirectory **mgnet/papers/Gupta** and accepted in the J. Comp. Phys.

[5] M.M. Gupta, R.P. Manohar and J.W. Stephenson, 1984, *High-Order Difference Schemes of Two-Dimensional Elliptic Equations*, Numerical Methods for Partial Differential Equations, **1**, pp:71-80.

[6] M.M. Gupta, 1984, *A Fourth-Order Poisson Solver*, J. Comp. Phys., **55**, pp:166-172.

[7] W. Hackbusch, 1980, *Multi-Grids Methods and Applications*, Springer Series in Computational Mathematics, 4, Springer-Verlag, Berlin.

[8] R.K. Iyengar and A. Goyal, 1990, *A Note on Multigrid for the Three-Dimensional Poisson Equation in Cylindrical Coordinates*, J. Comput. Appl. Math., **33**, pp:163-169.

[9] M. Othman and A.R. Abdullah, 1997, *The Halfsweeps Multigrid Method As A Fast Multigrid Poisson Solver*, submitted to Int. J. Comp. Math.

[10] M. Othman and A.R. Abdullah, 1997, *A High-Order Rotated Nine Points Difference Solutions for Poisson's Equations*, UKM-FTSM-TR1xxx/97, Department of Industrial Computing, University Kebangsaan Malaysia.

[11] K. Stüben and U. Trottenberg, 1982, *Multigrid Methods: Fundamental Algorithms, Model Problem Analysis and Applications*, in Multigrid Methods edited by A. Dold and B. Eckmann, Lecture Notes in Mathematics, **960**, Springer-Verlag, Berlin, pp:1-176.

[12] P. M. de Zeeuw and E.J. van Asselt, 1985, *The Convergence Rate of Multilevel Algorithms Applied to the Convection-Diffusion Equation*, SIAM J. Sci. Stat. Comp., **6**, pp:492-503.

[13] P.W. Hemker, R. Kettler, P.Wesseling and P.M. de Zeeuw, 1983, *Multigrid Methods: Development of Fast Solvers*, Applied Mathematics and Computation, **13**, pp:311-326.

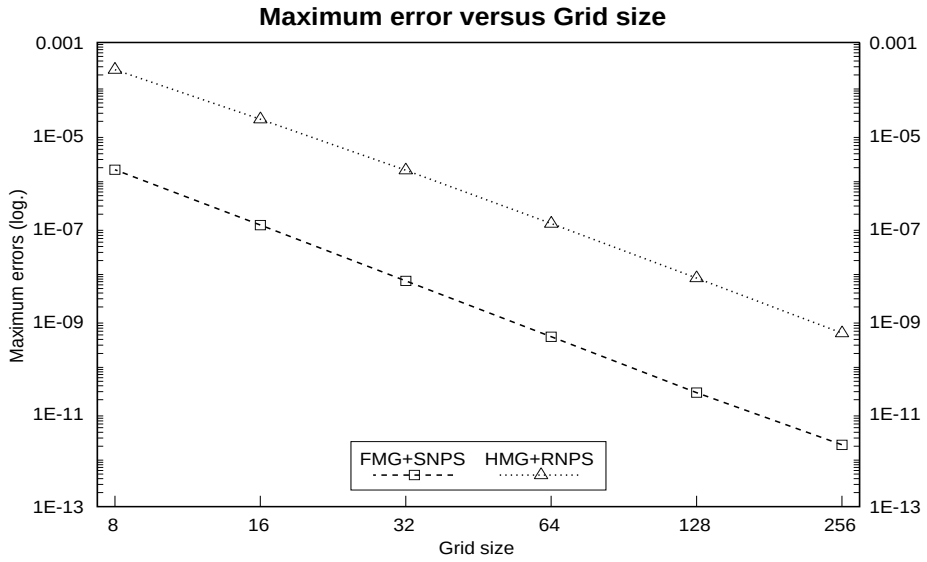

Figure 2: Maximum errors versus grid size.

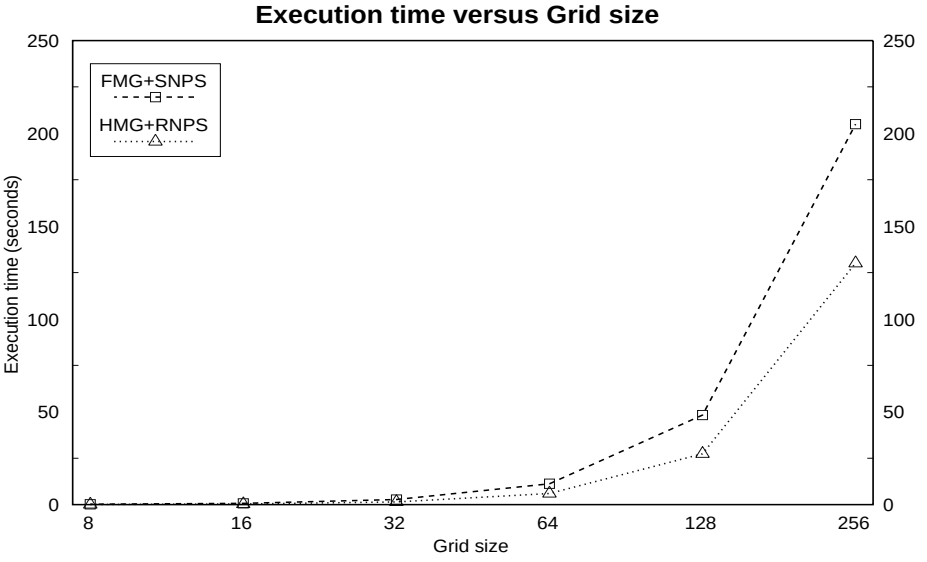

Figure 3: Execution time (second) versus grid size.