Studying Synthetic Geometry with Technology

Jen-chung Chuan

Department of Mathematics

National Tsing Hua University

Hsinchu, Taiwan 300

jcchuan@math.nthu.edu.tw

Introduction

Synthetic geometry is a branch of geometry that relies on the visual observation of figures and can be developed independently of number concepts and of algebraic notions. In the late nineteenth century synthetic geometry was developed to full perfection. During the twentieth century, however, the analytic aspect of geometry became the mainstream while the synthetic method fell into oblivion. Now a new kind of software, collectively referred to as dynamic geometry, is again arousing people's interest towards synthetic geometry.

In what follows we shall show how synthetic geometry can be investigated profitably under the environment of one dynamic geometry software, the Geometer's Sketchpad. Three examples, one from the area of elementary mathematics, one from the area of advanced mathematics, and one from the area of the linkage design, are given.

Horner's Method Visualized

The mathematical text "Arithmetic in Nine Sections", dated before 213 B.C., contains many interesting algorithms. One of them later led to the method closely related to Horner's method [1]. This important device for the numerical equation was officially invented in

1819, although Newton knew about it more than a century earlier [5]. The idea of the algorithm is this: in order to evaluate a polynomial function such as

$$p(x) = 2x^3 - 4x^2 + 7x - 1$$

at the point $x = 5$, it suffices to perform the series of substitutions

$$x_0 = 2$$

$$x_1 = x_0 \cdot 5 - 4$$

$$x_2 = x_1 \cdot 5 + 7$$

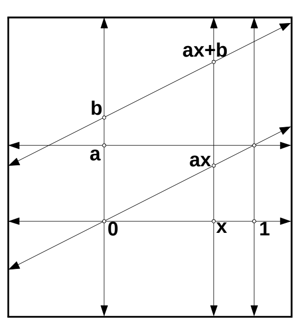
$$x_3 = x_2 \cdot 5 - 1$$

to obtain the result

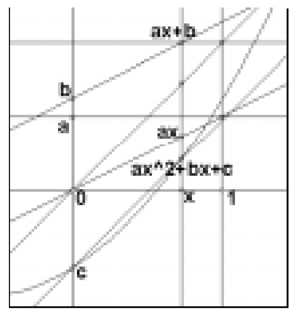
$$p(5) = x_3.$$

Thus the main feature of Horner's method is "substitution". We all know how frequently the process of substitution comes into mathematics, say in algebra or trigonometry or analytic geometry, which can be said to have substitution as its basal and central principle, or in calculus or the higher branches of analysis. Less obvious but no less vitally substitution enters pure geometry. We now show how geometric substitution leads to the synthetic construction of the graph of a polynomial function.

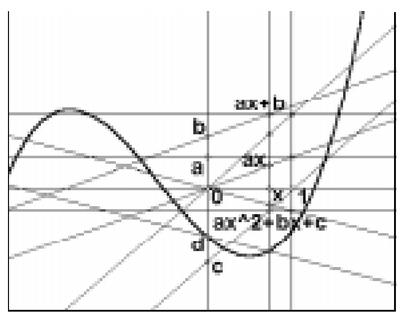
The basic component in the construction is this figure:



If we "substitute geometrically" the horizontal line passing through $ax + b$ by the horizontal line through a in the basic component, the graph of a general quadratic polynomial ax^2+bx+c can then be produced as a locus:



When the process is repeated again, the graph of an arbitrary cubic real polynomial $ax^3 + bx^2 + cx + d$ appears.

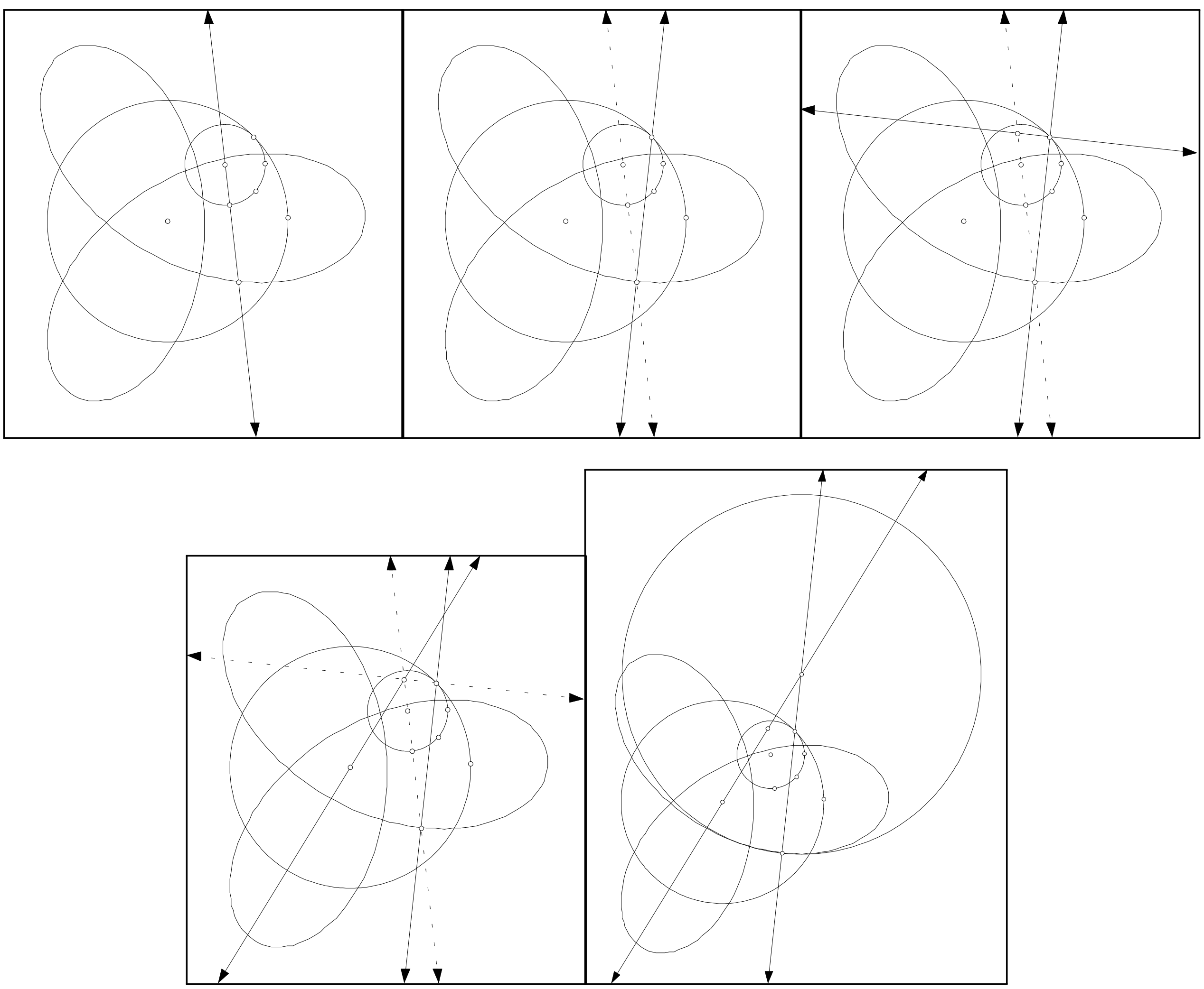


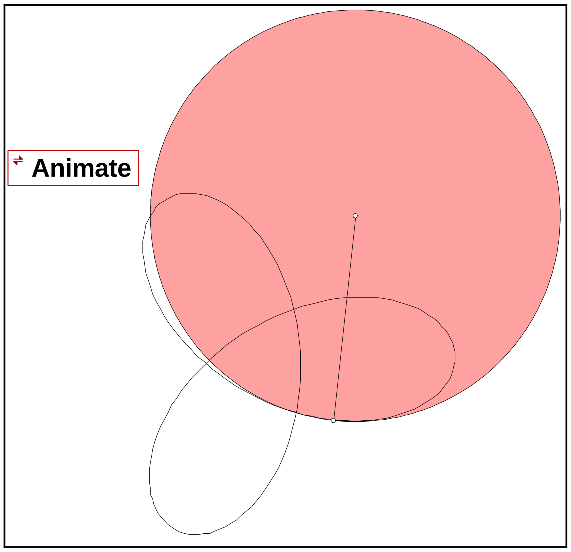
Each time the geometric substitution is performed the degree of the resulting polynomial increases by 1. It follows that any real polynomial can be generated this way. This method of polynomial construction, by no means the fastest, conveys a sense of transparency not to be found in the traditional black-box method offered by most mathematical software.

Osculating Circle

The falling of an apple inspired ideas not only in physics but also in mathematics, since the phenomenon involves the concept of the second derivative. Outside physics, the best way of visualizing the second derivative is to observe the dynamic appearance of the osculating circle of a curve other than the circle and the straight line. In our previous paper [2], the synthetic construction of the cyloidal curves was discussed. We now show that the construction may be extended to a wider class of curves known as the trochoids. According to Horton [3] the method is due to Savary.

The trochoid (also called the roulette) is the locus of a point rigidly attached to a curve that rolls upon a fixed curve. Epitrochoid (hypotrochoid) is the path of a point rigidly attached to a circle rolling upon a fixed circle. Clearly the epicycloids and the hypocycloids are particular varieties of trochoids. A generalization of the construction for hypocycloids may be used to construct the center of curvature of a hypotrochoid. These are the steps of the construction of the circle of curvature for the roulette traced by a point carried by a circle rolling on a fixed circle with diameter thrice as big:

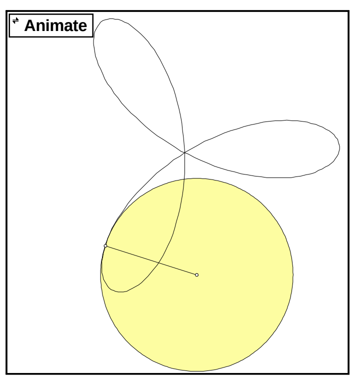




Many interesting curves can be realized as trochoids. For example, the ellipse and the rosette of three leaves, given in polar coordinates as

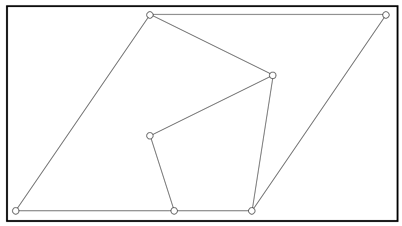
$$r = \cos 3\theta, 0 \leq \theta \leq 2\pi.$$

The above construction can be modified without toil for such curves.

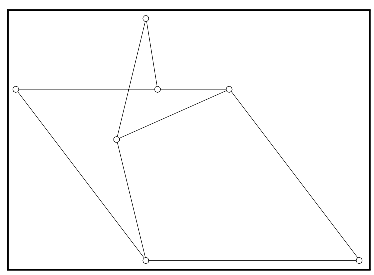


Linkage Formed by Kites and Darts

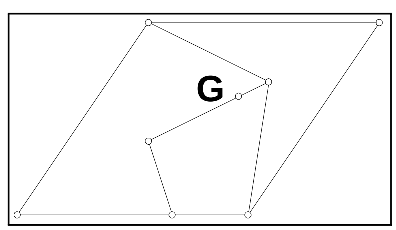
This interesting design of a linkage was announced by A.B. Kempe's lecture on linkages with the title "How to Draw a Straight Line", delivered in 1877 [4]:



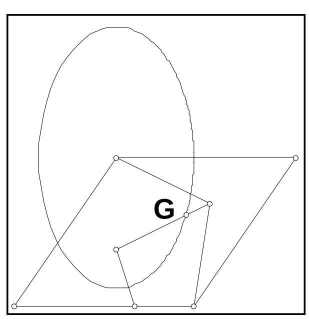
The figure consists of two similar "kites" together with a "dart" completing a parallelogram. Without performing the computer experiment it is hard to imagine that the same static figure can be deformed into one shown below, while the length of each line segment remains fixed:



If we mark some arbitrary point such as the point G,

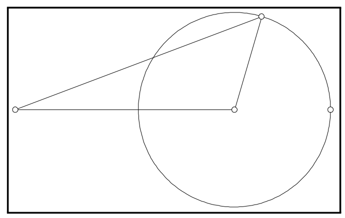


and explore its locus while the figure is altering its position, this picture emerges:

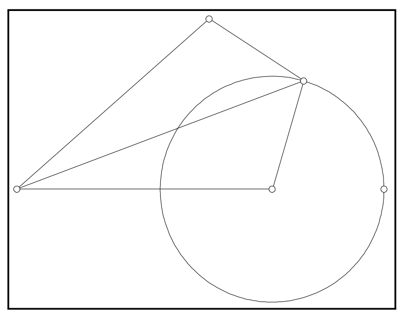


Yes, it is an ellipse. The construction of this linkage with Geometer's Sketchpad applies the full capabilities of the fundamental Euclidean transformations: reflection, rotation and translation. These are the steps of the construction

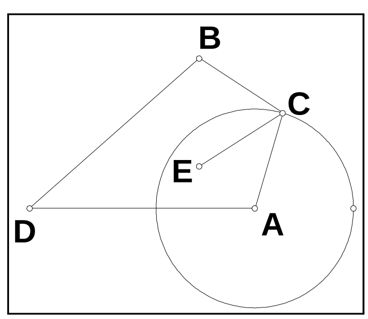
1. One-half of the bigger kite is formed by a triangle with fixed base and a variable third vertex located on the circumference of a circle.



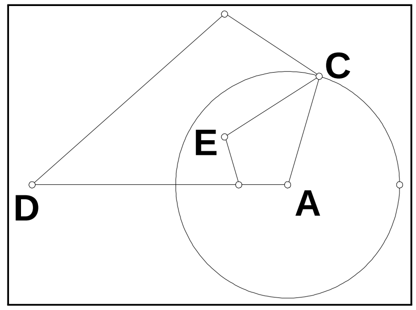
2. The bigger kite is formed by reflecting the triangle in the previous step.



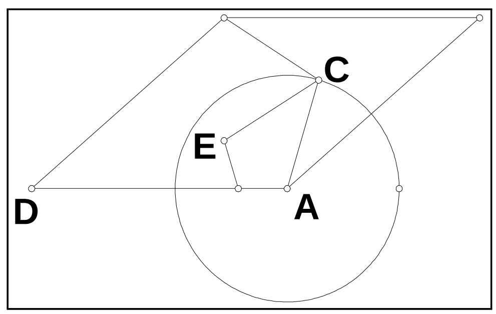
3. With the point C as center, rotate CA by the angle BDA to obtain the side CE of the smaller kite.



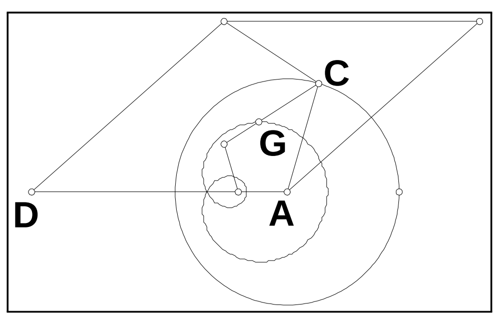
4. The fourth vertex of the smaller kite is found by taking the intersection of AD with the bisector of the angle CAE.



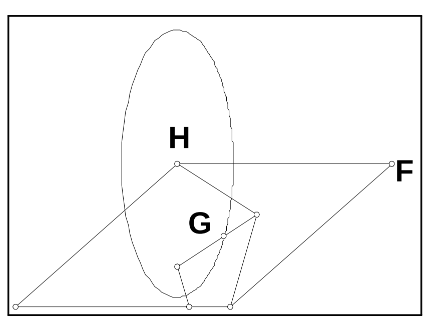
5. The dart is constructed by completing the parallelogram..



6. As the point C moves along the circle, the point G does NOT trace out an ellipse.



7. If, however, we allow the points H and F remain stationary while all other objects moving, the point G will trace out an ellipse as promised.



The last step can be achieved through a translation of the whole configuration.

References

1. F. Cajori, A History of Mathematics, Chelsea, 1991.

2. Jen-chung Chuan, Geometric Constructions with the Computer, Proceedings of the First Asian Technology Conference in Mathematics, 1995, pp. 329-338.

3. G. Horton, Concerning roulettes, Amer. Math. Monthly, 23, pp. 237-241.

4. A.B. Kempe, How to draw a straight line, a lecture on linkage, reprinted in Squaring the Circles and Other Monographs (edited byE.W. Hobson), 1953.

5. D. E. Knuth, The Art of Computer Programming, Vol. 2, Addison-Wesley, 1968.